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UNIVERSITES PARIS VI et VII
Projets Logique Mathématique et Théorie des Nombres
Institut de Mathématiques de Jussieu – UMR 7586

SEMINAIRE DE STRUCTURES ALGEBRIQUES ORDONNEES

Responsables: F. Delon, M. Dickmann, D. Gondard, T. Servi

2016-2017

Liste des exposés

31/01/17 **Salma KUHLMANN** (Universität Konstanz, Allemagne)
Quasi-ordered algebraic structures: a uniform approach to ordering and valuations.

28/02/17 **Maja RESMAN** (Université de Zagreb, Croatie)
Transseries and formal normal forms

28/03/17 **Journée thématique “Algèbres quasi-analytiques d'Ilyashenko et o-minimalité”**

Le problème de Dulac à paramètres concerne l’existence de bornes uniformes sur le nombre de cycles limites d’une famille de champs de vecteurs analytiques dans le plan. Une approche au problème de Dulac à travers la o-minimalité a été explorée, sous certaines hypothèses, par Kaiser, Rolin et Speissegger en 2009.
Cette journée est dédiée à l’exposition des travaux en cours des orateurs, dans l’esprit d’une solution du problème de Dulac à paramètres grâce à des techniques o-minimales.
Programme de la journée :

10:00 - 11:30 **Patrick SPEISSEGGER** (Universität Konstanz, Allemagne/McMaster University, Canada)
Quasianalytic Ilyashenko algebras I.

14:00 - 15:00 **Tobias KAISER** (Universität Passau, Allemagne)
Holomorphic extensions of germs definable in R_an,exp.

15:15 - 15:45 **Zeinab GALAL** (Université Paris Diderot)
Quasianalytic Ilyashenko algebras II.

02/05/17 **Fred WEHRUNG** (Université de Caen)
Espaces spectraux de groupes réticulés Abéliens.

16/05/17 **Françoise DELON** (Université Paris 7-Denis Diderot)
Dérivabilité des fonctions définissables dans un corps valué C-minimal.

23/05/17 **Christopher MILLER** (The Ohio State University, Columbus, Ohio, USA)
Component-closed expansions of the real line.

30/05/17 **Tim NETZER** (Universität Innsbruck, Autriche)
On non-commutative quantifier elimination in real algebra.

06/06/17 **Daniel PLAUMANN** (Universität Dortmund, Allemagne)
Hyperbolic polynomials and interlacers.

13/06/17 **Ayhan GÜNAYDIN** (Bogazici University, Turquie)
Topological study of pairs of algebraically closed fields.

UNIVERSITES PARIS VI et VII

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SEMINAIRE DE STRUCTURES ALGEBRIQUES ORDONNEES

Responsables: F. Delon, M. Dickmann, D. Gondard, T. Servi

2016-2017

Liste des contributions

Friedrich WEHRUNG (Université de Caen)

Spectral Spaces of Countable Abelian Lattice-Ordered Groups.
(Extended Abstract)

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Fonctions définissables dans un corps valué C-minimal.

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Component-closed Expansions of the Real Line. (Preliminary Report)

Tim NETZER (Universität Innsbruck, Autriche)

On Non-Commutative Quantifier Elimination in Real Algebra.
(Extended Abstract)

December 6, 2017

COMPATIBILITY OF QUASI-ORDERINGS AND VALUATIONS; A BAER-KRULL THEOREM FOR QUASI-ORDERED RINGS

SALMA KUHLMANN, SIMON MÜLLER

Abstract. In his work of 1969, Merle E. Manis introduced valuations on commutative rings. Recently, the class of totally quasi-ordered rings was developped in [10]. In the present paper, given a quasi-ordered ring $(R, \preceq)$ and a valuation v on R, we establish the notion of compatibility between v and $\preceq$, leading to a definition of the rank of $(R, \preceq)$.
Moreover, we prove a Baer-Krull Theorem for quasi-ordered rings: fixing a Manis valuation v on R, we characterize all v-compatible quasi-orders of R by lifting the quasi-orders from the residue class ring to R itself.

1. Introduction

There have been several attempts to find a uniform approach to orders and valuations. In [2] for instance, Ido Efrat simply defined localities on a field to be either orders or valuations. S.M. Fakhruddin introduced the notion of (totally) quasi-ordered fields $(K, \preceq)$ and proved the dichotomy, that any such field is either an ordered field or else there exists a valuation v on K such that $x \preceq y$ if and only if $v(y) \leq v(x)$ ([4, Theorem 2.1]). Thus, Fakhruddin found a way to treat these two classes simultaneously. Inspired by this result, the second author of this paper established the said dichotomy for commutative ring with 1 ([10, Theorem 4.6].

The aim of the present paper is to continue our study of quasi-ordered rings. For this purpose we consider important results from real algebra, which are also meaningful if the order is replaced by a quasi-order. The paper is organized as follows:

In section 2 we briefly recall ordered and valued rings, and give our definition of quasi-ordered rings (see Definition 2.5). Moreover, we quote the two theorems that we want to establish for this class (see Theorems 2.7 and 2.8).

Section 3 deals with the notion of compatibility between quasi-orders and valuations. Given a quasi-ordered ring $(R, \preceq)$, we first give a characterization of all Manis valuations (i.e. surjective valuations) v on R that are compatible with $\preceq$ (see Theorem 3.12). In case where $\preceq$ also comes from a Manis valuation, say w, we will show that v is compatible with $\preceq$ if and only if v is a coarsening of w (see Lemma 3.16), leading to a characterization of all the Manis coarsenings v of w (see Theorem 3.17). We conclude this section by developing a notion of rank of a quasi-ordered ring (see Definition 3.27).

In the fourth and final section we establish a Baer-Krull Theorem for quasi-ordered rings (see Theorem 4.10, respectively Corollary 4.11). Once this is proven, we can not only generalize the classical Baer-Krull Theorem to ordered rings (see Corollary 4.13), but also characterize all Manis refinements w of a valued ring (R, v), given that v is also Manis (see Corollary 4.19).

The authors want to thank Vicky Powers, who aroused our interest in valued rings.

2. Preliminaries

Here we briefly introduce some basic results concerning valued, ordered and quasi-ordered rings. Moreover, we introduce the theorems, which we aim to establish for quasi-ordered rings, in the ordered field case (see Theorems 2.7 and 2.8). In this section let R always denote a commutative ring with 1.

Definition 2.1. (see [1, VI. 3.1]) Let $(\Gamma, +, \leq)$ be an ordered abelian group and ∞ a symbol such that $\gamma < \infty$ and $\infty = \infty + \infty = \gamma + \infty = \infty + \gamma$ for all $\gamma \in \Gamma$. A map $v : R \to \Gamma \cup \{\infty\}$ is called a **valuation** on R if $\forall x, y \in R$:

(V1) $v(0) = \infty$,
(V2) $v(1) = 0$,
(V3) $v(xy) = v(x) + v(y)$,
(V4) $v(x + y) \geq \min\{v(x), v(y)\}$.

We always assume that Γ is the group generated by $\{v(x) : x \in v^{-1}(\Gamma)\}$ and call it the **value group** of R. We also denote it by Γ_v. We call v **trivial** if Γ_v is trivial, i.e. if $\Gamma_v = \{0\}$. The set $\mathfrak{q}_v := \text{supp}(v) := v^{-1}(\infty)$ is called the **support** of v.

Facts 2.2.
(1) An easy consequence of these axioms is that $\mathfrak{q}_v$ is a prime ideal.
(2) In general, v is not surjective, as $v(R \backslash \mathfrak{q}_v)$ is not necessarily closed under additive inverses. However, if x is a unit, then $v(x^{-1}) = -v(x)$.
(3) The subring $R_v := \{x \in R : v(x) \geq 0\}$ of R is said to be the **valuation ring** of v. The prime ideal $I_v := \{x \in R : v(x) > 0\}$ of R is called the **valuation ideal**. If R is a field, R_v is a local ring with maximal ideal I_v.

We conclude our introduction of valuations with a simple but very helpful lemma.

Lemma 2.3. *Let (R, v) be a valued ring and $x, y \in R$ such that $v(x) \neq v(y)$. Then $v(x + y) = \min\{v(x), v(y)\}$.*

Proof. Completely analogue as in the field case, see for instance [3, (1.3.4)]. □

Orders are often identified with positive cones $P \subset R$, where $x \in P$ means that x is non-negative. However, recall from the introduction that some quasi-orders are induced by a valuation v via $x \preceq y$ if and only if $v(y) \leq v(x)$. But then all elements are non-negative. Hence, positive cones are inappropriate to deal with quasi-orders. Therefore, we decided to give a binary definition of ordered rings here.

Definition 2.4. (see [10, Definition 2.3]) Let $\leq$ be a binary, reflexive, transitive and total relation on R. Then $(R, \leq)$ is called an **ordered ring** if $\forall x, y, z \in R$:

(O1) $0 < 1$,
(O2) $xy \leq 0 \Rightarrow x \leq 0 \vee y \leq 0$,
(O3) $x \leq y,\ 0 \leq z \Rightarrow xz \leq yz$,
(O4) $x \leq y \Rightarrow x + z \leq y + z$.

While (O3) and (O4) express the usual compatibility of $\leq$ with $\cdot$ and $+$, the axioms (O1) and (O2) yield that the support $\mathfrak{p}_{\leq} := \{x \in R : 0 \leq x \leq 0\}$ is a prime ideal.

Next, we recall quasi-ordered rings, which were developped in [10].

Definition 2.5. (see [10, Definition 3.2]) Let R be a commutative ring with 1 and $\preceq$ a binary, reflexive, transitive and total relation on R. If $x, y \in R$, we write $x \sim y$ if $x \preceq y$ and $y \preceq x$, and we write $x \prec y$ if $x \preceq y$ but $y \npreceq x$.
The pair $(R, \preceq)$ is called a **quasi-ordered ring** if $\forall x, y, z \in R$:

(QR1) $0 \prec 1$,
(QR2) $xy \preceq 0 \Rightarrow x \preceq 0 \vee y \preceq 0$,
(QR3) $x \preceq y,\ 0 \preceq z \Rightarrow xz \preceq yz$,
(QR4) $x \preceq y,\ z \nsim y \Rightarrow x + z \preceq y + z$,

(QR5) If $0 \prec z$, then $xz \preceq yz \Rightarrow x \preceq y$.

We write E_x for the equivalence class of x w.r.t. $\sim$. E_0 is called the **support** of $\preceq$.

In [10, Theorem 4.6], the second author proved that a quasi-ordered ring $(R, \preceq)$ is either an ordered ring or a valued ring (R, v) such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$. Thus, via quasi-ordered rings, we can treat ordered and valued rings simultaneously.

Remark 2.6.

(1) If $(R, \preceq)$ is a quasi-ordered ring with $x \sim 0$ and $y \nsim 0$, then $x + y \sim y$ (see [10, Lemma 3.6]). This result will be useful later on.
(2) As indicated in the previous theorem, the support E_0 is a prime ideal of R (see [10, Proposition 3.8]).
(3) The "new" axiom (QR5) is crucial for the dichotomy, see [10, Proposition 3.1]. Moreover, note that it easily implies (QR2). Indeed, if $xy \preceq 0$ and $0 \prec x$, then (QR2) yields $y \preceq 0$. However, we decided to keep axiom (QR2) in order to preserve the analogy between ordered and quasi-ordered rings.
(4) Later on we will also need the following variant of axiom (QR5): For $x, y, z \in R$, if $z \nsim 0$, then $xz \sim yz \Rightarrow x \sim y$ (see [10, Lemma 3.7]).

We conclude this introductive section by recalling the Theorems 2.7 and 2.8 below, which we will establish for quasi-ordered rings in this paper. So let $(K, \leq)$ be an ordered field. Recall that a valuation v on K is said to be compatible with $\leq$, if $0 \leq x \leq y$ implies $v(y) \leq v(x)$ (see for instance [8, Definition 2.4]). A subset $S \subseteq K$ is convex (w.r.t. $\leq$), if from $x \leq y \leq z$ and $x, z \in S$ follows $y \in S$.

Theorem 2.7. (see [8, Theorem 2.3 and Proposition 2.9] or [3, Proposition 2.2.4]) *Let $(K, \leq)$ be an ordered field and let v be a valuation on K. The following are equivalent:*

(1) v is compatible with $\leq$,
(2) the valuation ring K_v is convex,
(3) the maximal ideal I_v is convex,
(4) $I_v < 1$,
(5) $\leq$ induces canonically via the residue map $\varphi_v : K_v \to Kv := K_v/I_v$, $x \mapsto x + I_v$ an order $\leq'$ on the residue field Kv.

Condition (5) is crucial for the second theorem, the so called Baer-Krull Theorem (see [3, p.37]). Let K again be a field and v a valuation on K with value group Γ_v. Note that $\overline{\Gamma_v} = \Gamma_v/2\Gamma_v$ is in a canonical way an $\mathbb{F}_2$-vector space. Hence, we find a subset $\{\pi_i : i \in I\} \subset K$, such that $\{\overline{v(\pi_i)} : i \in I\}$ is an $\mathbb{F}_2$-basis of $\overline{\Gamma_v}$.

Theorem 2.8. (*Baer-Krull Theorem for ordered fields;* see [3, Theorem 2.2.5]) *Let K be a field and v a valuation on K. Moreover, let $\mathcal{X}(K)$ and $\mathcal{X}(Kv)$ denote the set of all orderings on K, respectively Kv. There exists a bijective map*

$$\psi : \{\leq \in \mathcal{X}(K) : \leq \text{ is } v\text{-compatible}\} \to \{-1, 1\}^I \times \mathcal{X}(Kv),$$

described as follows: given an ordering $\leq$ in the domain of ψ, let $\eta_{\leq} : I \to \{-1, 1\}$, where $\eta_{\leq}(i) = 1 \Leftrightarrow 0 \leq \pi_i$. Then the map $\leq \mapsto (\eta_{\leq}, \leq')$ is the above bijection, where $\leq'$ denotes the order on Kv from Theorem 2.7(5).

3. Compatibility between quasi-orders and valuations

The aim of this section is to prove an analogue of Theorem 2.7 for quasi-ordered rings. First we convince ourselves that for this end, we have to restrict our attention to surjective valuations (see Example 3.2), also called Manis valuations. Then we establish that the conditions (1) - (3) and (5) from the said theorem are equivalent for quasi-ordered rings, if v is Manis (see Theorem 3.12). This gives rise to a characterization of all Manis valuations w on R, which are coarser than v (see Theorem 3.17). Afterwards, we prove that $I_v \prec 1$ is no equivalent condition anymore, no matter of which of the two kinds the quasi-order is (see Examples 3.19 and 3.20). Furthermore, we show that Theorem 2.7 holds to the full extend, if we additionally demand that v is local (see Lemma 3.23). We conclude this section by establishing the notion of rank of a quasi-ordered ring (see Definition 3.27).

Notation 3.1. Let R always denote a commutative ring with 1. If a quasi-order $\preceq$ on R is induced by some valuation w on R, we also write $\preceq_w$ instead of $\preceq$ and call it a **proper quasi-order** (p.q.o). The symbol $\leq$ is reserved for orders.
If v is a valuation on R, we denote by $R_v := \{x \in R : v(x) \geq 0\}$ the **valuation ring** of v, by $I_v := \{x \in R : v(x) > 0\}$ the valuation ideal, and by $U_v := R_v \backslash I_v := \{x \in R : v(x) = 0\}$. Last but not least, $Rv := R_v/I_v$ denotes the **residue class ring** of v and $\varphi_v : R_v \to Rv,\ x \mapsto x + I_v$ the **residue map**.

In general, we cannot expect that Theorem 2.7 holds even for ordered rings:

Example 3.2. Consider $v : \mathbb{Z}[X] \to \mathbb{Z} \cup \{\infty\},\ f = \sum_{i\in\mathbb{N}} a_i X^i \mapsto -\deg f$. It is easy to verify that v is a valuation on R. We can extend the unique order on $\mathbb{Z}$ to $\mathbb{Z}[X]$ by declaring $0 \leq f :\Leftrightarrow 0 \leq f(0)$. Note that $R_v = \mathbb{Z}$ and $I_v = \{0\}$, so obviously the conditions (4) and (5) of Theorem 2.7 are satisfied. However, the inequalities $0 \leq X \leq 0$ yield that neither I_v nor R_v is convex with respect to $\leq$. Moreover, $0 \leq X + 1 \leq 1$, but $v(X + 1) = -1 < 0 = v(1)$, so (1) is also not satisfied.

Such counterexamples can be prevented by demanding surjectivity of v.

Definition 3.3. (see [9, p.193]) Let v be a valuation on R. Then v is said to be a **Manis valuation**, if v is surjective.

We now turn towards the proof of Theorem 2.7 for quasi-ordered rings.

Definition 3.4. Let $(R, \preceq)$ be a quasi-ordered ring. A valuation v on R is said to be **compatible** with $\preceq$ (or $\preceq$-compatible), if $\forall y, z \in R : 0 \preceq y \preceq z \Rightarrow v(z) \leq v(y)$.

Definition 3.5. Let $(R, \preceq)$ be a quasi-ordered ring and $S \subseteq R$ a subset of R. Then S is said to be **convex**, if $x \preceq y \preceq z$ and $x, z \in S$ implies $y \in S$.

The following lemma simplifies convexity in a usual manner and holds particularly for the valuation ring R_v and its prime ideal I_v, as $v(x) = v(-x)$ for all $x \in R$.

Lemma 3.6. *Let $(R, \preceq)$ be a quasi-ordered ring. A subset $S \subseteq R$ with $0 \in S$ and $S = -S$ is convex, if and only if $0 \preceq y \preceq z$ and $z \in S$ implies $y \in S$.*

Proof. The implication $\Rightarrow$ is trivial. So suppose that the right hand side holds and let $x \preceq y \preceq z$ with $x, z \in S$. If $0 \preceq y$, it follows immediately by assumption that $y \in S$. So suppose that $y \prec 0$. Then $x \preceq y \prec 0$. We will show $0 \prec -y \preceq -x$. Note that $-x \in S$ because $S = -S$. Hence, we obtain $-y \in S$, but then also $y \in S$.
Clearly $0 \prec -x, -y$ by axiom (QR4) and the fact that E_0 is an ideal (see Remark 2.6(2)). It remains to show that $-y \preceq -x$. Assume for a contradiction $-y \not\preceq -x$, so $-x \prec -y$. Note that $y \prec 0 \prec -x, -y$, therefore $-x \not\sim y$ and $y \not\sim -y$. Via (QR4), it follows from $x \preceq y$ that $0 \preceq y - x$ and from $-x \preceq -y$ that $y - x \preceq 0$. Thus, $y - x \in E_0$. This implies $-y \sim -x$ (see Remark 2.6(2)), a contradiction. □

The most difficult part of the proof will be to show that if v is a $\preceq$-compatible valuation on R, then $\preceq$ induces a quasi-order on the residue class ring Rv. For this implication we want to exploit convexity of I_v.

Lemma 3.7. *Let $(R, \preceq)$ be a quasi-ordered ring, v a valuation on R such that I_v is convex, and $u \in U_v$.*

(1) If $c \in I_v$, then $c \not\sim u$.
(2) If $0 \prec u$, then $0 \prec u + c$ for all $c \in I_v$.
(3) If $u \prec 0$, then $u + c \prec 0$ for all $c \in I_v$.

Proof. For (1) suppose that $c \sim u$. Then $c \preceq u \preceq c$, so convexity of I_v yields $u \in I_v$, a contradiction. Next, assume that $0 \prec u$, but $0 \not\prec u+c$ for some $c \in I_v$. Then $0 \prec u$ and $u + c \preceq 0$. This implies $c \notin E_0$, as otherwise $u \sim u + c$ (see Remark 2.6(1)). Hence, we obtain $u \preceq -c$. So it holds $0 \prec u \preceq -c$. Convexity of I_v yields $u \in I_v$, a contradiction. Thus, (2) is proven. Finally suppose that $0 \preceq u + c$ for some $c \in I_v$. Then $u \prec 0 \preceq u+c$. It remains to show that $-u \not\sim u+c$. Then $0 \prec -u \preceq c$ and one may conclude by convexity of I_v. So assume for a contradiction that $-u \preceq u + c$. From Lemma 2.3 follows $u + c \in U_v$, so (1) yields that $-c \not\sim u + c$. Thus, one obtains $-u - c \preceq u$. Now note that $0 \prec -u \in U_v$. So (2) yields $0 \prec -u - c$. Therefore $0 \prec -u - c \preceq u \prec 0$, a contradiction. This finishes the proof. □

Moreover, we require a couple of results that Fakhruddin established in the more specific setting of quasi-ordered fields (see [4]).

Lemma 3.8. *Let $(R, \preceq)$ be a quasi-ordered ring and $x \in R$. Then $x \sim -x$ if and only if $0 \preceq x, -x$.*

Proof. Just as in the case of quasi-ordered fields, see [4, Lemma 3.1]. □

Lemma 3.9. *Let $(R, \preceq)$ be a quasi-ordered ring and $x, y \in R$. If $x \sim y$, then $x \sim -y$ or $0 \sim x - y$.*

Proof. If $x, y \sim 0$, then $x \sim -y$, as E_0 is an ideal. So suppose that $x, y \not\sim 0$. We show $0 \sim x - y$. Note that $y \preceq x \not\sim -y$. Therefore $0 \preceq x - y$. Moreover, $x \preceq y \sim x \not\sim -y$, so $y \not\sim -y$, and therewith $x - y \preceq 0$. Thus, $0 \sim x - y$. □

Corollary 3.10. *Let $(R, \preceq)$ be a quasi-ordered ring. Then $\sim$ is preserved under multiplication, i.e. if $x, y, a \in R$ such that $x \sim y$, then $ax \sim ay$.*

Proof. The cases $0 \preceq a$ (axiom (QR3)) and x, y in E_0 (E_0 is an ideal) are both trivial. So suppose that $0 \not\preceq a$ and $x, y \not\sim 0$. Then $0 \preceq -a$. The previous lemma gives rise to a case distinction. First suppose $0 \sim x - y$. Since $-x \not\sim 0$ it holds $-x \not\sim x - y$. Hence, $0 \sim x - y$ yields $-x \sim -y$. Since $0 \preceq -a$, axiom (QR3) yields $ax \sim ay$. Now suppose that $0 \not\sim x - y$. Then also $0 \not\sim y - x$. The previous lemma implies $x \sim -y$ and $y \sim -x$. Therefore $-y \sim x \sim y \sim -x$. Since $0 \preceq -a$, we obtain $ay = (-a)(-y) \sim (-a)(-x) = ax$. □

Lemma 3.11. *Let $(R, \preceq)$ be a quasi-ordered ring such that $0 \prec -1$. Then it holds $x + y \preceq \max\{x, y\}$ for all $x, y \in R$.*

Proof. Basically as in the field case, see [4, Lemma 4.1]. Suppose $x \preceq y$. Assume for a contradiction that $y \prec x + y$. Note that $0 \preceq 1$ by axiom (QR1). Lemma 3.8 implies $-1 \sim 1$, so the previous corollary yields $-r \sim r$ for all $r \in R$. It follows $-x \sim x \preceq y \prec x + y$. Particularly, $-y \not\sim x + y$, since $y \not\sim x + y$. So, by applying (QR4), we obtain $x + y \sim -x - y \preceq x \preceq y$, a contradiction. □

Finally, we can prove the main theorem of this section:

Theorem 3.12. *Let $(R, \preceq)$ be a quasi-ordered ring and v a Manis valuation on R.*
(a) *The following are equivalent:*
  (1) *v is compatible with $\preceq$.*
  (2) *I_v is convex.*
  (3) *$\preceq$ induces canonically via the residue map $x \mapsto x + I_v$ a quasi-order $\preceq'$ with support $\{0\}$ on the residue class ring Rv.*

Moreover, any of these conditions implies that R_v is convex.

(b) *If v is non-trivial, the following is equivalent to* $(1)-(3)$:

  (4) *R_v is convex*

Proof. **(a)** We first prove that (1) and (2) are equivalent. If $0 \preceq y \preceq z$ with $z \in I_v$, then (1) yields that $0 < v(z) \leq v(y)$, and therefore $y \in I_v$. Conversely, assume for a contradiction that there exist some $0 \preceq y \preceq z$ such that $v(y) < v(z)$. Then $y \notin \mathfrak{q}_v$. Since v is Manis, we find some $0 \preceq a$ such that $v(a) = -v(y)$ (if $a \prec 0$, then $0 \preceq -a$ and $v(a) = v(-a)$). Via axiom (QR3) follows $0 \preceq ay \preceq az$ with $v(ay) = 0$ and $v(az) = v(z) - v(y) > 0$, so $az \in I_v$ but $ay \notin I_v$. This contradicts (2).

Next, we show that (2) and (3) are equivalent. First suppose that (3) holds and let $0 \preceq y \preceq z$ with $z \in I_v$. Assume for a contradiction $y \notin I_v$. Choose $a \in R$ with $0 \preceq a$ and $v(a) = -v(y)$. Then $0 \preceq ay \preceq az$ with $v(ay) = 0$ and $v(az) > 0$. Taking residues, it follows $0 \preceq' \overline{ay} \preceq' 0$. Since the support of $\preceq'$ is trivial, this yields that $\overline{ay} = 0$, contradicting $v(ay) = 0$, i.e. $ay \notin I_v$. Therefore, $y \in I_v$.

Now suppose that (2) holds. The quasi-order induced by the residue map is given by $\overline{x} \preceq' \overline{y} :\Leftrightarrow \exists c_1, c_2 \in I_v : x + c_1 \preceq y + c_2$. First of all we verify that $\preceq'$ is well-defined. So assume that $\overline{x} \preceq' \overline{y}$, and let $\overline{x} = \overline{x_1}$ and $\overline{y} = \overline{y_1}$, say $x = x_1 + c_1$ and $y = y_1 + c_2$ for some $c_1, c_2 \in I_v$. There exist some $c_3, c_4 \in I_v$ such that $x + c_3 \preceq y + c_4$. But then $x_1 + (c_1 + c_3) \preceq y_1 + (c_2 + c_4)$, thus, $\overline{x_1} \preceq' \overline{y_1}$.
Clearly, $\preceq'$ is reflexive and total. To show transitivity, assume that $x + c_1 \preceq y + c_2$ and $y + d_1 \preceq z + d_2$ for some $c_1, c_2, d_1, d_2 \in I_v$. We argue by case distinction. First suppose that $y \in U_v$. Assume for a contradiction that $x + e_1 \succ z + e_2$ for all $e_1, e_2 \in I_v$. In particular, $x + c_1 + d_1 - c_2 \succ z + d_2$. Note that $y + c_2 \in U_v$ and $d_1 - c_2 \in I_v$, so Lemma 3.7(1) yields $y + c_2 \not\sim d_1 - c_2$. So from the inequality $x + c_1 \preceq y + c_2$ follows $x + c_1 + d_1 - c_2 \preceq y + c_2 + d_1 - c_2 = y + d_1 \preceq z + d_2$, a contradiction. If $y \in I_v$, then $y + c_2$ and $y + d_1 \in I_v$. By convexity and Lemma 3.7(2) and (3), this yields that x is either a negative unit or in I_v, and that z is either a positive unit or in I_v. We only have to consider the case where both elements are in the ideal. But then $x + (z - x) \preceq z + 0$, and thus $\overline{x} \preceq' \overline{z}$.
Now we show that the support of $\preceq'$ equals $\{0\}$. So let $\overline{x} \sim 0$ and assume for a contradiction that $x \in R_v \backslash I_v = U_v$. Then there exist $c_1, c_2 \in I_v$ such that $x + c_1 \preceq c_2$ and there exist $d_1, d_2 \in I_v$ such that $d_1 \preceq x + d_2$. If $0 \prec x$, then $0 \prec x + c_1 \preceq c_2$ by Lemma 3.7(2), and we have $x + c_1 \in I_v$ by convexity, a contradiction. Likewise, if $x \prec 0$, then $d_1 \preceq x + d_2 \prec 0$ by Lemma 3.7(3), again contradicting the convexity.
It remains to check the axioms (QR1) and (QR3) - (QR5) (see Remark 2.6(3)).
For **(QR1)**, assume for a contradiction $\overline{1} \preceq' \overline{0}$. Then there exist $c_1, c_2 \in I_v$ such that $0 \prec 1 + c_1 \preceq c_2$ (Lemma 3.7(2)). Convexity of I_v yields $1 + c_1 \in I_v$, and therefore $1 \in I_v$, a contradiction. Thus, $\overline{0} \prec' \overline{1}$.
To prove **(QR3)**, let $0 \preceq' \overline{x}$ and $\overline{y} \preceq' \overline{z}$ For $\overline{x} = 0$, there is nothing to show, so we suppose that $x \in U_v$. From $0 \preceq' \overline{x}$ follows that there are some $c_1, c_2 \in I_v$ such that $c_1 \preceq x + c_2$. Applying Lemma 3.7(1) yields that $c_1 - c_2 \preceq x$. So convexity of I_v gives us $0 \preceq x$. Moreover, $\overline{y} \preceq' \overline{z}$ means $y + d_1 \preceq z + d_2$ for some $d_1, d_2 \in I_v$. (QR3) implies $xy + xd_1 \preceq xz + xd_2$, and therefore $\overline{xy} \preceq \overline{xz}$.
For **(QR4)** we have to prove that $\overline{x} \preceq' \overline{y}$ and $\overline{y} \not\sim' \overline{z}$ yields $\overline{x+z} \preceq' \overline{y+z}$. Let $c_1, c_2 \in I_v$ such that $x + c_1 \preceq y + c_2$. Note that $\overline{y} \not\sim' \overline{z}$ implies either $\forall e_1, e_2 :$ $y + e_1 \prec z + e_2$ or $\forall e_1, e_2 : y + e_1 \succ z + e_2$. Either way, $z \not\sim y + c_2$. But then $x + z + c_1 \preceq y + z + c_2$ by (QR4), i.e. $\overline{x+z} \preceq' \overline{y+z}$.
Let us finally prove **(QR5)**. We have to show that if $0 \prec' \overline{a}$, then $\overline{ax} \preceq' \overline{ay}$ implies $\overline{x} \preceq' \overline{y}$. Note that if $ax \preceq ay$, then $x \preceq y$ by axiom (QR5), hence $\overline{x} \preceq' \overline{y}$. So from now on assume that $ay \prec ax$. First we show that one may also assume that $x, y \in U_v$. Indeed, suppose that $\overline{x} = 0$ and assume for a contradiction that $\overline{y} \prec' 0$. Then $\overline{ay} \preceq' 0$ by axiom (QR3). But $=$ cannot hold because neither $a \in I_v$, nor $y \in I_v$.

Thus, $\overline{ay} \prec' 0 = \overline{ax}$, contradicting the assumption. Now suppose that $\overline{y} = 0$ and assume for a contradiction that $0 \prec' \overline{x}$. Then $\overline{ay} = 0 \prec \overline{ax}$, again a contradiction. Hence, one may assume that both x and y lie in U_v. So from $\overline{ax} \preceq' \overline{ay}$ follows that there exists some $c \in I_v$ such that $ax \preceq ay + c$. Thus, it holds $ay \prec ax \preceq ay + c$. The rest of the proof is done by case distinction.
If $0 \prec -1$, then $0 \preceq -r$ for all $r \in R$ with $0 \preceq r$ by (QR3). Lemma 3.8 yields that all elements are non-negative. Particularly, since ay is a unit and I_v is convex, it holds $c \prec ay$ (otherwise $0 \prec ay \preceq c \in I_v$). From Lemma 3.11 follows $ay \prec ay + c \preceq \max\{ay + c\} = ay$, a contradiction.
Finally suppose $-1 \prec 0$. Consider the inequalities $ay \prec ax \preceq ay + c$. By Lemma 3.7(2) and (3), ay and $ay + c$ have the same sign, and so ax has also the same sign, which is contrary to the sign of $-ay$. Particularly, we may add $-ay$ to these two inequalities and obtain $0 \preceq a(x-y) \preceq c$. By convexity of I_v follows $a(y-x) \in I_v$ and since I_v is a prime ideal with $a \notin I_v$, one obtains $\overline{x} = \overline{y}$. Thus, $\overline{x} \preceq' \overline{y}$.

The convexity of R_v follows immediately from (1), just like the convexity of I_v.

(b) It suffices to show that (4) implies (2). So let $0 \preceq y \preceq z$ with $z \in I_v$. Assume for a contradiction that $y \notin I_v$, so by convexity of R_v it holds $y \in R_v - I_v = U_v$, i.e. $v(y) = 0$. Since $z \in I_v$, it holds $\gamma := v(z) > 0$. If $z \notin \mathfrak{q}_v$, there exists some $0 \preceq a \in R$ such that $v(a) = -\gamma < 0$. Axiom (QR3) yields $0 \preceq ay \preceq az$. As 0 and az lie in R_v, it follows by convexity of R_v that $ay \in R_v$, i.e. $v(ay) \geq 0$. However, $v(ay) = v(a) + v(y) < 0$, a contradiction. If $z \in \mathfrak{q}_v$, choose some $0 \preceq a \in R$ with $v(a) < 0$, which exists since v is a non-trivial Manis valuation. Then $0 \preceq ay \preceq az$ with $az \in R_v$. However, $ay \notin R_v$, contradicting the convexity of R_v. □

Remark 3.13. (1) The assumption in (b) that v is non-trivial is crucial, no matter which kind of a quasi-order $\preceq$ is. For the ordered case consider $\mathbb{Z}$ with its unique order and the trivial valuation v mapping the even integers to ∞ and the odd integers to 0. Then $R_v = \mathbb{Z}$ is convex, while $I_v = 2\mathbb{Z}$ is not. In the case $\preceq = \preceq_w$, take the same v and let w be the p-adic valuation on $\mathbb{Z}$ for some prime $p > 2$. Then R_v is clearly convex, however $0 = w(2) \leq w(1) = 0$ and $0 < v(2) = \infty$, but $0 = v(1)$.

(2) Instead of v non-trivial, we may have also demanded that $\mathfrak{q}_v = E_0$ for (b). Then $z \in \mathfrak{q}_v$ yields $z \in E_0$, so also $y \in E_0 = \mathfrak{q}_v \subseteq I_v$ by transitivity of $\preceq$.

(3) $I_v \prec 1$ (compare Theorem 2.7) is an easy consequence of these conditions. It follows for instance immediately from the convexity I_v.

(4) If $\preceq$ is an order (respectively a proper quasi-order), then $\preceq'$ is also an order (respectively a proper quasi-order).

Proof. The ordered case is easy to verify (compare the Definitions 2.4 and 2.5). So suppose that $\preceq = \preceq_w$ for some valuation w on R. We consider the map $w/v : Rv \to \Gamma_{w/v} \cup \{\infty\}$ given by

$$w/v(a + I_v) := \begin{cases} \infty & \text{if } a \in I_v \\ w(a) & \text{else} \end{cases} .$$

(compare [3, p.45] for the field case). We prove that w/v is well-defined. For $a \in I_v$ this is clear by definition. So suppose that $a \in U_v$ and $c \in I_v$. We have to show that $w(a) = w(a + c)$. From condition (1) of the previous theorem we obtain for all $x, y \in R$ that if $w(x) \leq w(y)$, then $v(x) \leq v(y)$. Hence, it follows from $v(a) = 0 < v(c)$ that also $w(a) < w(c)$. Lemma 2.3 yields $w(a + c) = \min\{w(a), w(c)\} = w(a)$.
It is easy to see that w/v satisfies the axioms (V1) and (V2) from Definition 2.1. For (V3) note that $ab \in I_v$ if and only if $a \in I_v$ or $b \in I_v$, since I_v is

prime, so $w/v(ab+I_v)=\infty$ if and only if $w/v(a+I_v)+w/v(b+I_v)=\infty$. From this observation (V3) is easily deduced. The prove of (V4) is done by a similar case distinction. Hence, w/v defines a valuation on Rv. Its support is $\{0\}$, as $\mathfrak{q}_w \subseteq \mathfrak{q}_v \subseteq I_v$, which again follows from Theorem 3.12(1). Moreover, for $x, y \in U_v$ (i.e. $\overline{x}, \overline{y} \neq 0$) it holds

$$\begin{aligned}\overline{x} \preceq'_w \overline{y} &\Leftrightarrow x+c_1 \preceq_w y+c_2 \text{ for some } c_1, c_2 \in I_v\\ &\Leftrightarrow w(y+c_2) \leq w(x+c_1) \text{ for some } c_1, c_2 \in I_v\\ &\Leftrightarrow w(y) \leq w(x)\\ &\Leftrightarrow w/v(\overline{y}) \leq w/v(\overline{x}),\end{aligned}$$

where the third equivalence follows precisely as in the proof of the well-definedness of w/v, while the last equivalence is just the definition of w/v. This proves that $\preceq'_w = \preceq_{w/v}$. □

If $\preceq$ is an order, then Theorem 3.12 generalizes Theorem 2.7 from ordered fields to ordered rings. Next we show that if $\preceq = \preceq_w$ for some Manis valuation w, then Theorem 3.12 characterizes the Manis valuations v on R that are coarser than w.

Definition 3.14. (see [5, p.415]) Let v, w be valuations on R. Then v is said to be a **coarsening** of w (or w a **refinement** of v), in short, $v \leq w$, iff there exists a order homomorphism $\varphi : \Gamma_v \to \Gamma_w$ such that $w = \varphi \circ v$, or equivalently, iff $R_w \subseteq R_v$ and $I_v \subseteq I_w$.

Lemma 3.15. *Let $v \leq w$ be non-trivial Manis valuations on R. Then $q_v = q_w$.*

Proof. This is part of [11, Proposition 3.1]. □

Actually, Power's proof of the previous result only uses that v is non-trivial. However, from v non-trivial and $v \leq w$ follows immediately that w is also non-trivial.

Lemma 3.16. *Let v and w be non-trivial Manis valuations on R. The following are equivalent:*

(1) v is $\preceq_w$-compatible (i.e. $w(y) \leq w(z) \Rightarrow v(y) \leq v(z)$).
(2) v is a coarsening of w.

Proof. We first show that (1) implies (2). Let $x \in R_w$. Then $0 = w(1) \leq w(x)$, so also $0 = v(1) \leq v(x)$, thus $x \in R_v$. Likewise, if $x \notin I_w$, then $w(x) \leq w(1) = 0$, which yields that $v(x) \leq v(1) = 0$. Therefore $x \notin I_v$.
Conversely, assume that (2) holds and suppose that $w(y) \leq w(z)$. By the previous lemma we get $\mathfrak{q}_w = \mathfrak{q}_v$, so we may assume that y is not in the support of these valuations. Moreover note that $U_w \subseteq U_v$; indeed, if $u \in U_w$, then $u \in R_w$ and $u \notin I_w$, thus $u \in R_v$ and $u \notin I_v$. Therefore, $u \in U_v$. As $w(y) \in \Gamma_w$ and w is Manis, there exists some $a \in R$ such that $w(a) = -w(y)$. It follows $ay \in U_w$ and $az \in R_w$. Therefore, $ay \in U_v$ and $az \in R_v$. It is easy to see that this implies $v(y) \leq v(z)$. □

Theorem 3.12 and Lemma 3.16 yield the following characterization of coarsenings:

Theorem 3.17. *Let v, w be non-trivial Manis valuations on R. Then v is a coarsening of w, if and only if one of the following equivalent conditions is satisfied for all $x, y \in R$:*

(1) $w(x) \leq w(y) \Rightarrow v(x) \leq v(y)$,
(2) $w(x) \leq w(y), 0 \leq v(x) \Rightarrow 0 \leq v(y)$,
(3) $w(x) \leq w(y), 0 < v(x) \Rightarrow 0 < v(y)$,
(4) $w/v : Rv \to \Gamma_{w/v} \cup \{\infty\},\ x + I_v \mapsto \begin{cases}\infty & \text{if } x \in I_v\\ w(x) & \text{else}\end{cases}$ defines a valuation with support $\{0\}$.

Proof. This is precisely Theorem 3.12 in the special case where the quasi-order $\preceq$ comes from a Manis valuation w, and Lemma 3.16. Moreover, we simplified the convexity of R_v and I_v (in (2) and (3)) according to Lemma 3.6. □

Next we show that $I_v \prec 1$ is not equivalent to all the other conditions of Theorem 3.12, regardless of whether $\preceq$ is a proper quasi-order (Example 3.19) or an order (Example 3.20), even if v is non-trivial.

Theorem 3.18. (see [3, Theorem 2.2.1]) *Let K be a field, $\Gamma \subseteq \Gamma'$ ordered abelian groups, $u : K \to \Gamma \cup \{\infty\}$ a valuation on K, and $\gamma \in \Gamma'$. For $f = \sum_{i=0}^n a_i X^i \in K[X]$ define*

$$v(f) = \begin{cases} \infty & \text{if } f = 0 \\ \min\limits_{0 \leq i \leq n} \{u(a_i) + i\gamma\} & \text{otherwise} \end{cases}.$$

Then $v : K[X] \to \Gamma' \cup \{\infty\}$ is a valuation that extends u.

Example 3.19. Let $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ denote the p-adic valuation for some prime number $p \in \mathbb{N}$ (see [3, p.18]). Apply the previous theorem with $\gamma = 1$ to extend v_p to a valuation $v : \mathbb{Q}[X] \to \mathbb{Z} \cup \{\infty\}$. The valuation v is Manis, as v_p is Manis and they have the same value group. We do the same procedure with w instead of v, except that this time $\gamma = 0$. Note that $v = w$ on $\mathbb{Q}$ and $v(f) = w(f) + i$ for some $i \geq 0$ if $f \in \mathbb{Q}[X] \backslash \mathbb{Q}$. This implies $I_v \prec_w 1$. However, v is not compatible with $\preceq_w$. For instance $w(X^2) = 0 < w(p) = 1 < w(0) = \infty$, but $v(p) = 1 < v(X^2) = 2$.

Example 3.20. Consider the trivial valuation $v(x) = 0$ for $x \neq 0$ on $\mathbb{Z}$. Extend v via Theorem 3.18 to a valuation v' on $\mathbb{Z}[X, Y]$ with $\gamma = 1$ (for X), respectively $\gamma = -1$ (for Y). Thus, for any $0 \neq f = \sum_{i,j} a_{ij} X^i Y^j \in \mathbb{Z}[X, Y]$, we have $v'(f) = \min_{i,j}\{v(a_{ij}) + i - j\}$. Note that v' is a Manis valuation with value group $\mathbb{Z}$, for if m is an integer, then either $v'(X^m) = m$ or $v'(Y^{-m}) = m$. Order $\mathbb{Z}[X, Y]$ by $f \geq 0 :\Leftrightarrow f(0) \geq 0$. Then $v(f) \leq 0$ if $a_{00} \neq 0$. Therefore, $I_v \subseteq \langle X, Y\rangle \subseteq E_0$, so $I_v < 1$. However, I_v is not convex since $0 \leq Y \leq 0$, but $Y \notin I_v$.

Remark 3.21. In the case of ordered fields $(K, \leq)$, the condition $I_v < 1$ is often times replaced with the equivalent condition $1 + I_v \geq 0$ (see for instance [8, Definition 2.4] or [3, Proposition 2.2.4]). Note, however, that this is inappropriate for proper quasi-orders, as $1 + I_v \succeq_w 0$ is then trivially satisfied.

We continue this section by imposing a suitable extra condition on v, such that $I_v \prec 1$ is equivalent to (1) - (3) from Theorem 3.12.

Definition 3.22. (see [6, Ch. I, Definition 5]) A valuation v on R is called **local**, if the pair (R_v, I_v) is local, i.e. if I_v is the unique maximal ideal of R_v.

A characterization of local valuations is given in [6, Ch. I, Proposition 1.3] and [5, Proposition 5], respectively. If R is a field, then v is always a local Manis valuation.

Lemma 3.23. *Let $(R, \preceq)$ be a quasi-ordered ring and v a local Manis valuation on R. The following are equivalent:*

(1) v is compatible with $\preceq$.
(2) $I_v \prec 1$.

Proof. (1) implies (2) is clear, see Remark 3.13(3). Now suppose that (2) holds, and assume for a contradiction that there are some $y, z \in R$ such that $0 \preceq y \preceq z$, but $v(y) < v(z)$. The latter implies $y \notin \mathfrak{q}_v$. Since v is Manis, we find some $0 \preceq a$ such that $v(a) = -v(y)$. We obtain $0 = v(ay) < v(az)$, so $ay \in U_v$ and $az \in I_v$. As v is local and $ay \in U_v$, ay is a unit. It follows $0 < v(az) - v(ay) = v(az(ay)^{-1})$, i.e. $\frac{az}{ay} \in I_v$. Hence, (2) yields $az(ay)^{-1} \prec 1$. This implies $az \prec ay$. On the other hand, it follows from $y \preceq z$ and $0 \preceq a$ that $ay \preceq az$, a contraction. □

Corollary 3.24. *Let v, w be non-trivial Manis valuations on R such that v is local. Then v is coarser than w if and only if $I_v \subseteq I_w$.*

Proof. This is an immediate consequence of Lemma 3.16 and Lemma 3.23 in the case where $\preceq = \preceq_w$ for some non-trivial Manis valuation w. □

We conclude this section by establishing a notion of rank of a quasi-ordered ring. For the sake of convenience we first recall this notion in the field case. There it follows easily from [7, Theorem 2.2] and the fact that valuation rings in the field case are local:

Proposition 3.25. *Let $(K, \preceq)$ be a quasi-ordered field. The set*

$$\mathcal{R} := \{w : w \text{ is a non-trivial } \preceq \text{-compatible valuation on } K\}$$

is totally ordered by $\leq$ ("coarser")

Hence, the rank of a quasi-ordered field is the order type of $\mathcal{R}$. For a further discussion of the rank of a quasi-ordered field we refer to [7, p. 403].
We now consider the ring case. For the following result we use that $(R, \preceq)$ is a quasi-ordered ring if and only if $(R/E_0, \preceq')$ is a quasi-ordered ring, where $\overline{x} \preceq' \overline{y} \Leftrightarrow x \preceq y$ (see [10, Lemma 4.1]). Moreover, we exploit that the quasi-order $\preceq'$ uniquely extends to a quasi-order $\trianglelefteq$ on $K := \mathrm{Quot}(R/E_0)$ via $\overline{x}/\overline{y} \trianglelefteq \overline{a}/\overline{b} :\Leftrightarrow \overline{xyb^2} \preceq' \overline{aby^2}$ (see [10, Proposition 4.3]). If v is a valuation on R, let $\overline{v'}$ denote the respective extension to K. We now can prove:

Lemma 3.26. *Let $(R, \preceq)$ be a quasi-ordered ring and v a valuation on R with support $\mathfrak{q}_v = E_0$. Then v is $\preceq$-compatible if and only if $\overline{v'}$ is $\trianglelefteq$-compatible.*

Proof. Clearly $\preceq$ is compatible with v in R if and only if $\preceq'$ is compatible with v' in R/E_0. Hence, we may without loss of generality assume that E_0 is trivial. It is clear that compatibility in K reduces to compatibility in R as it is a universal statement. For the contrary, let $0 \trianglelefteq \frac{x}{y} \trianglelefteq \frac{a}{b}$. Then $0 \preceq xyb^2 \preceq aby^2$. By compatibility with v, we obtain that $v(aby^2) \leq v(xyb^2)$, i.e. that $v(ay) \leq v(xb)$. Thus, $\overline{v'}(\frac{a}{b}) \leq \overline{v'}(\frac{x}{y})$. □

This lemma justifies to define:

Definition 3.27. The **rank of a quasi-ordered ring** $(R, \preceq)$ is the rank of the naturally associated quasi-ordered field $(\mathrm{Quot}(R/E_0), \trianglelefteq)$.

4. The Baer-Krull Theorem

In the previous section we fixed a quasi-ordered ring $(R, \preceq)$ and characterized all $\preceq$-compatible Manis valuations on R (see Theorem 3.12). It is natural to ask what happens the other way round, i.e. if we fix a valued ring (R, v) with v Manis, can we describe all the quasi-orders on R that are compatible with v? A positive answer is given by the Baer-Krull Theorem (see Theorem 4.10, respectively Corollary 4.11 or Corollary 4.12). After establishing this result for quasi-ordered rings, we deduce a version for ordered, respectively proper quasi-ordered, rings (see Corollary 4.13, respectively Corollary 4.19). The first one yields a generalization of the classical Baer-Krull Theorem (see Theorem 2.8), while the latter characterizes all Manis valuations on R that are finer than v.

For quasi-ordered rings, the Baer-Krull theorem is more complicated than for ordered fields (see Theorem 2.8). Note that the map η there is completely determined by the signs of the elements π_i. If the quasi-order is an order, then all $\eta \in \{-1, 1\}^I$ are realizeable and one gets a bijective correspondence as in Theorem 2.8. However, if it is a proper quasi-order, then all elements are non-negative, so the only η

possible is $\eta = 1$. Therefore, the best we can hope for is that ψ is an injective map such that the image of ψ contains all possible tuples $(\eta_{\preceq}, \preceq')$ as just described.

Notation 4.1. Let R always be a commutative ring with 1 and $v\colon R \to \Gamma_v \cup \{\infty\}$ a Manis valuation on R. We define $\tilde{R} := R \backslash \mathfrak{q}_v = v^{-1}(\Gamma_v)$. Moreover, we fix some $\mathbb{F}_2$-basis $\{\overline{\gamma_i} : i \in I\}$ of $\overline{\Gamma_v} = \Gamma_v/2\Gamma_v$, and let $\{\pi_i : i \in I\} \subseteq \tilde{R}$ be such that $v(\pi_i) = \gamma_i$. Given a v-compatible quasi-order on R, we denote by $\preceq'$ the induced quasi-order on Rv (see Theorem 3.12(3)). By $\eta_{\preceq}$ we denote the map $I \to \{-1, 1\}$ defined by $\eta_{\preceq}(i) = 1$ iff $0 \preceq \pi_i$.

Now we fix some tuple $(\eta^*, \preceq^*)$ from the disjoint union

$$\{-1,1\}^I \times \{\text{orders on } Rv \text{ with support } \{0\}\} \sqcup \{1\}^I \times \{\text{p.q.o. on } Rv \text{ with support } \{0\}\}$$

The main part of the proof of the Baer-Krull Theorem is to construct a quasi-order on R that is mapped to $(\eta^*, \preceq^*)$ under the analogue of the map ψ from Theorem 2.8. For that purpose we define a binary relation $\preceq$ on R as a function of $\preceq^*$ and η^* as follows: If $x, y \in \mathfrak{q}_v$, declare $x \preceq y$. Otherwise, if $x \in \tilde{R}$ or $y \in \tilde{R}$, consider

$$\gamma := \gamma_{x,y} := \max\{-v(x), -v(y)\} \in \Gamma_v.$$

Write $\overline{\gamma} = \sum_i \overline{\gamma_i}$. Then $\gamma = \sum_i \gamma_i + 2v(a)$ for some $a \in \tilde{R}$, which is uniquely determined up to its value. Consider $x \prod_i \pi_i a^2$ and $y \prod_i \pi_i a^2$.

Lemma 4.2. *Let $x, y \in \tilde{R}$ and I, π_i, a as above. Then $x \prod_i \pi_i a^2$, $y \prod_i \pi_i a^2 \in R_v$. Moreover, $\overline{x \prod_i \pi_i a^2} = 0$ if and only if $v(x) > v(y)$.*

Proof. Note that

$$\begin{aligned} v\left(x \prod_i \pi_i a^2\right) &= v(x) + \sum_i v(\pi_i) + 2v(a) = v(x) + \gamma \\ &= v(x) + \max\{-v(x), -v(y)\} \geq 0, \end{aligned}$$

and likewise for $y \prod_i \pi_i a^2$, so both are in R_v. Moreover,

$$\overline{x \prod_i \pi_i a^2} = 0 \Leftrightarrow v(x) + \max\{-v(x), -v(y)\} > 0 \Leftrightarrow v(x) > v(y).$$

□

Particularly, we can take residues of both $x \prod_i \pi_i a^2$ and $y \prod_i \pi_i a^2$. The moreover part of the statement will be of great importance in the proof of Main Lemma 4.5. For the latter, we also require the following two lemmas, which extend the statements from axiom (QR3), respectively (QR5).

Lemma 4.3. *Let $(R, \preceq)$ be a quasi-ordered ring. If $x \preceq y$ and $z \preceq 0$, then $yz \preceq xz$.*

Proof. As E_0 is an ideal, we may without loss of generality assume that $z \nsim 0$. Moreover, note that if $y \sim 0$, then $x, z \preceq 0$, thus $0 \preceq -x, -z$. It follows via (QR3) that $yz \sim 0 \preceq xz$. So we may also assume that $y \notin E_0$. From $x \preceq y$ and $z \preceq 0$ follows $-xz \preceq -yz$. We claim that $yz \nsim -yz$. Once this is shown, it follows from $-xz \preceq -yz$ that $yz - xz \preceq 0$. The latter implies $yz \preceq xz$. Indeed, either $x \nsim 0$ and therefore $xz \nsim 0$ (E_0 is a prime ideal), so that we can apply (QR4); or $x \sim 0$, i.e. $xz \sim 0$, and therefore $yz - xz \sim yz \preceq 0 \sim xz$ (see Remark 2.6(1)). So assume for a contradiction that $yz \sim -yz$. Lemma 3.8 yields $0 \preceq yz, -yz$. As $y \notin E_0$, either $0 \prec y$ or $0 \prec -y$. So via (QR5) it follows either from $0 \preceq yz$ (if $0 \prec y$) or from $0 \preceq -yz$ (if $0 \prec -y$) that $0 \preceq z$. Hence $z \sim 0$, a contradiction. □

Lemma 4.4. *Let $(R, \preceq)$ be a quasi-ordered ring and $x, y, z \in R$. If $xz \preceq yz$ and $z \prec 0$, then $y \preceq x$.*

Proof. Assume for a contradiction $x \prec y$. The previous lemma yields $yz \preceq xz$. Hence $xz \sim yz$. Remark 2.6(4) yields $x \sim y$, a contradiction. □

Main Lemma 4.5. *With the notation from above, define for $x \in \tilde{R}$ or $y \in \tilde{R}$ that*

$$x \preceq y :\Leftrightarrow \begin{cases} \text{Either} \quad \overline{x \prod_i \pi_i a^2} \preceq^* \overline{y \prod_i \pi_i a^2} \quad \text{and} \quad \prod_i \eta^*(i) = 1 \\ \\ \text{or} \quad \overline{y \prod_i \pi_i a^2} \preceq^* \overline{x \prod_i \pi_i a^2} \quad \text{and} \quad \prod_i \eta^*(i) = -1. \end{cases}$$

Moreover, declare $x \preceq y$ for $x, y \in \mathfrak{q}_v$. Then $\preceq$ defines a quasi-order on R with support $E_0 = \mathfrak{q}_v$.

Proof. The proof of the Main Lemma is extensive, however, the methods are widely the same. Notably, the moreover part from Lemma 4.2 is frequently exploited. We always use the notation from above. For the sake of convenience and uniformity, we treat $\preceq^*$ and η^* as an arbitrary quasi-order on Rv with support $\{0\}$, respectively an arbitrary map from I to $\{-1, 1\}$, for as long as possible. In fact, the distinction whether $\preceq^*$ is an order or induced by a valuation (in which case the map η^* is trivial) is only necessary at some points when we verify axiom (QR4).
First we show that $\preceq$ is well-defined. Recall that $a \in \tilde{R}$ was only determined up to its value. So let $b \in \tilde{R}$ with $v(a) = v(b)$, and suppose that $\overline{x \prod_i \pi_i a^2} \preceq^* \overline{y \prod_i \pi_i a^2}$. As v is Manis, there exists some $z \in \tilde{R}$ with $v(z) = -v(b)$, so $v(bz) = 0$, i.e. $\overline{bz} \neq 0$. Particularly, $0 \prec^* \overline{bz}^2$. With axiom (QR3) follows, after rearranging, that $\overline{x \prod_i \pi_i b^2}\ \overline{az}^2 \preceq^* \overline{y \prod_i \pi_i b^2}\ \overline{az}^2$. We conclude by eliminating $\overline{az}^2$ via (QR5).
Clearly $\preceq$ is reflexive and total. At next we prove transitivity. So let $x \preceq y$ and $y \preceq z$, without loss of generality $x \in \tilde{R}$ or $z \in \tilde{R}$. Denote by I the index set to compare x and y, by J the one to compare y and z, and by L the one to compare x and z, with corresponding squares a^2, b^2 and c^2, respectively. The proof is done by distinguishing four cases. First of all assume that $v(p) = v(q) \leq v(r)$ with $p, q, r \in \{x, y, z\}$ pairwise distinct. Then $\gamma_{x,y} = \gamma_{x,z} = \gamma_{y,z} \in \Gamma_v$ all coincide, so $I = J = L$ and $a = b = c$. Hence, transitivity of $\preceq$ follows immediately by transitivity of $\preceq^*$. It remains to verify the cases where there is a unique smallest element among $v(x), v(y)$ and $v(z)$. We do the case $v(x) < v(y), v(z)$, leaving the other ones to the reader. Then $\gamma_{x,y} = -v(x) = \gamma_{x,z}$, i.e. $I = L$ and $a = c$. We assume without loss of generality $\prod_i \eta^*(i) = -1$, as the easier case $\prod_i \eta^*(i) = 1$ is proven likewise. From $x \preceq y$ and $v(x) < v(y)$ then follows $\overline{y \prod_i \pi_i a^2} = 0 \preceq^* \overline{x \prod_i \pi_i a^2}$ (Lemma 4.2). Now $v(x) < v(z)$ and Lemma 4.2 imply that $\overline{z \prod_i \pi_i a^2} = 0$. Therefore, $x \preceq z$.
Now we establish that the support of $\preceq$ is $\mathfrak{q}_v$. Assume there is some $x \in E_0$ with $x \notin \mathfrak{q}_v$. Then $\overline{x \prod_i \pi_i a^2} \sim 0$. As the support of $\preceq^*$ is $\{0\}$, this yields $x \prod_i \pi_i a^2 \in I_v$. However, as $v(x) < v(0) = \infty$, this contradicts Lemma 4.2. We obtain that $E_0 \subseteq \mathfrak{q}_v$. The other implication follows immeediately from the definition of $\preceq$.
It remains to verify the axioms (QR1) - (QR5) and compatibility with v. For the proof of **(QR1)** assume for a contradiction that $1 \preceq 0$. Note that $\gamma_{0,1} = 0$, so $I = \emptyset$ and $\prod_i \eta^*(i) = 1$. It follows from $1 \preceq 0$ that $\overline{a}^2 \preceq^* 0$ for some $a \in R$ with $v(a) = 0$. This contradicts the facts that squares are non-negative and that the support of $\preceq^*$ is trivial.
For **(QR2)** is nothing to show by Remark 2.6(3). In order to prove **(QR3)**, first note that one may without loss of generality assume that $z \notin \mathfrak{q}_v$, and that not both x and y are in $\mathfrak{q}_v$. Further note that

$$\begin{aligned}\gamma_{xz,yz} &= \max\{-v(xz), -v(yz)\} = \max\{-v(z), -v(0)\} + \max\{-v(x), -v(y)\} \\ &= \gamma_{0,z} + \gamma_{x,y} \in \Gamma_v.\end{aligned}$$

Write $\gamma_{x,y} = \sum_i \gamma_i + 2v(a)$ and $\gamma_{0,z} = \sum_j \gamma_j + 2v(b)$. Set $L = I \sqcup J$, the (wlog) disjoint union of the index sets I and J. Then $\gamma_{xz,yz} = \gamma_{y,z} + \gamma_{0,z} = \sum_l \gamma_l + 2v(ab)$.

So to compare xz and yz with respect to $\preceq$, one has to consider $xz\prod_l \pi_l a^2b^2$ and $yz\prod_l \pi_l a^2b^2$. Evidently $\prod_l \eta^*(l) = \prod_i \eta^*(i)\cdot\prod_j \eta^*(j)$. First consider the case $\prod_j \eta^*(j) = 1$. This yields $0 \preceq^* \overline{z\prod_j \pi_j b^2}$. Suppose $\prod_i \eta^*(i) = -1 = \prod_l \eta^*(l)$. From $x \preceq y$ then follows $\overline{y\prod_i \pi_i a^2} \preceq^* \overline{x\prod_i \pi_i a^2}$. Applying (QR3) yields $\overline{yz\prod_l \pi_l a^2b^2} \preceq^* \overline{xz\prod_l \pi_l a^2b^2}$. Therefore, $xz \preceq yz$. The case $\prod_i \eta^*(i) = -1 = \prod_l \eta^*(l)$ is analogue. The proof for $\prod_j \eta^*(j) = -1$ is also almost the same; we just apply Lemma 4.3 instead of axiom (QR3).

The proof of axiom **(QR4)** is divided into five subcases. Let I, J, L and a, b, c as in the verification of transitivity. First suppose that $v(x) < v(z)$ or $v(y) < v(z)$. Either way, $\gamma_{x,y} = \gamma_{x+z,y+z}$. Moreover, in both cases $\overline{z\prod_i \pi_i a^2} = 0$. From this observation, the claim follows immediately. Further note that if $\preceq^*$ is an order and $x \prec y$, we obtain $x + z \prec y + z$, because orders preserve strict inequalities under addition. We will exploit this fact to prove the difficult case $v(x) = v(y) = v(z)$. The cases $v(z) < v(x), v(y)$, and $v(x) = v(z) < v(y)$, and $v(y) = v(z) < v(x)$ are left to the reader. So assume that $v(x) = v(y) = v(z) \in \Gamma_v$. It holds

$$\gamma_{x+z,y+z} = \max\{-v(x+z), -v(y+z)\} \leq -v(z).$$

First suppose that equality holds. Then $\max\{-v(x+z), -v(y+z)\} = -v(z)$, i.e. all $\gamma' s$ coincide. If $\prod_i \eta^*(i) = 1$, the claim follows immediately from (QR4) and the fact that $y \nsim z$ by simply adding $\overline{z\prod_i \pi_i a^2}$ to both sides of the inequality $\overline{x\prod_i \pi_i a^2} \preceq^* \overline{y\prod_i \pi_i a^2}$. Contrary, if $\prod_i \eta^*(i) = -1$, then $\preceq^*$ must be an order and we may simply add $\overline{z\prod_i \pi_i a^2}$ on both sides anyway.

Last but not least assume that $<$ holds, i.e. $\max\{-v(x+z), -v(y+z)\} < -v(z)$. Then $v(x+z), v(y+z) < v(z)$. Lemma 4.2 yields that $\overline{x\prod_i \pi_i a^2}, \overline{y\prod_i \pi_i a^2}$ and $\overline{z\prod_i \pi_i a^2}$ are all non-zero, whereas $\overline{(x+z)\prod_i \pi_i a^2} = 0 = \overline{(y+z)\prod_i \pi_i a^2}$. This yields $\overline{x\prod_i \pi_i a^2} = \overline{y\prod_i \pi_i a^2} = -\overline{z\prod_i \pi_i a^2}$. Particularly, we may assume that $\preceq^*$ is an order, since in the proper quasi-ordered case $\overline{y\prod_i \pi_i a^2} \sim -\overline{y\prod_i \pi_i a^2} = \overline{z\prod_i \pi_i a^2}$, contradicting the assumption $y \nsim z$. We claim that $x + z \sim 0 \sim y + c$, which clearly implies $x + z \preceq y + z$. Assume for a contradiction that $x + z \nsim 0$. If $x + z \prec 0$, it follows from the case "$v(x) < v(z)$" (where $x + z$ plays the role of x, 0 the one of y and $-z$ the one of z; recall that $v(x + z) < v(z)$) above and the fact that $\preceq^*$ is an order, that $x \prec -z$, contradicting $x \sim -z$. Likewise, if $0 \prec x + z$, it follows from the case "$v(y) < v(z)$" that $-z \prec x$, again a contradiction. Therefore $x + z \sim 0$. The same reasoning shows that $y + z \sim 0$ as well.

Last but not least we prove axiom **(QR5)**. Suppose $xz \preceq yz$ and $0 \prec z$. Clearly $z \in \tilde{R}$, as $0 \nsim z$. Moreover, without loss of generality $x \in \tilde{R}$ or $y \in \tilde{R}$. Note that $\gamma_{xz,yz} = \gamma_{x,y} + \gamma_{z,0}$. Let I denote the index set to compare x and y, J the one to compare z and 0, and L the one to compare xz and yz, with squares a^2, b^2 and $(ab)^2$, respectively. Note that $\prod_l \eta^*(l) = \prod_i \eta^*(i)\prod_j \eta^*(j)$.

First consider the case $\prod_j \eta^*(j) = 1$. Then $0 \prec^* \overline{z\prod_j \pi_j b^2}$. If $\prod_l \eta^*(l) = -1$, also $\prod_i \eta^*(i) = -1$. It holds $\overline{yz\prod_l \pi_l a^2b^2} \preceq^* \overline{xz\prod_l \pi_l a^2b^2}$. Eliminating $\overline{z\prod_j \pi_j b^2}$ via (QR5) yields $\overline{y\prod_i \pi_i a^2} \preceq^* \overline{x\prod_i \pi_i a^2}$, and therefore $x \preceq y$. On the other hand, if $\prod_l \eta^*(l) = 1$, then also $\prod_i \eta^*(i) = 1$, and the prove is analogoue.

If $\prod_j \eta^*(j) = -1$, then $\overline{z\prod_j \pi_j b^2} \prec^* 0$. If $\prod_l \eta^*(l) = -1$, then $\prod_i \eta^*(i) = 1$. It follows $\overline{yz\prod_l \pi_l a^2b^2} \preceq^* \overline{xz\prod_l \pi_l a^2b^2}$. Applying Lemma 4.4 yields $\overline{x\prod_i \pi_i a^2} \preceq^* \overline{y\prod_i \pi_i a^2}$. Therefore $x \preceq y$. The case $\prod_l \eta^*(l) = 1$ is analogue.

We conclude by showing that $\preceq$ is v-compatible. Suppose $0 \preceq x \preceq y$ but $v(x) < v(y)$ for some $x, y \in R$. Note that $\gamma_{0,x} = -v(x) = \gamma_{x,y}$. If $\prod_i \eta^*(i) = -1$, then $0 \preceq x$ yields $\overline{x\prod_i \pi_i a^2} \prec^* 0 = \overline{y\prod_i \pi_i a^2}$, i.e. $y \prec x$, a contradiction. The same argument works for $\prod_i \eta^*(i) = 1$. □

Remark 4.6. The quasi-order $\preceq$ from the Main Lemma becomes very simple in the case where $x \in U_v$ and $y \in R_v$ (or vice versa). Note that then $\gamma_{x,y} = 0$. This implies $I = \emptyset$. Hence, $\prod_i \eta^*(i) = 1$. Moreover, the element a satisfies $v(a) = 0$, so by well-definedness of $\preceq$ we may simply choose $a = 1$. Therefore $x \preceq y \Leftrightarrow \overline{x} \preceq^* \overline{y}$.

For the proof of the Baer-Krull Theorem we require two more lemmas. They will be used to compare the "size" of two quasi-orders on R.

Lemma 4.7. *Let $(R, \preceq)$ be a quasi-ordered ring and $x \in R$. Then $E_0 + \{x\} \subseteq E_x$.*

Proof. For $x \in E_0$ there is nothing to show. So let $y \in R \backslash E_0$ such that $y = c + x$ for some $c \in E_0$. Remark 2.6(1) yields $c + x \sim x$, so $y \in E_x$. □

Lemma 4.8. *Let $(R, \preceq)$ be a quasi-ordered ring and $x \in R$. If $E_0 + \{x\} \subsetneq E_x$, then $E_x = -E_x$.*

Proof. Let $z \in E_x$ be arbitrary and $y \in E_x \backslash (E_0 + \{x\})$. We will show that $-y \in E_x$. From $z \sim x \sim y$ and Corollary 3.10 then follows $-z \sim -y \sim x$, i.e. also $-z \in E_x$, what proves that $E_x = -E_x$. The proof that $-y \in E_x$ is like in [4, p.208]. Assume for a contradiction that $-y \notin E_x$. Then $y \preceq x \not\sim -y$, thus $0 \preceq x - y$. Likewise, it follows from $x \preceq y \not\sim -y$ that $x - y \preceq 0$. Therefore, $x - y \in E_0$, i.e. $y \in E_0 + \{x\}$, a contradiction. Hence, $-y \in E_x$, i.e. $E_x = -E_x$. □

Notation 4.9. For a prime ideal $\mathfrak{p}$ of R denote by $\mathcal{X}_{\mathfrak{p}}(R)$ the set of all quasi-orders on R with support $\mathfrak{p}$. Analogously, denote by $\mathcal{X}_{o,\mathfrak{p}}(R)$ (respectively $\mathcal{X}_{p,\mathfrak{p}}(R)$) the set of all orders (respectively proper quasi-orders) on R with support $\mathfrak{p}$.

Theorem 4.10. (*Baer-Krull Theorem for quasi-ordered Rings I*)
Let R be a commutative ring with 1 and v a Manis valuation on R. Then

$$\psi \colon \{\preceq \in \mathcal{X}_{\mathfrak{q}_v}(R) \colon \preceq \textit{ is } v\textit{-compatible}\} \to \{-1,1\}^I \times \mathcal{X}_{\{0\}}(Rv), \qquad \preceq \mapsto (\eta_{\preceq}, \preceq')$$

is a well-defined map such that $\psi \restriction \psi^{-1}(\mathcal{A}) : \psi^{-1}(\mathcal{A}) \to \mathcal{A}$ is a bijection, where $\mathcal{A} := \{-1,1\}^I \times \mathcal{X}_{o,\{0\}}(Rv) \sqcup \{1\}^I \times \mathcal{X}_{p,\{0\}}(Rv)$.

Proof. By Theorem 3.12(3) the map ψ is well-defined. Next, let $(\eta^*, \preceq^*) \in \mathcal{A}$ be arbitrary. We prove that ψ maps the quasi-order $\preceq$ constructed in the Main Lemma to the tuple $(\eta^*, \preceq^*)$. First we verify that $\eta_{\preceq} = \eta^*$. To compare π_i and 0 w.r.t. $\preceq$, let $\gamma := \max\{-v(\pi_i), -v(0)\} = -\gamma_i$, i.e. $\gamma = v(\pi_i a^2)$ for some $a \in \tilde{R}$. Hence, we have to consider 0 and $\pi_i \pi_i a^2 = (\pi_i a)^2$, and to distinguish whether $\eta^*(i)$ equals 1 or -1. Note that $0 \prec^* \overline{\pi_i a}^2$, as it is a square and $\preceq^*$ has trivial support. From this observation we obtain $\eta_{\preceq}(i) = 1 \Leftrightarrow 0 \preceq \pi_i \Leftrightarrow \eta^*(i) = 1$, and therefore $\eta_{\preceq} = \eta^*$.
Next we prove that $\preceq' = \preceq^*$. Assume without loss of generality that not both $x, y \in I_v$. Then also $x + c$ and $y + d$ are not both in I_v for all $c, d \in I_v$. It follows from Remark 4.6 that $x + c \preceq y + d \Leftrightarrow \overline{x + c} \preceq^* \overline{y + d}$. Thus,

$$\begin{aligned} \overline{x} \preceq' \overline{y} &\Leftrightarrow \exists c_1, c_2 \in I_v : x + c_1 \preceq y + c_2 \\ &\Leftrightarrow \exists c_1, c_2 \in I_v : \overline{x + c_1} \preceq^* \overline{y + c_2} \\ &\Leftrightarrow \overline{x} \preceq^* \overline{y}, \end{aligned}$$

where the first equivalence is simply the definition of $\preceq'$.
We conclude by showing that $\psi \restriction \psi^{-1}(\mathcal{A})$ is injective. Let $\preceq_1 \in \psi^{-1}(\mathcal{A})$ be arbitrary, and denote by $\preceq_2$ the quasi-order on R defined by $\eta_{\preceq_1}$ and $\preceq_1'$ (see Main Lemma). We prove that $\preceq_1 = \preceq_2$. First of all we claim that $\preceq_1 \subseteq \preceq_2$. So let $x, y \in R$. Since $\preceq_1$ and $\preceq_2$ have both support $\mathfrak{q}_v$, we may without loss of generality assume that $x \notin \mathfrak{q}_v$ or $y \notin \mathfrak{q}_v$. Let I, π_i and a be as in the definition of the quasi-order $\preceq_2$. First

suppose that $\prod_i \eta_{\preceq_1}(i) = -1$, i.e. $\prod_i \pi_i a^2 \prec_1 0$. With Lemma 4.3 and Lemma 4.4, we obtain

$$\begin{aligned} x \preceq_1 y &\Leftrightarrow y \prod_i \pi_i a^2 \preceq_1 x \prod_i \pi_i a^2 \\ &\Rightarrow \overline{y \prod_i \pi_i a^2} \preceq_1' \overline{x \prod_i \pi_i a^2} \\ &\Leftrightarrow x \preceq_2 y. \end{aligned}$$

Likewise, if $\prod_i \eta_{\preceq_1}(i) = 1$, we just apply (QR3) instead of Lemma 4.3 and (QR5) instead of Lemma 4.4 to get the same result. Thus, $\preceq_1 \subseteq \preceq_2$. For the rest of the proof we distinguish the cases $-1 \nsim_{\preceq_2} 1$ and $-1 \sim_{\preceq_2} 1$.
If $-1 \nsim_{\preceq_2} 1$, then Remark 2.6(4) yields $-x \nsim_{\preceq_2} x$ for all $x \in \tilde{R}$, so $E_{x,\preceq_2} \neq -E_{x,\preceq_2}$ for all such x. From Lemma 4.7 and Lemma 4.8 follows $E_{x,\preceq_2} = \mathfrak{q}_v + \{x\}$ for all $x \in R$. So Lemma 4.7 yields that $\preceq_2$ is the smallest quasi-order with support $\mathfrak{q}_v$ possible. Therefore, $\preceq_1 \subseteq \preceq_2$ implies equality, as desired. So suppose for the rest of this proof that $-1 \sim_{\preceq_2} 1$. We distinguish the subcases $v(x) \neq v(y)$ and $v(x) = v(y)$.
If $v(x) \neq v(y)$, then Lemma 4.2 states $\overline{x \prod_i \pi_i a^2} \neq 0$ and $\overline{y \prod_i \pi_i a^2} = 0$, or vice versa. We show $\preceq_1 = \preceq_2$ by proving that the only $\Rightarrow$ above is also an equivalence. First suppose that $\overline{y \prod_i \pi_i a^2} = 0$. Assume for a contradiction that

$$0 = \overline{y \prod_i \pi_i a^2} \preceq_1' \overline{x \prod_i \pi_i a^2} \text{ but } x \prod_i \pi_i a^2 \prec_1 y \prod_i \pi_i a^2.$$

Then we find some $c_1, c_2 \in I_v$ such that $c_1 \preceq_1 x \prod_i \pi_i a^2 + c_2$. With Lemma 3.7(1) follows $c_1 - c_2 \preceq_1 x \prod_i \pi_i a^2 \prec_1 y \prod_i \pi_i a^2$, thus convexity of I_v yields $x \prod_i \pi_i a^2 \in I_v$, contradicting $\overline{x \prod_i \pi_i a^2} \neq 0$. Now suppose that $\overline{x \prod_i \pi_i a^2} = 0$ and assume the same contradiction. Then we obtain that

$$y \prod_i \pi_i a^2 + c \preceq_1 x \prod_i \pi_i a^2 \prec_1 y \prod_i \pi_i a^2,$$

and taking residues yields that $\overline{y \prod_i \pi_i a^2} = 0$, since the support of $\preceq'$ is trivial, a contradiction.
So finally suppose that $v(x) = v(y)$ and assume for a contradiction that $x \sim_{\preceq_2} y$, but $x \prec_1 y$. Choose $a \in \tilde{R}$ such that $0 \prec_1 a$ (and hence $0 \prec_2 a$) and $v(a) = -\gamma$. Note that $ax \prec_1 ay$ if and only if $x \prec_1 y$ (by (QR5) and (QR3)), and also $ax \sim_{\preceq_2} ay$ if and only if $x \sim_{\preceq_2} y$ (Remark 2.6(4) and Corollary 3.10). So we may replace x and y with ax and ay. In other words, we may without loss of generality assume that $v(x) = v(y) = 0$. It holds $y \preceq_2 x$. So by definition of $\preceq_2$ and the fact that $v(x) = v(y) = 0$, we get that $\overline{y} \preceq_1' \overline{x}$ (see Remark 4.6). Thus, there exist some $c_1, c_2 \in I_v$ such that $y + c_1 \preceq_1 x + c_2$, respectively, $y \preceq_1 x + c$ for $c := c_2 - c_1$ (see Lemma 3.7(1)). Recall that $-1 \sim_{\preceq_2} 1$. But then also $-1 \sim_{\preceq_1} 1$. Otherwise $-1 \preceq_1 0$, but $-1 \npreceq_2 0$, contradicting the fact that $\preceq_1 \subseteq \preceq_2$. Therefore, Corollary 3.10 and Lemma 3.8 yield that all elements in R are non-negative with respect to $\sim_1$. Particularly, $0 \prec_1 -1$. So Lemma 3.11 implies $y \preceq_1 x + c \preceq_1 \max\{x, c\} \prec_1 y$, a contradiction (note that $y \preceq c$ would contradict the convexity of I_v, as $0 \prec_1 y$). This finishes the proof of the Baer-Krull Theorem. □

Note that for the sake of uniformity, we so far avoided the dichotomy that every quasi-ordered ring is either an ordered or else a valued ring. Taking it into consideration, the Baer-Krull Theorem simplifies as follows:

Corollary 4.11. (*Baer-Krull Theorem for quasi-ordered Rings II*)
Let R be a commutative ring with 1 and v a Manis valuation on R. Then the map

$$\psi\colon \{\preceq \in \mathcal{X}_{\mathfrak{q}_v}(R)\colon \preceq \text{ is } v\text{-compatible}\} \to \{-1,1\}^I \times \mathcal{X}_{\{0\}}(Rv), \quad \preceq \mapsto (\eta_\preceq, \preceq')$$

is an embedding with $\{-1,1\}^I \times \mathcal{X}_{o,\{0\}}(Rv) \sqcup \{1\}^I \times \mathcal{X}_{p,\{0\}}(Rv) \subseteq Im(\psi)$.

Proof. The dichotomy and Remark 3.13(4) yield that $\psi^{-1}(\mathcal{A})$ coincides with the domain of ψ. The claim follows now immediately from Theorem 4.10. □

The Baer-Krull Theorem simplifies even much further in the case where the value group Γ_v is 2-divisible, because then $\overline{\Gamma_v} = \Gamma_v/2\Gamma_v$ is trivial and therefore $I = \emptyset$.

Corollary 4.12. (*Baer-Krull Theorem for quasi-ordered Rings III*)
Let R be a commutative ring with 1 and v a Manis valuation on R such that its value group Γ_v is 2-divisible. The following map is a bijection:

$$\psi\colon \{\preceq \in \mathcal{X}_{\mathfrak{q}_v}(R)\colon \preceq \text{ is } v\text{-compatible}\} \to \mathcal{X}_{\{0\}}(Rv), \quad \preceq \mapsto \preceq'$$

Proof. This follows immediately from the previous corollary and the 2-divisibility of Γ_v, see the explanation above. □

We conclude this paper by deducing Baer-Krull Theorems for ordered, respectively proper quasi-ordered, rings, from Corollary 4.11.

Corollary 4.13. (*Baer-Krull Theorem for ordered Rings*)
Let R be a commutative ring with 1 and v a Manis valuation on R. Then the map

$$\psi\colon \{\leq \in \mathcal{X}_{\mathrm{o},\mathfrak{q}_v}(R)\colon \leq \text{ is } v\text{-compatible}\} \to \{-1,1\}^I \times \mathcal{X}_{\mathrm{o},\{0\}}(Rv), \quad \preceq \mapsto \preceq'$$

is a bijection.

If R is a field, then this result coincides with Theorem 2.8. Further note that if Γ_v is 2-divisible, then Corollary 4.13 simplifies in the same manner as Corollary 4.12. Moreover, the statement becomes evidently much easier if the domain Rv is uniquely ordered.

Lemma 4.14. *For a domain R, the following are equivalent:*

(1) *R is uniquely ordered.*
(2) *0 is not a sum of non-zero squares and for each $a \in R$, there exists some non-zero b such that either ab^2 or $-ab^2$ is a sum of squares.*

Proof. In the proof we exploit the fact that R is uniquely ordered if and only if $K := \mathrm{Quot}(R)$ is uniquely ordered. Note that the latter is equivalent to the fact that for any $a \in K^*$, either a or $-a$ (and not both) is a sum of squares.
We first show that (2) implies (1). So let $x/y \in K^*$ with $x, y \in R$. Then $xy \in R$. So there exists some $0 \neq b$ such that (wlog) xyb^2 is a sum of squares in R, say $xyb^2 = \sum p_i^2$, with $p_i \in R$. Then $x/y = \sum_i (p_i/yb)^2$ is a sum of squares in K. Moreover, $-x/y$ is not a sum of squares in K, since otherwise 0 would be a sum of non-zero squares in R.
We conclude by showing that (1) implies (2). So suppose that R is uniquely ordered, i.e. also K is uniquely ordered. Hence, 0 is not a sum of non-zero squares in K, but then this is also the case in R. Now let $a \in R \subseteq K$. Then a or $-a$ is a sum of squares, say $\pm a = \sum_i (x_i/y_i)^2 \quad (x_i, y_i \in R)$. This yields $\pm a \prod_i y_i^2 = \sum_i \left(x_i \prod_{j\neq i} y_j\right)^2$. Hence, $b := \prod_i y_i$ satisfies (2). □

Our Baer-Krull Theorem allows us to transfer [3, Corollary 2.2.6] to the ring case. In analogy to the field case, we call an ordered ring $(R, \leq)$ Archimedean, if for any $x \in R$ there exists some $n \in \mathbb{N}$ such that $x < n$, and otherwise non-Archimedean.

Corollary 4.15. *(1) If R carries a non-trivial Manis valuation with real residue class ring, then R admits a non-Archimedean ordering.*

(2) Conversely, if R carries a non-Archimedean ordering, then R admits a non-trivial valuation with real residue class ring.

Proof. The non-Archimedean ordering in (1) is derived exactly as in the field case, see [3, Corollary 2.2.6]. Note that $R_v \subsetneq R$, since v is a non-trivial and Manis. For the proof of (2) we first suppose that R is a domain. Let $\leq$ denote a non-Archimedean ordering on R. Then $\leq$ uniquely extends to a non-Archimedean ordering on $K := \mathrm{Quot}(R)$. [3, Corollary 2.2.6] yields that K carries a non-trivial valuation w with real residue class field Kw. Note that the restriction v of w to R is a non-trivial (not necessarily Manis) valuation on R. Moreover, the map $\varphi : R_v \to Kw, x \mapsto x + I_w$ is a ring homomorphism with kernel I_v, so Rv is real as a subring of the real ring Kw. For the general case, note that if R carries a non-Archimedean ordering $\leq$, then $\overline{x} \leq' \overline{y} :\Leftrightarrow x \leq y$ defines a non-Archimedean ordering on the domain R/E_0 (see [10, Lemma 4.1]). Hence, there exists a non-trivial valuation w on R/E_0 such that $(R/E_0)w$ is real. As was shown in [10, Lemma 4.4], this yields a valuation v on R with support E_0 via $v(x) = w(\overline{x})$, and the value groups of v and w coincide, i.e. v is non-trivial as well. By definition of v, it is easy to see that Rv inherits the order from $(R/E_0)w$. □

Remark 4.16. In the first statement of the previous corollary, the assumption that the valuation is Manis is crucial, since we want to apply the Baer-Krull Theorem. However, for the converse, we can not derive surjectivity, because the restriction of a field valuation to a subring is in general not Manis. For instance any field valuation restricted to the integers is either trivial or not Manis. since $\mathbb{Z}$ admits no non-trivial Manis valuation. The latter is due to the fact that the triangle inequality yields $v(n) \geq 0$ for any natural number n.

The Baer-Krull Theorem for quasi-ordered rings also gives rise to a characterization of all Manis valuations w on R, that are finer than v, if we additionally assume that v is non-trivial.

Corollary 4.17. (*Baer-Krull Theorem for proper quasi-ordered Rings I*)
Let R be a commutative ring with 1 and v a Manis valuation on R. Then the map

$$\psi\colon \{\preceq_w \in \mathcal{X}_{\mathrm{p},\mathfrak{q}_v}(R)\colon\ \preceq_w \text{ is } v\text{-compatible}\} \to \mathcal{X}_{\mathrm{p},\{0\}}(Rv),$$
$$\preceq \mapsto \preceq'$$

is a bijection.

Now recall from Theorem 3.12 and Remark 3.13(4) that if $\preceq = \preceq_w$ is v-compatible, then $\preceq' = \preceq_{w/v}$ (see Remark 3.13(4) for the proof and a definition of w/v). This allows us to reformulate the previous corollary more precisely (see Corollary 4.19).

Lemma 4.18. *Let (R, v) be a valued ring for some Manis valuation v on R, and let w be a valuation on R such that $\preceq_w$ is v-compatible and $\mathfrak{q}_v = \mathfrak{q}_w$. Then w is Manis if and only if w/v is Manis.*

Proof. If u is some arbitrary valuation of R, then $u(R \backslash \mathfrak{q}_u)$ is additively closed by axiom (V3) of Definition 2.1. So in order to show that u is Manis, it suffices to prove that $u(R \backslash \mathfrak{q}_u)$ is closed under additive inverses.
Suppose that w is Manis. Let $\gamma := w/v(\overline{a}) \in \Gamma_{w/v}$ be arbitrary, $a \in U_v$. Then $w/v(\overline{a}) = w(a)$. Since w is Manis, there exists some $b \in R$ such that $w(b) = -w(a)$.

Thus, $w(ab) = 0 = w(1)$. By v-compatibility of $\preceq_w$, we obtain that also $v(ab) = 0$. Since $a \in U_v$, also $b \in U_v$. Therefore, it holds $w/v(\bar{b}) = w(b) = -\gamma \in \Gamma_{w/v}$.
Now assume that w/v is Manis and let $a \in R$ such that $w(a) =: \gamma \in \Gamma_w$. We show that there exists some $b \in R$ with $w(b) = -\gamma$. Note that $a \notin \mathfrak{q}_v$, since $\mathfrak{q}_v = \mathfrak{q}_w$. Since v is Manis, we find some $y \in R$ such that $ay \in U_v$. So $w/v(\overline{ay}) = w(ay) =: \gamma_1$. By surjectivity of w/v, there exists some $z \in R$ such that $w/v(\overline{z}) = w(z) = -\gamma_1$. Therefore, $w(z) = -w(a) - w(y)$. This yields $w(yz) = -w(a) = -\gamma$, i.e. $b = yz$. □

Corollary 4.19. (*Baer-Krull Theorem for proper quasi-ordered Rings II*)
Let R be a commutative ring with 1 and v a Manis valuation on R. Then the map

$$\psi \colon \{w \colon w \text{ Manis}, \preceq_w v\text{-comp.}, \mathfrak{q}_w = \mathfrak{q}_v\} \to \{u \colon u \text{ Manis val. on } Rv, \mathfrak{q}_u = \{0\}\},$$
$$w \mapsto w/v$$

is a bijection.

Proof. We deduce this corollary from Corollary 4.17. As mentioned above, if $\preceq = \preceq_w$ is a proper quasi-order compatible with v, then $\preceq' = \preceq'_{w/v}$. Moreover we have shown in the previous lemma that w is Manis if and only if w/v is Manis. So we may restrict both the domain and co-domain of ψ to proper quasi-orders that come from a Manis valuation. □

Since v and w are both Manis and $\preceq_w$ is compatible with v, it follows via Lemma 3.16 that the previous corollary characterizes precisely all Manis refinements w of v, if the valuation v (and then also w) is non-trivial.

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DULAC TRANSSERIES AND FORMAL NORMAL FORMS

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Introduction. *A transseries is a generalized formal series whose monomials involve not only powers, but also exponentials and logarithms. Transseries play an important role in dynamical systems as asymptotic expansions of certain functions. Classical results show how for a given holomorphic dynamical system one can determine its simple normal form. We show how to extend the formal version of this result to certain transseries - the so-called* Dulac transseries*. Consideration of this class of transseries is motivated by the study of Hilbert's 16th problem. This note is mostly an overview of results from Mardešić, Resman, Rolin, Županović* [7].

1. Motivation: the first return maps of limit periodic sets of planar vector fields

A very important open problem in dynamical systems is the 16^{th} *Hilbert problem.* It asks if there exists an upper bound for the number of limit cycles of polynomial systems, dependent only on the degree of the system. A local version of the problem is the *cyclicity* problem. The cyclicity is the number of *limit cycles*, that is, closed periodic orbits, that appear in generic bifurcations of an invariant set of the system.

The basic invariant sets in the original planar system which bifurcate into limit cycles are monodromic *elliptic points*, *limit cycles* itself and *hyperbolic polycycles* (consisting of nondegenerate saddle or saddle-node singular points and separatrices joining them). The term monodromic refers to accumulation of spiral trajectories onto the invariant sets. For an overview, see [9].

One way to measure the complexity of such sets is to consider the one-dimensional *first return* (*Poincaré*) *germ* $P(s)$, which is defined locally on a transversal to the invariant set. Take a point s on a transversal, close to the set. Then $P(s)$ is defined as the next intersection of the spiral trajectory through s with the transversal. The origin (the fixed point zero) is the intersection of the transversal with the given invariant set. Closed periodic orbits are recognized as the fixed points of the Poincaré germ.

By [9], the first return map of an elliptic point or a limit cycle is a germ *analytic* at the origin. In the analytic case, the multiplicity of the fixed point zero of a Poincaré map is related to the cyclicity of the invariant set by an explicit formula: the bigger the multiplicity, the bigger the cyclicity. Note that the multiplicity of the fixed point zero of a Poincaré map of a monodromic set in fact describes how fast its iterates tend to zero, that is, the density of accumulation of a spiral trajectory to the invariant set. However, the first return map of a hyperbolic polycycle with saddle points is by [2], [6] a *non-analytic germ* at the origin, analytic on some open positive neighborhood $(0, d)$. It admits an asymptotic expansion in the power-logarithm scale called the *Dulac expansion*:

there exists a sequence of *polynomials* and a strictly increasing, finitely generated sequence (α_i), $\alpha_i \geq 1$, tending to $+\infty$ or finite, such that

$$P - \sum_{i=1}^{n} P_i(-\log s)s^{\alpha_i} = o(s^{\alpha_n}),\ s \to 0,\ n \in \mathbb{N}. \tag{1.1}$$

The formal series $\widehat{P}(s) = \sum_{i=1}^{\infty} P_i(-\log s)s^{\alpha_i}$ is called the *Dulac series*.

We will consider here only the *parabolic* (*tangent to the identity*) Dulac germs, for which $\alpha_1 = 1$ and $P_1 \equiv 1$:

$$P(s) = s + o(s).$$

Compared to the *hyperbolic* cases when $\alpha_1 > 1$ or $\alpha_1 = 1$ and $P_1 \equiv c$, $0 < c < 1$, the convergence of the iterates of the parabolic Poincaré maps to the origin is slower. In hyperbolic cases, see [9], the cyclicity is 1. Due to non-trivial cyclicities (whose upper bounds for general polycycles are not known), the parabolic case is the most interesting case.

One important question in dynamical systems is a reduction of a germ or a vector field to its *normal forms*. That is, by a change of variables (formal, analytic, continuous) we reduce a germ to a simpler germ - its (formal, analytic, topological) normal form. We concentrate here on the *formal normal forms.*

Let us recall the standard results about formal classification of analytic germs. Let f be an analytic germ and let $\widehat{f}$ be its Taylor expansion. By *formal changes of variables*, we mean a sequence of *elementary changes of variables*, $\varphi_1(x) = ax$, $\varphi_n(x) = x + c_n x^n$, $c_n \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$, which are used to eliminate, power-by-power, all powers from $\widehat{f}$ that can be eliminated. That is, we construct a sequence $(\widehat{f_n})$ of *partial normal forms*, where $f_0 = \widehat{f}$ and $f_\infty = \widetilde{f}$ is a formal normal form, by:

$$\widehat{f_n} = \widehat{\varphi}_n^{-1} \circ \widehat{f}_{n-1} \circ \widehat{\varphi}_n,\ n \in \mathbb{N}.$$

The choice of the constant $c_n \in \mathbb{N}$ is such that the conjugation by φ_n eliminates from $\widehat{f}_{n-1}$ the first possible term. At the same time, the conjugation does not change previous terms.

The formal change of variables is thus a formal composition

$$\widehat{\varphi}(x) = \sum_{k=1}^{\infty} a_k x^k,\ a_k \in \mathbb{R},$$

where we do not pose the question of convergence of the above series.

It is a well-known classical result (see e.g. [3], [4], [5]) that by formal change of variables the analytic germs may be reduced to the following *finite* normal forms:

- *strongly hyperbolic* case
$$f(x) = ax^k + o(x^k),\ a > 0,\ k \geq 2\ \Rightarrow\ \widetilde{f}(x) = x^k$$
- *weakly hyperbolic* case
$$f(x) = \lambda x + o(x),\ 0 < \lambda < 1\ \Rightarrow \widetilde{f}(x) = \lambda x$$
- *parabolic* case
$$f(x) = x - ax^{k+1} + o(x^{k+1}),\ a > 0,\ k \in \mathbb{N}$$
$$\Rightarrow \tilde{f}(x) = x - x^{k+1} + \rho x^{2k+1},\ \rho \in \mathbb{R}.$$

The term x^{2k+1} cannot be eliminated. It is called the *residual term.* The formal class is described by two numbers: (ρ, k). Here, $k+1$ is the multiplicity of the fixed point zero of f and $\rho = \mathrm{Res}_0(\frac{1}{x-f(x)})$ is the residual invariant.

In the parabolic case, there is yet another formal normal form which is given as a formal time-one map of a vector field. It is thus related to another important question in dynamical systems: of embedding (formal, analytic) of a germ in a flow or in a vector field. Such normal form is:

$$\overline{f}(x) = \mathrm{Exp}\Big(\frac{-x^{k+1}}{1-(\frac{k+1}{2}-\rho)x^k}\frac{d}{dx}\Big).\mathrm{id}.$$

Here, Exp denotes the *exponential of an operator.* Here, it means the exponential of the formal differential operator $\widehat{\xi}\frac{d}{dx}$, where $\widehat{\xi}$ is the Taylor expansion of $\xi(x) := \frac{-x^{k+1}}{1-(\frac{k+1}{2}-\rho)x^k}$. It is therefore equal to the following formal series:

$$\overline{f}(x) = (\widehat{\xi}\frac{d}{dx}).\mathrm{id} = \mathrm{id} + \widehat{\xi} + \frac{1}{2!}\widehat{\xi}'\widehat{\xi} + \ldots = x - x^{k+1} + \rho x^{2k+1} + \mathrm{h.o.t.}$$

2. Formal reduction of a Dulac series - appearance of the transseries

The goal is to reduce the Dulac series $\widehat{P}$ from (1.1) to a simple formal normal form which is:
(i) finite,
(ii) a formal time-one map of a vector field.
We generalize the standard procedure described above for the formal Taylor series.

We first want to define an appropriate class of formal series containing the Dulac series and which is closed to reduction to formal normal forms.

The admissible set of *elementary* changes of variables is naturally expanded from the simplest formal analytic diffeomorphisms to the simplest power-logarithmic diffeomorphisms:

$$\varphi_{1,0}(x) = cx,\ c \in \mathbb{R},$$
$$\varphi_{\beta,k}(x) = x + c_{\beta,k}x^{\beta}\boldsymbol{\ell}^k,\ c_{\beta,k} \in \mathbb{R},\ (\beta,k) \in \mathbb{R}_{>0}\times\mathbb{Z},\ (\beta,k) \succ (1,0).$$

Here, $\boldsymbol{\ell} := -\frac{1}{\log x}$. Note that $\boldsymbol{\ell} \to 0$, as $x \to 0$. On $\mathbb{R}^2$ we consider the natural *lexicographic order* $\succ$.

In Example 1, starting from a very simple Dulac germ, we perform term-by-term eliminations to reduce it to a finite normal form. We use a family of elementary changes of variables of the type $\varphi_{\beta,k}$. We use Lemma 2.1 which describes how the formal conjugation by an elementary change affects the initial formal series. The notation *h.o.t.* denotes the monomials of the higher (lexicographic) order. We show in Example 1 that during the reduction we leave the class of the Dulac series.

Lemma 2.1. *Let* $\widehat{f}(x) = x + ax^{\alpha}\boldsymbol{\ell}^m + h.o.t.$ *be a formal series consisting of monomials of the type* $x^{\gamma}\boldsymbol{\ell}^{\ell}$, $(\gamma,\ell) \succeq (1,0)$. *Let* $\varphi_{\beta,k}$, $(\beta,k) \succ (1,0)$, *be an elementary* *change of variables. Then:*

$$\widehat{\varphi}_{\beta_k}^{-1} \circ \widehat{f} \circ \widehat{\varphi}_{\beta_k} - \widehat{f} = \left\{ cx^{\beta}\boldsymbol{\ell}^k, ax^{\alpha}\boldsymbol{\ell}^m \right\} + h.o.t. =$$
$$= c(\alpha - \beta)x^{\alpha+\beta-1}\boldsymbol{\ell}^{m+k} + c(k-m)x^{\alpha+\beta-1}\boldsymbol{\ell}^{m+k+1} + h.o.t.$$

Here, $\{.,.\}$ *denotes the* Lie bracket *of two monomials, defined as:*

$$\{m_1(x), m_2(x)\} := m_1'(x)m_2(x) - m_1 m_2'(x).$$

Proof. The proof of the lemma is a simple formal computation. The formal inversion and composition are done monomial by monomial, using the binomial expansions. For details, see [1] or [7, Sections 2, 3]. □

Example 1. ([7, Example 6.2]) Take the *normalized* Dulac germ

$$f(x) = x - x^2\boldsymbol{\ell}^{-1} + x^2.$$

By normalized, we mean that we already have $a = 1$. We do not perform the first homothecy change $\varphi_{1,0}(x) = cx,\ c \in \mathbb{R}$ that normalizes the germ.

Take a change of variables $\varphi_{\beta,k}(x) = x + c_{\beta,k}x^{\beta}\boldsymbol{\ell}^k,\ c_{\beta,k} \in \mathbb{R},\ (\beta, k) \in \mathbb{R}_{>0} \times \mathbb{Z},\ (\beta, k) \succ (1, 0)$. By choosing appropriate β, k, $c_{\beta,k}$, we eliminate monomial by monomial all possible terms from $\widehat{f}$. By Lemma 2.1,

$$\widehat{\varphi}_{\beta,k}^{-1} \circ \widehat{f} \circ \widehat{\varphi}_{\beta,k} = \widehat{f} + c_{\beta,k}(\beta - 2)x^{\beta+1}\boldsymbol{\ell}^{k-1} + c(k+1)x^{\beta+1}\boldsymbol{\ell}^k + \text{h.o.t.}$$

We conclude:

1. We can eliminate, by appropriate choices of constants $c_{\beta,k}$, all terms except for the first $x^2\boldsymbol{\ell}^{-1}$ (since $(\beta, k) \succ (1, 0)$) and the *residual* $x^3\boldsymbol{\ell}^{-1}$ (corresponds to $(\beta, k) = (2, 0)$, but stays intact in the change).

2. To eliminate x^2, we apply a series of changes $\varphi_{1,m}$, $m \in \mathbb{N}$. By the change $\varphi_{1,m}(x) = x + c_{1,m}x\boldsymbol{\ell}^m,\ c_{1,m} \in \mathbb{R}$, we eliminate the term $x^2\boldsymbol{\ell}^{m-1}$, but at the same time generate the *next one*: $x^2\boldsymbol{\ell}^m$, $m \in \mathbb{N}$.

$$f(x) = x + x^2\boldsymbol{\ell}^{-1} + x^2 \overset{\varphi_{1,1}(x)=x+c_{1,1}x\boldsymbol{\ell}}{\longrightarrow} \widehat{f}_1(x) = x + x^2\boldsymbol{\ell}^{-1} + a_1x^2\boldsymbol{\ell} + h.o.t.$$
$$\overset{\varphi_{1,2}(x)=x+c_{1,2}x\boldsymbol{\ell}^2}{\longrightarrow} \widehat{f}_2(x) = x + x^2\boldsymbol{\ell}^{-1} + a_2x^2\boldsymbol{\ell}^2 + h.o.t. \longrightarrow \ldots$$

By Example 1, to reduce a Dulac germ to its finite formal normal form, a simple sequence of changes of variables (indexed by ω) is not sufficient. In general we need a *transfinite sequence of changes of variables*. This motivates the definition of the universal class $\widehat{\mathcal{L}}$ of *power-logarithmic transseries* and of a subclass $\widehat{\mathcal{L}}_D \subset \widehat{\mathcal{L}}$ of *Dulac transseries* in the next section. Both classes contain Dulac series and are closed to reductions to normal forms.

There are three questions to be addressed in the next section, to prove the main Theorem about the formal normal forms for power-logarithmic transseries.

(1) Determining the exact normal form in $\widehat{\mathcal{L}}$.
(2) Ensuring that at each point of the elimination the successor step is uniquely determined. That is, that the changes of variables needed for the reduction are indexed by a well-ordered set. Then we can perform the algorithm using a *transfinite sequence of changes.*
(3) Ensuring that the transfinite composition of the changes of variables converges in some formal topology in $\widehat{\mathcal{L}}$.

3. Main results: normal forms for power-log transseries

3.1. The class $\widehat{\mathcal{L}}$ of formal power-logarithmic transseries. Three formal topologies.

Definition 3.1. *The class $\widehat{\mathcal{L}}$ is the set of all formal transseries of the form:*

$$\widehat{f}(x)=\sum_{\alpha\in S}\sum_{k\in\mathbb{Z}}a_{\alpha,k}x^{\alpha}\boldsymbol{\ell}^{k},\ \ a_{\alpha,k}\in\mathbb{R}, \tag{3.1}$$

where the support $\mathcal{S}(f)=\{(\alpha,k)\in\mathbb{R}_{>0}\times\mathbb{Z}\colon a_{\alpha,k}\neq 0\}$ is a well-ordered subset of $\mathbb{R}_{>0}\times\mathbb{Z}$, equipped with the lexicographic order $\prec$. *In other words, S is well-ordered with respect to the usual order $<$ on $\mathbb{R}$, and for every $\alpha\in S$ there exists $n_\alpha\in\mathbb{Z}$ such that $a_{\alpha,n}=0$, for every $n<n_\alpha$.*

The class of Dulac transseries *is a subset $\widehat{\mathcal{L}}_D\subset\widehat{\mathcal{L}}$ of transseries of the form* (3.1) *for which the set S is finitely generated.*

By $\widehat{\mathcal{L}}_0$ resp. $\widehat{\mathcal{L}}_D^0$ we denote the formal diffeomorphisms, that is, the transseries from $\widehat{\mathcal{L}}$ resp. $\widehat{\mathcal{L}}_D^0$ of the form $\widehat{f}(x)=ax+h.o.t.,\ a\in\mathbb{R}$.

The classes $\widehat{\mathcal{L}}$, as well as $\widehat{\mathcal{L}}_D$, contain the Dulac series. The formal changes of variables as composition of elementary changes belong to $\widehat{\mathcal{L}}_0$ resp. $\widehat{\mathcal{L}}_D^0$.

Note that the lexicographic order on pairs (α,k) corresponds to the order of the growth of the monomials $x^\alpha\boldsymbol{\ell}^m$, as $x\to 0$:

$$(\alpha,k)\prec(\alpha',k')\Longleftrightarrow\lim_{x\to 0^+}\frac{x^{\alpha'}\boldsymbol{\ell}^{k'}}{x^{\alpha}\boldsymbol{\ell}^{k}}=0.$$

Let $\widehat{f}\in\widehat{\mathcal{L}}$, $\widehat{f}=ax^\gamma\boldsymbol{\ell}^r+\text{h.o.t.}$ We define the *order of $\widehat{f}$* as the order of its leading term, and denote it by $\operatorname{ord}(\widehat{f})=(\gamma,r)$.

The class $\widehat{\mathcal{L}}$ of power-logarithm transseries is a subclass of the general field $\mathbb{R}((x^{-1}))^{LE}$ of *logarithmic-exponential series* defined in [1]. Additionally, it has been shown in [7] that the classes $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_D$ are algebras without unity with respect to operations of formal addition, multiplication and composition inherited from $\mathbb{R}((x^{-1}))^{LE}$. Recall that the operations of multiplication and of composition are well-defined due to the *Neumann's lemma* from [8], which ensures the *convergence* of the operations. In simple words: formal multiplication and composition are done termwise (in the composition using binomial expansions in each term), and then grouping the terms with the same monomials. The Neumann's lemma ensures that only finitely many monomials of each factor contribute to each monomial of the product/composition. Therefore, the coefficient of every monomial in the product/composition is well-defined.

Remark 3.2. The transseries from $\widehat{\mathcal{L}}$ are indexed by *ordinals*. In the case of a Dulac transseries $\widehat{f}\in\widehat{\mathcal{L}}_D$, the possible limit ordinals are $(\alpha_i,\,n_{\alpha_i})$ (and the final), where $\alpha_i\in S$ and n_{α_i} is as in Definition 3.1. The others (except the smallest) are necessarily successor ordinals. For example, for

$$\widehat{f}(x)=x+x^2\sum_{k=1}^{\infty}a_k\boldsymbol{\ell}^k+x^3(\boldsymbol{\ell}+2\boldsymbol{\ell}^2)+x^4\sum_{k=3}^{\infty}b_k\boldsymbol{\ell}^k+\sum_{k=5}^{\infty}x^k\in\widehat{\mathcal{L}}_D,\ a_k,\ b_k\neq 0,$$

the limit ordinals are just $(3,1),(5,0)$ (and the final).

3.1.1. *The formal convergence in* $\widehat{\mathcal{L}}$.
Suppose that a topology is imposed on $\widehat{\mathcal{L}}$. Let θ be an ordinal. We say that a transfinite sequence $(\widehat{f}_\beta)_{\beta<\theta}$, $\widehat{f}_\beta \in \widehat{\mathcal{L}}$, *converges to* $\widehat{f} \in \widehat{\mathcal{L}}$ as $\beta \to \theta$, and write

$$\widehat{f} := \lim_{\beta\to\theta} \widehat{f}_\beta,$$

if, for every open neighborhood U of $\widehat{f}$, there exists an ordinal $\beta_0 < \theta$ such that $f_\beta \in U$ for all β such that $\beta_0 < \beta < \theta$.

The proofs of convergence in $\widehat{\mathcal{L}}$ rely on the *principle of transfinite induction.*

3.1.2. *Three topologies on* $\widehat{\mathcal{L}}$. We impose three topologies on $\widehat{\mathcal{L}}$, mentioned by decreasing strength:

(1) *The strong formal topology.* For $\widehat{f} \in \widehat{\mathcal{L}}$, the fundamental system of neighborhoods is given by the balls:

$$B_{(\alpha,k)}(\widehat{f}) = \left\{\widehat{g} \in \widehat{\mathcal{L}} \colon \mathrm{ord}\big(\widehat{g} - \widehat{f}\big) \succ (\alpha, k)\right\}, (\alpha, k) \in \mathbb{R}_{>0} \times \mathbb{Z}.$$

In the strong topology, $(\widehat{f}_\beta)_{\beta<\omega} \to f$ if, for every order $(\gamma, m) \in \mathbb{R}_{>0} \times \mathbb{Z}$, there exists an ordinal $\beta_0 < \omega$ such that $\mathrm{ord}(\widehat{f} - \widehat{f}_\beta) > (\gamma, m)$ for all β such that $\beta_0 < \beta < \omega$.

(2) *The product topology with respect to the discrete topology on* $\mathbb{R}$. We consider the class $\widehat{\mathcal{L}}$ as a subspace of the Cartesian product $\mathbb{R}^{\mathbb{R}_{>0}\times\mathbb{Z}}$. This Cartesian product is the set of functions from $\mathbb{R}_{>0} \times \mathbb{Z}$ to $\mathbb{R}$, which to every pair $(\alpha, m) \in \mathbb{R}_{>0} \times \mathbb{Z}$ attributes the coefficient of the monomial $x^\alpha \boldsymbol{\ell}^m$ in $\widehat{f}$, or zero if $x^\alpha \boldsymbol{\ell}^m \notin \mathcal{S}(\widehat{f})$.

Obviously, $(\widehat{f}_\beta)_{\beta<\omega} \to \widehat{f}$ in this topology if for every pair $(\alpha, m) \in \mathbb{R}_{>0} \times \mathbb{Z}$ there exists an ordinal $\beta_0 < \omega$ such that the coefficient of the monomial $x^\alpha \boldsymbol{\ell}^m$ in the sequence $(\widehat{f}_\beta)_{\beta_0<\beta<\omega}$ *stabilises* (is constant).

(3) *The product topology with respect to the euclidean topology on* $\mathbb{R}$. This is the standard convergence by components. The sequence $(\widehat{f}_\beta)_{\beta<\omega} \to f$ in this topology if, for every pair $(\alpha, m) \in \mathbb{R}_{>0} \times \mathbb{Z}$, it holds that $\big(\widehat{f}_\beta(\alpha, m)\big)_{\beta<\omega} \to \widehat{f}(\alpha, m)$ in the euclidean topology in $\mathbb{R}$.

Example 2.

(1) In the case of formal Taylor series $\widehat{f}$ and formal power series changes of variables, the convergence of the sequence partial normal forms $\widehat{f}_n = \widehat{\varphi}_n^{-1} \circ \widehat{f} \circ \widehat{\varphi}_n$ to the formal normal form $\widetilde{f}$, as well as of the sequence of partial compositions $\widehat{\varphi}_n = \circ_{k=1}^n \widehat{\varphi}_k$ to formal composition $\widehat{\varphi}$ such that $\widetilde{f} = \widehat{\varphi}^{-1} \circ \widehat{f} \circ \widehat{\varphi}$ is in the *strong topology*. Indeed, $\mathrm{ord}(\widehat{\varphi} - \widehat{\varphi}_n)$, $\mathrm{ord}(\widetilde{f} - \widehat{f}_n) \to \infty$, as $n \to \infty$.

(2) The sequence $(\widehat{\varphi}_{1,n})_{n\in\mathbb{N}} \subset \widehat{\mathcal{L}}$ needed to eliminate the term x^2 from Dulac series f in Example 1 converges to $\widehat{\varphi} := \circ_{n\in\mathbb{N}} \widehat{\varphi}_{1,n} \in \widehat{\mathcal{L}}$ in the topology (2), but not in the strong topology. In general, we cannot expect that the transfinite sequence of the changes of variables needed for formal reduction of a Dulac series will converge in the strong topology. However, we have proven in [7] that it converges in the weaker topology (2).

Similarly, the sequence $\widehat{f}_n(x) = x^{2-\frac{1}{n}}$ converges to 0 in topology (2), but not in the strong topology on $\widehat{\mathcal{L}}$.

(3) The sequence $\widehat{f}_n(x) = \frac{x}{n}$ does not converge in the strong or in topology (2) in $\widehat{\mathcal{L}}$, but it converges to zero transseries in the weakest topology (3).

3.2. **Formal normal forms for transseries in $\widehat{\mathcal{L}}$ or $\widehat{\mathcal{L}}_D$.**

Theorem. ([7, Theorem A]) *Let $\widehat{f} \in \widehat{\mathcal{L}}$ resp. $\widehat{f} \in \widehat{\mathcal{L}}_D$, without a logarithm in the leading term. Then $\widehat{f}$ is formally equivalent in $\widehat{\mathcal{L}}_0$ resp. $\widehat{\mathcal{L}}_D^0$ to the following finite normal forms $\widetilde{f}$:*

(1) (parabolic case)

$$\widehat{f}(x) = x + ax^{\alpha}\boldsymbol{\ell}^k + \text{h.o.t},\ (\alpha,k) \in \mathbb{R}_{>0} \times \mathbb{Z},\ (\alpha,k) \succ (1,0);\ a \in \mathbb{R},\ a \neq 0;$$
$$\widetilde{f}(x) = x + ax^{\alpha}\boldsymbol{\ell}^k + \rho x^{2\alpha-1}\boldsymbol{\ell}^{2k+1},\ \rho \in \mathbb{R}.$$

(2) (hyperbolic case)

$$\widehat{f}(x) = \lambda x + ax\boldsymbol{\ell} + \text{h.o.t},\ a \in \mathbb{R},\ \lambda > 0,\ \lambda \neq 1;$$
$$\widetilde{f}(x) = \lambda x + ax\boldsymbol{\ell}.$$

(3) (strongly hyperbolic case)

$$\widehat{f}(x) = \lambda x^{\alpha} + \text{h.o.t},\ \lambda > 0,\ \alpha \neq 1; \qquad \widetilde{f}(x) = x^{\alpha}.$$

Let $\widehat{f}$ be hyperbolic or parabolic. Then $\widehat{f}$ is formally equivalent in $\mathcal{L}^0$ resp. in $\mathcal{L}^0_{\mathfrak{D}}$ to the formal normal form $\overline{f} \in \widehat{\mathcal{L}}$ resp. *$\overline{f} \in \widehat{\mathcal{L}}_{\mathfrak{D}}$ which is the formal time-one map of the following vector fields:*

(1) (parabolic case)

$$\overline{f}(x) = \exp(X_{\alpha,k,a,\rho}).\text{id},$$
$$X_{\alpha,k,a,b} = \frac{ax^{\alpha}\boldsymbol{\ell}^k}{1 + \frac{a\alpha}{2}x^{\alpha-1}\boldsymbol{\ell}^k - (\frac{ak}{2} + \frac{\rho}{a})x^{\alpha-1}\boldsymbol{\ell}^{k+1}}\frac{\mathrm{d}}{\mathrm{d}x}.$$

(2) (hyperbolic case)

$$\overline{f}(x) = \exp(X_{\lambda,a}).\text{id}, \quad X_{\lambda,a} = \frac{\ln\lambda \cdot x}{1 + \frac{a}{2(\lambda-1)}\boldsymbol{\ell}}\frac{\mathrm{d}}{\mathrm{d}x}.$$

In other words, the parabolic and hyperbolic transseries from $\widehat{\mathcal{L}}$ (resp. $\widehat{\mathcal{L}}_D$) can be formally embedded in a flow of a vector field *from $\widehat{\mathcal{L}}$ (resp. $\widehat{\mathcal{L}}_D$).*

In the parabolic case, the formal invariants are the quadruples:

$$(\alpha,k,a,\rho);\ \alpha \geq 1,\ k \in \mathbb{Z},\ \rho \in \mathbb{R},\ a \neq 0.$$

Proof. We only give a sketch of the proof in the parabolic case. The complete proof of the theorem can be found in [7, Section 4].

Let $\widehat{f} = x + ax^{\alpha}\boldsymbol{\ell}^m + \text{h.o.t.}$ be parabolic. To derive the finite normal form, we use the term-by term procedure and Lie brackets, as described in Lemma 2.1 and Example 1. By Lemma 2.1, the only two monomials that cannot be eliminated are: the first monomial $x^{\alpha}\boldsymbol{\ell}^m$ and the residual monomial $x^{2\alpha-1}\boldsymbol{\ell}^{2m+1}$.

By a meticulous analysis of terms that appear in the termwise eliminations, we see that the changes of variables are indexed by a well-ordered set ω. We prove by transfinite induction that the transfinite sequence of partial compositions $\widehat{\varphi}_\beta := \circ_{\gamma<\beta}\widehat{\varphi}_\gamma$, $\beta < \omega$, converges to $\widehat{\varphi}_\omega := \circ_{\gamma<\omega}\widehat{\varphi}_\gamma \in \widehat{\mathcal{L}}$ in the product topology with respect to the discrete topology. We also prove that the partial normal forms $\widehat{f}_\beta := \widehat{\varphi}_\beta^{-1} \circ f \circ \varphi_\beta$ converge to $\tilde{f} := \widehat{\varphi}_\omega^{-1} \circ \widehat{f} \circ \widehat{\varphi}_\omega$ in the same topology on $\widehat{\mathcal{L}}$.

□

3.3. **The convergence of the operator exponential.**

We illustrate on simple examples the type of the convergence in $\widehat{\mathcal{L}}$ (i.e. the appropriate topology) for the operator exponential series from the Theorem that defines the formal normal form $\overline{f}$. We distinguish the parabolic and the hyperbolic case. Let $\widehat{\xi} \in \widehat{\mathcal{L}}$ be such that in the Theorem

$$\overline{f} = \exp(\widehat{\xi}\frac{d}{dx}).\mathrm{id}. \tag{3.2}$$

(1) $\widehat{f}$ *parabolic*, $\widehat{f}(x) = x + ax^\alpha \boldsymbol{\ell}^k + \text{h.o.t.}$, $a \in \mathbb{R}$, $\alpha > 1$, $k \in \mathbb{Z}$. Then $\mathrm{ord}(\widehat{\xi}) = (\alpha, k)$. The series (3.2) converges in the strong formal topology.

Example 3. Take $\widehat{\xi}(x) = x^2\boldsymbol{\ell}$.

$$\exp(\widehat{\xi}\frac{d}{dx}).\mathrm{id} = x + x^2\boldsymbol{\ell} + \frac{1}{2}x^3(2\boldsymbol{\ell}^2 + \boldsymbol{\ell}^3) + \dots$$

In each summand the powers of x are getting bigger by some constant positive value, so the convergence is in the strong formal topology.

(2) $\widehat{f}$ *parabolic*, $\widehat{f}(x) = x + ax\boldsymbol{\ell}^k + \text{h.o.t.}$, $a \in \mathbb{R}$, $k \in \mathbb{N}$. Then $\mathrm{ord}(\widehat{\xi}) = (1, k)$. The convergence of the operator series (3.2) in $\widehat{\mathcal{L}}$ is in the product topology with respect to the discrete topology.

Example 4. Take $\widehat{\xi}(x) = x\boldsymbol{\ell}$.

$$\begin{aligned}\exp(\widehat{\xi}\frac{d}{dx}).\mathrm{id} &= x + x\boldsymbol{\ell} + \frac{1}{2!}(x\boldsymbol{\ell}^2 + x\boldsymbol{\ell}^3) + \frac{1}{3!}(x\boldsymbol{\ell}^3 + 3x\boldsymbol{\ell}^4 + 3x\boldsymbol{\ell}^5) + \dots = \\ &= x + x\boldsymbol{\ell} + \frac{1}{2!}x\boldsymbol{\ell}^2 + (\frac{1}{2!} + \frac{1}{3!})x\boldsymbol{\ell}^3 + \dots\end{aligned}$$

The sequence of partial sums does not converge in the strong topology since the orders of monomials are never getting bigger than 1 in x. Nevertheless, the coefficient in front of every monomial $x\boldsymbol{\ell}^k$ stabilizes (the monomial $x\boldsymbol{\ell}^k$ is present only in the first $k+1$ summands).

(3) $\widehat{f}$ *hyperbolic*, $\widehat{f}(x) = \lambda x + \text{h.o.t.}$, $\lambda \neq 0$. Then $\mathrm{ord}(\widehat{\xi}) = (1, 0)$. The series (3.2) converges in the product topology with respect to the euclidean topology.

Example 5. Take $\widehat{\xi}(x) = \lambda x$, $\lambda \neq 0$.

$$\exp(\widehat{\xi}\frac{d}{dx}).\mathrm{id} = x + \lambda x + \frac{1}{2!}\lambda^2 x + \frac{1}{3!}\lambda^3 x + \dots = e^\lambda x.$$

(4) $\widehat{f}$ *strongly hyperbolic*, $\widehat{f}(x) = ax^\alpha + \text{h.o.t.}$, $a \in \mathbb{R}$, $\alpha > 1$. Note that we do not claim in the Theorem that the normal form can be expressed as the formal time-one map of a vector field. The finite formal normal form is $\widetilde{f}(x) = x^\alpha$. It can nevertheless be checked that, as a germ, it is the time-one map of a vector field $X = \log\alpha \cdot x\log x\frac{d}{dx}$. However, the formula (3.2) for

$\widehat{\xi}(x) = \log\alpha\cdot x\log x$ does not converge in any of the three formal topologies, since the orders of the summands are decreasing instead of increasing:

$$\exp\big(\log\alpha\cdot x\log x\frac{d}{dx}\big).id = x + \log\alpha{\cdot}x\log x + \frac{\log^2\alpha}{2!}(x\log^2 x + x\log x) + \\ + \frac{\log^3\alpha}{3!}(x\log^3 x + 3x\log^2 x + x\log x) + \dots$$

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ILYASHENKO ALGEBRAS BASED ON log-exp-ANALYTIC MONOMIALS

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ABSTRACT. We construct a Hardy field that contains Ilyashenko's class of germs at $+\infty$ of almost regular functions found in [2] as well as all log-exp-analytic germs. In addition, each germ in this Hardy field is uniquely characterized by an asymptotic expansion that is an LE-series as defined by van den Dries et al. [14]. As these series generally have support of order type larger than ω, the notion of asymptotic expansion itself needs to be generalized.

INTRODUCTION

We construct a Hardy field that contains Ilyashenko's class of germs at $+\infty$ of almost regular functions found in [2] as well as all log-exp-analytic germs. In addition, each germ in this Hardy field is uniquely characterized by an asymptotic expansion that is an LE-series as defined by van den Dries et al. [14]. As these series generally have support of order type larger than ω, the notion of asymptotic expansion itself needs to be generalized. This can be done naturally in the context of a quasianalytic algebra, leading to our definition of *quasianalytic asymptotic algebra*, or *qaa algebra* for short. Any qaa algebra constructed by generalizing Ilyashenko's construction will be called an **Ilyashenko algebra**. The last author's paper [11] contains a first attempt at constructing an Ilyashenko field.

Our main motivation for generalizing Ilyashenko's construction in this way is the conjecture that the class of almost regular germs generates an o-minimal structure over the field of real numbers. This

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conjecture, in turn, might lead to locally uniform bounds on the number of limit cycles in subanalytic families of real analytic planar vector fields all of whose singularities are hyperbolic; see Kaiser et al. [6] for explanations and a positive answer in the special case where all singularities are, in addition, non-resonant. (For a different treatment of the general hyperbolic case, see Mourtada [9].) These almost regular functions also play a role in the description of Riemann maps and solutions of Dirichlet's problem on semianalytic domains; see Kaiser [4, 5] for details.

Our hope is to eventually settle the general hyperbolic case by adapting the procedure in [6], which requires three main steps:

(1) extend the class of almost regular germs into an Ilyashenko field;
(2) construct corresponding algebras of germs of functions in several variables, such that the resulting system of algebras is stable under various operations (such as blowings-up, say);
(3) obtain o-minimality using a normalization procedure.

While [11] contains a first successful attempt at Step (1), Step (2) poses some challenges. For instance, it is not immediately obvious what the nature of logarithmic generalized power series (as defined in [11] and used as asymptotic expansions there) in several variables should be; they should at least be stable under all the operations required for Step (3). They should also contain the series used in [9] to characterize parametric transition maps in the hyperbolic case, which use *Ecalle-Roussarie compensators* as monomials.

Our approach to this problem is to enlarge the set of monomials used in asymptotic expansions. A first candidate for such a set of monomials is the set of all (germs of) functions definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$; see van den Dries and Miller [15] and van den Dries et al. [12]. This set of germs is obviously closed under the required operations, because the latter are all definable, and it contains the Ecalle-Roussarie compensators. However, it is too large to be meaningful for use as monomials in asymptotic expansions, as it is clearly not $\mathbb{R}$-linearly independent (neither in the additive nor the multiplicative sense) and contains many germs that have "similar asymptotic behavior" such as, in the case of unary germs, belonging to the same archimedean class. More suitable would be to find a minimal subclass $\mathcal{L}_n$ of all definable n-variable germs such that every definable n-variable germ is piecewise given by a convergent Laurent series (or, if necessary, a convergent *generalized* Laurent series, see van den Dries and Speissegger [16]) in a finite tuple of germs in $\mathcal{L}_n$.

Thus, the purpose of this paper is to determine such a minimal set of monomials $\mathcal{L} = \mathcal{L}_1$ contained in the set $\mathcal{H}$ of all unary germs at $+\infty$ definable in $\mathbb{R}_{\text{an,exp}}$, and to further adapt the construction in [11] to corresponding generalized series in one variable. Recalling that $\mathcal{H}$ is a Hardy field, we can summarize the results of this paper (Theorems 1.12 and 1.13 below) as follows:

Main Theorem. *There is a multiplicative subgroup $\mathcal{L}$ of $\mathcal{H}$ such that the following hold:*

(1) *no two germs in $\mathcal{L}$ belong to the same archimedean class;*
(2) *every germ in $\mathcal{H}$ is given by composing a convergent Laurent series with a tuple of germs in $\mathcal{L}$;*
(3) *the construction in* [11] *generalizes, after replacing the finite iterates of* log *with germs in $\mathcal{L}$, to obtain a corresponding Ilyashenko field $\mathcal{K}$.*

The resulting Ilyashenko field $\mathcal{K}$ is a Hardy field extending $\mathcal{H}$ as well as the Ilyashenko field $\mathcal{F}$ constructed in [11].

Remark. The Ilyashenko field $\mathcal{F}$ constructed in [11] does not extend $\mathcal{H}$.

We obtain this set $\mathcal{L}$ of monomials by giving an explicit description of the Hardy field $\mathcal{H}$ as the set of all *convergent LE-series*, as suggested in [14, Remark 6.31], with $\mathcal{L}$ being the corresponding set of *convergent LE-monomials*. The proof that the construction in [11] generalizes to this set $\mathcal{L}$ relies heavily on our recent paper [7]; indeed, our construction here was the main motivation for [7]. In the next two sections, we give a more detailed overview of the definitions and results of this paper (Section 1) and their proof (Section 2).

1. Main definitions and results

We let $\mathcal{C}$ be the ring of all germs at $+\infty$ of continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. A germ $f \in \mathcal{C}$ is **small** if $\lim_{x\to+\infty} f(x) = 0$ and **large** if $\lim_{x\to+\infty} |f(x)| = \infty$. To compare elements of $\mathcal{C}$, we use the dominance relation $\prec$ as found in Aschenbrenner and van den Dries [1, Section 1], defined by $f \prec g$ if and only if $g(x)$ is ultimately nonzero and $\lim_{x\to+\infty} f(x)/g(x) = 0$, or equivalently, if and only if $g(x)$ is ultimately nonzero and $f(x) = o(g(x))$ as $x \to +\infty$. Thus, $f \preceq g$ if and only if $f(x) = O(g(x))$ as $x \to +\infty$, and we write $f \asymp g$ if and only if $f \preceq g$ and $g \preceq f$. Note that the relation $\asymp$ is an equivalence relation on $\mathcal{C}$, and the corresponding equivalence classes are the **archimedean classes** of $\mathcal{C}$; we denote by $\Pi_\asymp : \mathcal{C} \longrightarrow \mathcal{C}/_\asymp$ the corresponding projection map.

We denote by $\mathcal{H} \subseteq \mathcal{C}$ the Hardy field of all germs of unary functions definable in $\mathbb{R}_{\text{an,exp}}$. Below, we let K be a commutative ring of characteristic 0 with unit 1.

Recall from [16] that a **generalized power series** over K is a power series $G = \sum_{\alpha \in [0,\infty)^n} a_\alpha X^\alpha$, where $X = (X_1, \dots, X_n)$, each $a_\alpha \in K$ and the **support** of G,

$$\operatorname{supp}(G) := \{\alpha \in [0,\infty)^n : \ a_\alpha \neq 0\},$$

is contained in a cartesian product of well-ordered subsets of $\mathbb{R}$. Moreover, we call the support of G **natural** (see Kaiser et al. [6]) if, for every $a > 0$, the intersection $[0,a) \cap \Pi_{X_i}(\operatorname{supp}(G))$ is finite, where $\Pi_{X_i} : \mathbb{R}^n \longrightarrow \mathbb{R}$ denotes the projection on the ith coordinate.

Throughout this paper, we work with the following series: we fix a multiplicative $\mathbb{R}$-vector subspace M of

$$\mathcal{H}^{>0} := \{h \in \mathcal{H} : \ h > 0\}.$$

Definition 1.1. An M**-generalized Laurent series (over** K**)** is a series of the form $n \cdot G(m_0, \dots, m_k)$, where $k \in \mathbb{N}$, $G(X_0, \dots, X_k)$ is a generalized power series with natural support, $m_0, \dots, m_k \in M$ are small and n belongs to the $\mathbb{R}$-multiplicative vector space $\langle m_0, \dots, m_k \rangle^\times$ generated by $m_0, \dots, m_k$. In this situation, we say that F has **generating monomials** $m_0, \dots, m_k$.

Example 1.2. Every logarithmic generalized power series, as defined in [11, Introduction], is an L-generalized Laurent series, where

$$L := \langle \exp, x, \log, \log_2, \dots \rangle^\times$$

is the multiplicative $\mathbb{R}$-vector space generated by $\{\exp, x, \log, \log_2, \dots\}$ and $\log_i$ denotes the i-th compositional iterate of log.

Every M-generalized Laurent series F belongs to the ring $K((M))$ of *generalized series*, as defined for instance in [13]. Correspondingly, we write $F = \sum_{m \in M} a_m m$, and the **support** of F is the reverse-well ordered set

$$\operatorname{supp}(F) := \{m \in M : \ a_m \neq 0\}.$$

We show that the set $K((M))^{\text{ls}}$ of all M-generalized Laurent series is a subring of $K((M))$ in general, and is a subfield if K is a field.

Remarks 1.3. Let $F \in K((M))^{\text{ls}}$, and let $m_0, \dots, m_k \in M$ be small, $n \in \langle m_0, \dots, m_k \rangle^\times$ and a generalized power series G with natural support be such that $F = n \cdot G(m_0, \dots, m_k)$.

(1) We show that the support of F is of reverse-order type at most ω^{k+1}. For instance, the logarithmic generalized power series
$$\sum_{m,n\in\mathbb{N}} x^{-m}\exp^{-n}$$
has reverse-order type ω^2.

(2) The latter is not a unique representation of F as an M-generalized Laurent series: taking, say,
$$H(X_0,\dots,X_k) := G\left(X_0^2, X_1,\dots,X_k\right),$$
we have $F = n\cdot H\left(\sqrt{m_0}, m_1,\dots,m_k\right)$ as well.

To justify using M-generalized Laurent series as asymptotic expansions, we need some further notations.

Definition 1.4. (1) M is an **asymptotic scale** if $m \asymp 1$ implies $m = 1$, for $m \in M$ (or, equivalently, if every Archimedean class of $\mathcal{H}^{>0}$ has at most one representative in M).

(2) A set $S \subseteq M$ is called M**-natural** if, for all $a \in M$, the intersection $S \cap (a, +\infty)$ is finite.

Examples 1.5. (1) For $k \in \mathbb{N}$ we set
$$L_k := \langle \exp, x, \log, \dots, \log_{k-1}\rangle^\times \subset L.$$
It follows from basic calculus that L, and hence each L_k, is an asymptotic scale. Every *Dulac series* (see Ilyashenko and Yakovenko [3, Section 24]) belongs to $\mathbb{R}\left(\!\left(L_1\right)\!\right)$ and has L_1-natural support.

(2) Let $\mathcal{L}$ be the set of all *principal monomials* of $\mathcal{H}$ as defined in [7, Section 2]. Since every Archimedean class of $\mathcal{H}^{>0}$ has a unique representative in $\mathcal{L}$ [7, Proposition 2.18(2)], the latter is a maximal asymptotic scale.

(3) Two germs $g, h \in \mathcal{C}$ are **comparable** if there exist $r, s > 0$ such that $|f|^r < |g| < |f|^s$ (see Rosenlicht [10]). We show that, if $G(X_1,\dots,X_k)$ is a generalized power series with natural support and $m_1,\dots,m_k \in M$ are small and pairwise comparable and $n \in \langle m_1,\dots,m_k\rangle^\times$, the M-generalized Laurent series $nG(m_1,\dots,m_k)$ has $\langle m_1,\dots,m_k\rangle^\times$-natural support.

We assume from now on that M is an asymptotic scale.

Definition 1.6. Let $f \in \mathcal{C}$ and $F = \sum a_m m \in \mathbb{R}\left(\!\left(M\right)\!\right)$. We say that f has **asymptotic expansion** F (at $+\infty$) if $\mathrm{supp}(F)$ is M-natural and

$$(*) \qquad f - \sum_{m\geq n} a_m m \prec n$$

for every $n \in M$.

Example 1.7. Every *almost regular* $f \in \mathcal{C}$, in the sense defined in the introduction of [11], has an asymptotic expansion in $\mathbb{R}\left(\!\left(L_1\right)\!\right)^{\mathrm{ls}}$.

We denote by $\mathcal{C}(M)$ the set of all $f \in \mathcal{C}$ that have an asymptotic expansion in $\mathbb{R}((M))$. We show that $\mathcal{C}(M)$ is an $\mathbb{R}$-algebra, every $f \in \mathcal{C}(M)$ has a unique asymptotic expansion $T_M(f)$ in $\mathbb{R}((M))$, and the map $T_M : \mathcal{C}(M) \longrightarrow \mathbb{R}((M))$ is an $\mathbb{R}$-algebra homomorphism. In this paper, we are interested in the following kind of subalgebras of $\mathcal{C}(M)$:

Definition 1.8. We call a subalgebra $\mathcal{K}$ of $\mathcal{C}(M)$ **quasianalytic** if the restriction of T_M to $\mathcal{K}$ is injective.

Note that, since $\mathbb{R}((M))$ is a field, every quasianalytic subalgebra of $\mathcal{C}(M)$ is an integral domain.

We now want to extend the definition of asymptotic expansion to all series in $\mathbb{R}((M))$, not just the ones with natural support. However, for this generalization we cannot separate "asymptotic expansion" from "quasianalyticity"; both need to be defined simultaneously in the context of a ring of germs, in the spirit of [11, Definition 2].

For $F = \sum a_m m \in \mathbb{R}((M))$ and $n \in M$, we denote by

$$F_n := \sum_{m \geq n} a_m m$$

the **truncation** of F above n. A subset $S \subseteq \mathbb{R}((M))$ is **truncation closed** if, for every $F \in S$ and $n \in M$, the truncation F_n belongs to S.

Example 1.9. The set $T_M(\mathcal{C}(M))$ is truncation closed.

Definition 1.10. Let $\mathcal{K} \subseteq \mathcal{C}$ be an $\mathbb{R}$-subalgebra and $T : \mathcal{K} \longrightarrow \mathbb{R}((M))$ be an $\mathbb{R}$-algebra homomorphism. The triple $(\mathcal{K}, M, T)$ is a **quasianalytic asymptotic algebra** (or **qaa algebra** for short) if

(i) T is injective;
(ii) the image $T(\mathcal{K})$ is truncation closed;
(iii) for every $f \in \mathcal{K}$ and every $n \in M$, we have

$$f - T^{-1}((Tf)_n) \prec n.$$

Example 1.11. We show that there is a field homomorphism $S_{\mathcal{L}} : \mathcal{H} \longrightarrow \mathbb{R}((\mathcal{L}))$ such that $(\mathcal{H}, \mathcal{L}, S_{\mathcal{L}})$ is a qaa field. Moreover, the image of $S_{\mathcal{L}}$ is the set of all convergent $\mathcal{L}$-generalized Laurent series over $\mathbb{R}$.

Let $M' \subset \mathcal{H}^{>0}$ be another asymptotic scale, and let $(\mathcal{K}, M, T)$ and $(\mathcal{K}', M', T')$ be two qaa algebras. We say that $(\mathcal{K}, M, T)$ **extends** $(\mathcal{K}', M', T')$ if $\mathcal{K}'$ is a subalgebra of $\mathcal{K}$, M' is a multiplicative $\mathbb{R}$-vector subspace of M and $T\restriction_{\mathcal{K}'} = T'$.

Theorem 1.12 (Construction). *Let $h \subseteq \mathcal{L}$ be finite.*

1. *There exists a qaa field $(\mathcal{K}_h, \langle h\rangle^\times, T_h)$ such that $h \subseteq \mathcal{K}_h$.*
2. *If $g \subseteq \mathcal{L}$ is finite and $h \subseteq g$, then the qaa field $(\mathcal{K}_g, \langle g\rangle^\times, T_g)$ extends $(\mathcal{K}_h, \langle h\rangle^\times, T_h)$.*

Remark. For general $f \in \mathcal{K}_h$, the series $T_h(f)$ is not convergent.

In view of the Construction Theorem, we consider the set consisting of all qaa fields $(\mathcal{K}_h, \langle h\rangle^\times, T_h)$, for finite $h \subseteq \mathcal{L}$, partially ordered by the subset ordering on the tuples h, and we let $(\mathcal{K}, \mathcal{L}, T)$ be the direct limit of this partially ordered set.

Theorem 1.13 (Closure).

1. *$(\mathcal{K}, \mathcal{L}, T)$ is a qaa field extending each $(\mathcal{K}_h, \langle h\rangle^\times, T_h)$.*
2. *$(\mathcal{K}, \mathcal{L}, T)$ extends the qaa field $(\mathcal{F}, L, T)$ constructed in* [11, Theorem 3].
3. *$(\mathcal{K}, \mathcal{L}, T)$ extends the qaa field $(\mathcal{H}, \mathcal{L}, S_{\mathcal{L}})$ of Example 1.11 above.*
4. *$\mathcal{K}$ is closed under differentiation; in particular, $\mathcal{K}$ is a Hardy field.*

2. Outline of proof and the Extension Theorem

The proof of the Construction Theorem proceeds by adapting the construction in [11] to the more general setting here. The role of *standard quadratic domain* there is taken on by the following domains here: for $a \in \mathbb{R}$, we set

$$H(a) := \{z \in \mathbb{C} : \operatorname{Re} z > a\}.$$

Definition 2.1. A **standard power domain** is a set

$$U_C^\epsilon := \phi_C^\epsilon(H(0)),$$

where $C > 0$, $\epsilon \in (0,1)$ and $\phi_C^\epsilon : H(0) \longrightarrow U_C^\epsilon$ is the biholomorphic map defined by

$$\phi_C^\epsilon(z) := z + C(1+z)^\epsilon,$$

where $(\cdot)^\epsilon$ denotes the standard branch of the power function on $H(0)$.

Note that $\epsilon = \frac{1}{2}$ corresponds to the standard quadratic domains of [11]. We use the following consequence of the Phragmén-Lindelöf principle [3, Theorem 24.36]:

Uniqueness Principle. *Let $U \subseteq \mathbb{C}$ be a standard power domain and $\phi : U \longrightarrow \mathbb{C}$ be holomorphic. If ϕ is bounded and*

$$\phi\restriction_{\mathbb{R}} \prec \exp^{-n} \quad \textit{for each } n \in \mathbb{N},$$

then $\phi = 0$.

The Uniqueness Principle follows from [3, Lemma 24.37], because $x < \phi_C^\epsilon(x)$ for $x > 0$. (The reason for working with standard power domains in place of standard quadratic domains is technical).

Recall that the construction in [11] is for the tuples

$$\left(\frac{1}{\exp}, \frac{1}{x}, \dots, \frac{1}{\log_k}\right) = \exp\circ(-x, -\log, \dots, -\log_{k+1}),$$

and it proceeds by induction on k. To understand how we can generalize this construction to more general sequences $h \in \mathcal{D}^{k+1}$, where

$$\mathcal{D} := \left\{h \in \mathcal{H}^{>0} : \ h \prec 1\right\}$$

is the set of all positive **infinitesimal** germs, we let

$$\mathcal{I} := \left\{h \in \mathcal{H}^{>0} : \ h \succ 1\right\}$$

be the set of all **infinitely increasing** germs in $\mathcal{H}$ and write

$$h = \exp\circ(-f) = \frac{1}{\exp} \circ f,$$

where $f = (f_0, \dots, f_k)$ with each $f_i \in \mathcal{I}$. In this situation, we shall also write M_f for the multiplicative $\mathbb{R}$-vector subspace $\langle h\rangle^\times$.

We first recall how the induction on k works in [11]: assuming the qaa field $(\mathcal{F}_{k-1}, L, T_{k-1})$ has been constructed such that every germ in $\mathcal{F}_{k-1}$ has a complex analytic continuation on some standard quadratic domain, we "right shift" by log, that is, we

(i)$_{[11]}$ set $\mathcal{F}_k' := \mathcal{F}_{k-1} \circ \log$ and define $T_k' : \mathcal{F}_k' \longrightarrow L$ by $T_k'(h \circ \log) := (T_{k-1}h) \circ \log$.

Note that, since log has a complex analytic continuation on any standard quadratic domain with image contained in every standard quadratic domain [7, Example 3.13(2)], the tuple $(\mathcal{F}_k', L, T_k')$ is also a qaa field as defined in [11] such that every germ in $\mathcal{F}_k'$ has a complex analytic continuation on some standard quadratic domain. So we

(ii)$_{[11]}$ let $\mathcal{A}_k$ be the $\mathbb{R}$-algebra of all germs $h \in \mathcal{C}$ that have a bounded, complex analytic continuation on some standard quadratic domain U and an asymptotic expansion $F = \sum h_m m \in \mathcal{F}_k'((L_0))$ that holds not only in $\mathbb{R}$, but in all of U, and we set

$$T_k h := \sum (T_k' h_m) m \in \mathbb{R}((L_k)).$$

(Note that, in general, $T_k h$ is an L-series over $\mathbb{R}$, but not an L-generalized Laurent series over $\mathbb{R}$; this observation was not explicitely mentioned in [11].) The corresponding generalization of asymptotic expansion $(*)$ to allowing coefficients in $\mathcal{F}_k'$ works, because each germ in $\mathcal{F}_k'$ is polynomially bounded, and the quasianalyticity follows from the

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{F}_0 \\ e^{-x} \circ (x) \end{bmatrix} \\
& \swarrow \circ \log \swarrow & \\
\begin{bmatrix} \mathcal{F}'_1 \\ e^{-x} \circ (\log) \end{bmatrix} & \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{F}_1 \\ e^{-x} \circ (x, \log) \end{bmatrix} \\
& \swarrow \circ \log \swarrow & \\
& \vdots & \\
& \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{F}_{k-1} \\ e^{-x} \circ (x, \log, \dots, \log_{k-1}) \end{bmatrix} \\
& \swarrow \circ \log \swarrow & \\
\begin{bmatrix} \mathcal{F}'_k \\ e^{-x} \circ (\log, \dots, \log_k) \end{bmatrix} & \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{F}_k \\ e^{-x} \circ (x, \log, \dots, \log_k) \end{bmatrix}
\end{array}
$$

FIGURE 1. Schematic of the construction in [11]: going horizontally from left to right represents one use of the Uniqueness Principle (UP) and adds e^{-x} to the generating monomials on the left; going from the right to the next lower left represents a right shift by log.

Uniqueness Principle. Finally, since $\mathbb{R}((L_k))$ is a field, the ring $\mathcal{A}_k$ is an integral domain, and we

(iii)$_{[11]}$ let $\mathcal{F}_k$ be the fraction field of $\mathcal{A}_k$ and extend T_k accordingly.

We represent this construction by the schematic in Figure 1.

Throughout this construction, the following property of L is used:

Definition 2.2. Let M be a multiplicative $\mathbb{R}$-vector subspace of $\mathcal{H}^{>0}$. We call M a **strong asymptotic scale** if

(1) there is a basis $\{m_0, \dots, m_k\}$ of M consisting of pairwise incomparable small germs;
(2) for every standard power domain U, every bounded $m \in M$ has a bounded complex analytic continuation $\mathbf{m}$ on U.

Remark 2.3. If M is a strong asymptotic scale, then M is an asymptotic scale: let $\{m_0, \dots, m_k\}$ be a basis of M consisting of pairwise incomparable small germs such that $m_0 < \cdots < m_k$, and set $m_{-1} := 1$.

Let $m \in M$ be such that $m \asymp 1$, let $\alpha_0, \dots, \alpha_k \in \mathbb{R}$ be such that

$$m = m_0^{\alpha_0} \cdots m_k^{\alpha_k},$$

and set $\alpha_{-1} := 1$. Since the m_i are pairwise incomparable, m is comparable to m_j, where $j := \min\{i = -1, \dots, k : \ \alpha_i \neq 0\}$; hence $m \asymp 1$ implies $\alpha_0 = \cdots = \alpha_k = 0$, that is, $m = 1$.

The use of strong aymptotic scales is to extend the notion of asymptotic expansion to standard power domains, leading to the notion of **strong asymptotic expansion** (whose definition we omit here).

Examples 2.4. (1) L is a strong asymptotic scale by [11, Lemma 8].

(2) $\mathcal{L}$ is not a strong asymptotic scale: the germ $e^{-x} \circ x^2$ belongs to $\mathcal{L}$ and is bounded, but its complex analytic continuation on any standard power domain is unbounded.

(3) Not every tuple from $\mathcal{L}$ is a basis consisting of pairwise incomparable small germs: consider the germs $f_0 := x$, $f_1 = x - \log$ and $f_2 := \log + \log\log$ in $\mathcal{U}$, the set of all **purely infinite** germs in $\mathcal{H}$ [7, Section 2]; we have $\mathcal{L} = \exp \circ\, \mathcal{U}$. While $\{f_0, f_1, f_2\}$ is additively linearly independent, we have $f_0 \asymp f_1$. However, M_f has the basis

$$e^{-x} \circ (x, \log, \log_2)$$

consisting of pairwise incomparable small germs, because x, $\log$ and $\log_2$ belong to distinct archimedean classes.

(4) We show that, if each f_i belongs to $\mathcal{U}$, then the additive $\mathbb{R}$-vector space $\langle f \rangle$ generated by the f_i has a basis consisting of infinitely increasing germs belonging to pairwise distinct archimedean classes; hence, M_f has a basis consisting of pairwise incomparable small germs.

The most straightforward generalization of the construction in [11] is to any sequence f of the form

$$f = \left(g^{\circ 0}, g^{\circ 1}, \dots, g^{\circ k}\right),$$

where $k \in \mathbb{N}$, $g \in \mathcal{I}$ belongs to a strictly smaller Archimedean class than x, $g^{\circ i}$ denotes the i-th compositional iterate of g and M_f is an asymptotic scale on standard power domains, and such that the following holds:

$(\dagger)_1$ for every standard power domain V, the germ g has a complex anlytic continuation $\mathbf{g}$ on some standard power domain U such that $\mathbf{g}(U) \subseteq V$.

The additional assumption $(\dagger)_1$ means that we can compose on the right ("right shift") with g in place of log, as in the construction in [11].

In general, we assume that $k > 0$ and $f_0 > f_1 > \cdots > f_k$ belong to $\mathcal{I}$ and that M_f is an asymptotic scale with basis $e^{-x} \circ f$ consisting of pairwise incomparable small germs; this implies, in particular, that $f_0 \succ \cdots \succ f_k$. In this situation, we aim to adapt the construction in [11] as represented by the schematic pictured in Figure 2. The "right shifts" are now by germs of the form $f_i \circ f_{i-1}^{-1}$—which still belong to $\mathcal{H}$ since they are definable—and the monomials at the i-th step are $e^{-x} \circ f^{\langle i \rangle}$, where we set

$$f^{\langle i \rangle} := \left(x, f_{k-i+1} \circ f_{k-i}^{-1}, \ldots, f_k \circ f_{k-i}^{-1}\right)$$

and

$$f^{\langle i \rangle'} := \left(f_{k-i+1} \circ f_{k-i}^{-1}, \ldots, f_k \circ f_{k-i}^{-1}\right).$$

In particular, we have $f^{\langle 0 \rangle} = (x)$, so that the first step in the construction yielding $\mathcal{K}_0$ is the same as the first step of the construction in Figure 1, that is, $\mathcal{K}_0 = \mathcal{F}_0$.

To determine what additional conditions f has to satisfy in order for this adaptation to go through at the i-th step, we assume that $M_{f^{\langle i-1 \rangle}}$ is a strong asymptotic scale with basis $e^{-x} \circ f^{\langle i-1 \rangle}$ consisting of pairwise incomparable germs, and that we have constructed $\mathcal{K}_{i-1}$ such that every $h \in \mathcal{K}_{i-1}$ has an analytic continuation $\mathbf{h}$ on some standard power domain. Provided that

$(\dagger)_2$ for every standard power domain V, the germ $f_i \circ f_{i-1}^{-1}$ has a complex analytic continuation $\mathbf{f}_{i,i-1}$ on some standard power domain U such that $\mathbf{f}_{i,i-1}(U) \subseteq V$,

the set $M_{f^{\langle i \rangle'}}$ is also a strong asymptotic scale (because $\mathbf{f}_{i,i-1}^{-1}$ maps standard power domains *into* standard power domains) with basis $e^{-x} \circ f^{\langle i \rangle'}$ consisting of pairwise incomparable small germs. Therefore, we right shift by $f_i \circ f_{i-1}^{-1}$, that is, we

(i) set $\mathcal{K}_i' := \mathcal{K}_{i-1} \circ f_i \circ f_{i-1}^{-1}$ and define $T_i' : \mathcal{K}_i' \longrightarrow \mathbb{R}\left(\left(M_{f^{\langle i \rangle'}}\right)\right)$ by $T_i'\left(h \circ f_i \circ f_{i-1}^{-1}\right) := (T_{i-1}h) \circ f_i \circ f_{i-1}^{-1}$.

Again by assumption $(\dagger)_2$, the triple $\left(\mathcal{K}_i', M_{f^{\langle i \rangle'}}, T_i'\right)$ is a qaa field such that every germ in $\mathcal{K}_i'$ has a complex analytic continuation on some standard power domain. So we

(ii) let $\mathcal{A}_i$ be the set of all germs $h \in \mathcal{C}$ that have a bounded, complex analytic continuation on some standard power domain

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{K}_0 \\ e^{-x} \circ f^{\langle 0 \rangle} \end{bmatrix} \\
& \swarrow \circ \left(f_k \circ f_{k-1}^{-1} \right) \swarrow & \\
\begin{bmatrix} \mathcal{K}_1' \\ e^{-x} \circ f^{\langle 1 \rangle'} \end{bmatrix} & \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{K}_1 \\ e^{-x} \circ f^{\langle 1 \rangle} \end{bmatrix} \\
& \swarrow \circ \left(f_{k-1} \circ f_{k-2}^{-1} \right) \swarrow & \\
& \vdots & \\
& \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{K}_{i-1} \\ e^{-x} \circ f^{\langle i-1 \rangle} \end{bmatrix} \\
& \swarrow \circ \left(f_{k-i+1} \circ f_{k-i}^{-1} \right) \swarrow & \\
\begin{bmatrix} \mathcal{K}_i' \\ e^{-x} \circ f^{\langle i \rangle'} \end{bmatrix} & \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{K}_i \\ e^{-x} \circ f^{\langle i \rangle} \end{bmatrix} \\
& \vdots & \\
& \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{K}_{k-1} \\ e^{-x} \circ f^{\langle k-1 \rangle} \end{bmatrix} \\
& \swarrow \circ \left(f_1 \circ f_0^{-1} \right) \swarrow & \\
\begin{bmatrix} \mathcal{K}_k' \\ e^{-x} \circ f^{\langle k \rangle'} \end{bmatrix} & \xrightarrow{\text{(UP)}} & \begin{bmatrix} \mathcal{K}_k \\ e^{-x} \circ f^{\langle k \rangle} \end{bmatrix} \\
& \swarrow \circ f_0 \swarrow & \\
\begin{bmatrix} \mathcal{K}_f \\ e^{-x} \circ f \end{bmatrix} & &
\end{array}
$$

FIGURE 2. Schematic of the generalized construction: going horizontally from left to right represents one use of the Uniqueness Principle (UP) and adds e^{-x} to the generating monomials on the left; going from the right to the next lower left represents a "right shift" by $f_i \circ f_{i-1}^{-1}$. One final right shift by f_0 yields the desired qaa field $\mathcal{K}_f$.

U and a strong asymptotic expansion

$$H = \sum h_m m \in \mathcal{K}_i' \left(\!\left(M_x\right)\!\right)$$

in U (roughly speaking, this asymptotic expansion holds as $|z| \to \infty$ in U), and we set

$$T_i(h) := \sum T_i'(h_m) m \in \mathbb{R} \left(\!\left(M_{f\langle i \rangle}\right)\!\right)^{\mathrm{s}}.$$

As in [11], the corresponding generalization of asymptotic expansion $(*)$ to allowing coefficients in $\mathcal{K}_i'$ works, because each germ in $\mathcal{K}_i'$ has comparability class strictly smaller than that of e^x and strictly larger than that of e^{-x}, and the quasianalyticity follows from the Uniqueness Principle. Finally, we

(iii) let $\mathcal{K}_i$ be the fraction field of $\mathcal{A}_i$, and we extend T_i accordingly.

Iterating this construction leads to the schematic pictured in Figure 2. The final step, a right shift by f_0, leads to the desired qaa field $(\mathcal{K}_f, M_f, T_f)$. Note that, for this last step, we do not need any analytic continuation assumptions and, consequently, we do not expect analytic continuation of the germs in $\mathcal{K}_f$ on standard power domains.

The crucial additional assumption we need to make this work is $(\dagger)_2$ above, which we need for each i. Requiring this condition to be inherited by all subtuples of f, we shall consider the following stronger assumption:

($\dagger$) for $0 \le j < i \le k$ and every standard power domain V, the germ $f_i \circ f_j^{-1}$ has a complex analytic continuation on some standard power domain U with image contained in V.

This leads us to the following condition on general tuples f:

Definition 2.5. We call the tuple f **admissible** if ($\dagger$) holds and $M_{f \circ f_0^{-1}}$ is a strong asymptotic scale with basis $e^{-x} \circ \left(f \circ f_0^{-1}\right)$ consisting of pairwise incomparable small germs.

Note that if f is admissible and g is a subtuple of f, then g is admissible as well and, in this situation, the above construction implies that $(\mathcal{K}_f, M_f, T_f)$ extends $(\mathcal{K}_g, M_g, T_g)$.

Since not every germ in $\mathcal{I}$ satisfies $(\dagger)_2$, not every tuple f is admissible. To figure out what tuples f are admissible, recall from [7] that a germ $f \in \mathcal{H}$ is **simple** if $\mathrm{eh}(f) = \mathrm{level}(f)$, where $\mathrm{eh}(f)$ is the exponential height of f as defined in [7] and $\mathrm{level}(f)$ is the level of f as found in [8]. We use Corollaries 1.7 and 8.10 of [7] to establish the following:

Theorem 2.6 (Admissibility). *Assume the f_i are simple and have pairwise distinct Archimedean classes. Then f is admissible.*

Note that the Admissibility Theorem fails for non-simple germs in general:

Example 2.7. Consider the tuple $f = (f_0, f_1) := \left(x, x + e^{-x^2}\right)$: while f_0 has a bounded complex analytic continuation on every standard power domain, the germ f_1 does not have a bounded complex analytic continuation on any standard power domain. In fact, M_f is not a strong asymptotic scale.

Since every germ in $\mathcal{U}$ is simple [7, Example 8.7], we obtain the following from the Admissibility Theorem 2.6:

Corollary 2.8 (Admissibility). *If each f_i belongs to $\mathcal{U}$, then every subtuple of $\langle f \rangle$ consisting of infinitely increasing germs belonging to pairwise disjoint archimedean classes is admissible.* □

Therefore, if each $f_i \in \mathcal{U}$, we obtain the qaa field $(\mathcal{K}_f, M_f, T_f)$ as follows: by Example 2.4(4), $\langle f \rangle$ has a basis g consisting of infinitely increasing germs belonging to pairwise disjoint archimedean classes. By the Admissibility Corollary 2.8, our construction then produces the qaa field $(\mathcal{K}_g, M_g, T_g)$, and we set

$$\mathcal{K}_f := \mathcal{K}_g \quad \text{and} \quad T_f := T_g.$$

We show that the resulting $\mathcal{K}_f$ is independent of the chosen basis g.

Finally, we show that the direct limit $(\mathcal{K}, \mathcal{L}, T)$ is maximal in the following sense: if each f_i belongs to $\mathcal{U}$, the qaa field $(\mathcal{K}_f, M_f, T_f)$ constructed here is extended by $(\mathcal{K}, \mathcal{L}, T)$; this implies, in particular, parts (1) and (2) of the Closure Theorem. For part (3) of the latter, it suffices to verify that every germ given by a convergent $\mathcal{L}$-generalized power series in $\mathbb{R}\left(\!\left(M_f\right)\!\right)$ belongs to $\mathcal{K}_f$. The proof of part (4) of the Closure Theorem is adapted from the proof of [11, Theorem 3(2)].

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ANALYTIC CONTINUATION OF log-exp-ANALYTIC GERMS

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Abstract. We describe maximal, in a sense made precise, $\mathbb{L}$-analytic continuations of germs at $+\infty$ of unary functions definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ on the Riemann surface $\mathbb{L}$ of the logarithm. As one application, we give an upper bound on the logarithmic-exponential complexity of the compositional inverse of an infinitely increasing such germ, in terms of its own logarithmic-exponential complexity and its level. As a second application, we strengthen Wilkie's theorem on definable complex analytic continuations of germs belonging to the residue field $\mathcal{R}_{\text{poly}}$ of the valuation ring of all polynomially bounded definable germs.

Introduction

The o-minimal structure $\mathbb{R}_{\text{an,exp}}$, see van den Dries and Miller [8] or van den Dries, Macintyre and Marker [7], is one of the most important regarding applications, because it defines all elementary functions (with the necessary restriction on periodic ones such as sin or cos). Holomorphic functions definable in $\mathbb{R}_{\text{an,exp}}$ have turned out to be crucial in applications to diophantine geometry, see for instance Pila [5] and Peterzil and Starchenko [4].

It is known [8, 7] that every function definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ is piecewise analytic. This implies that, if f is the germ at $+\infty$ of a one-variable function definable in $\mathbb{R}_{\text{an,exp}}$ (equivalently, an element of the Hardy field $\mathcal{H} = \mathcal{H}_{\text{an,exp}}$), also called a log-exp-**analytic germ** here, there is an open domain $U \subseteq \mathbb{C}$ and a complex analytic continuation $\mathbf{f} : U \longrightarrow \mathbb{C}$ of f, or an open domain $\mathfrak{U} \subseteq \mathbb{L}$ and an $\mathbb{L}$-analytic continuation $\mathfrak{f} : \mathfrak{U} \longrightarrow \mathbb{L}$, where $\mathbb{L}$ is the Riemann surface of

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the logarithm. Concerning complex analytic continuations $\mathbf{f} : U \longrightarrow \mathbb{C}$ of f, it is shown by Kaiser [2, Theorem C] that $\mathbf{f}$ can be chosen to be definable. Wilkie [9, Theorem 1.11] characterizes those f for which $\mathbf{f}$ extends definably on some right translate of a sector properly containing a right half-plane of $\mathbb{C}$; he then applies this continuation result to a diophantine problem.

The aim of this paper is to describe $\mathbb{L}$-analytic continuations $\mathfrak{f} : \mathfrak{U} \longrightarrow \mathbb{L}$: we find a maximal $\mathfrak{U}$ (in a sense to be made precise) such that $\mathfrak{f}$ is **half-bounded**, that is, either $\mathfrak{f}$ or $1/\mathfrak{f}$ is bounded (see the Continuation Corollary, Theorem 2 below). We obtain this statement from a more precise Continuation Theorem, which only applies to **infinitely increasing** f, that is, those f for which $\lim_{x\to+\infty} f(x) = +\infty$ holds.

These $\mathbb{L}$-analytic continuations of f depend on two integer-valued quantities associated to f: the *exponential height* $\mathrm{eh}(f)$ of f and the *level* $\mathrm{level}(f)$ of f. The former measures the logarithmic-exponential complexity of f; roughly speaking, if f is unbounded, then $\mathrm{eh}(\exp\circ f) = \mathrm{eh}(f) + 1$, while if f is bounded, then $\mathrm{eh}(\exp\circ f) = \mathrm{eh}(f)$. The latter measures the exponential order of growth of the germ f; we refer the reader to Marker and Miller [3] for details. The level extends to all log-exp-analytic germs in an obvious manner.

Remark. We show that $\mathrm{level}(f) \le \mathrm{eh}(f)$, for all log-exp-analytic germs f. The two are not equal in general: we have $\mathrm{level}(x + e^{-x}) = 0 \ne 1 = \mathrm{eh}(x + e^{-x})$.

What we find in the Continuation Corollary is that, if $\mathfrak{f} : \mathfrak{U} \longrightarrow \mathbb{L}$ is a maximal, half-bounded $\mathbb{L}$-analytic continuation of f, then the size (in a sense to be made precise) of $\mathfrak{U}$ is determined by $\mathrm{eh}(f)$ and, conversely, that the size of $\mathfrak{U}$ determines an upper bound on $\mathrm{eh}(f)$. Moreover, if f is infinitely increasing, we also find that $\mathfrak{f}$ is injective and, in this case, $\mathrm{level}(f)$ determines the size of the image $\mathfrak{f}(\mathfrak{U})$; see the Simplified Continuation Theorem 1 below. The actual Continuation Theorem is more technical. Finally we also describe, in Corollary 6 below, the resulting maximal complex continuations of germs in $\mathcal{H}$.

We include two applications of the Continuation Theorem and its corollaries. In Application 3, we give an upper bound on $\mathrm{eh}(f^{-1})$, in terms of $\mathrm{eh}(f)$ and $\mathrm{level}(f)$, of an infinitely increasing log-exp-analytic germ f, where f^{-1} denotes the compositional inverse of f. In Application 5, we strengthen Wilkie's theorem [9, Theorem 1.11] on definable complex analytic continuations of germs belonging to the residue field $\mathcal{R}_{\mathrm{poly}}$ of the valuation ring of all polynomially bounded log-exp-analytic germs.

The main motivation for us to prove the Continuation Theorem, however, is to show that all *principal monomials* of $\mathcal{H}$ can be used in asymptotic expansions to obtain a quasianalytic Ilyashenko field $\mathcal{K}$ extending the Ilyashenko field $\mathcal{F}$ constructed in Speissegger [6]. The details of this application (which also relies on Application 3 below), its motivations and the construction of $\mathcal{K}$ are the subject of a forthcoming paper.

Statements of results and some ideas

One of the main issues is that the language needed to state our results has to be developed from scratch. To illustrate some of the notions involved, we now briefly describe that of an *η-domain*, the type of domain in $\mathbb{L}$ that the sets $\mathfrak{U}$ above are selected from, and we state the precise Continuation Corollary and a simplified version of the Continuation Theorem that we call the Simplified Continuation Theorem below.

We denote by $\mathbb{L} := \{(r,\theta):\ r>0,\ \theta\in\mathbb{R}\}$ the Riemann surface of the logarithm with its usual covering map $\pi:\mathbb{L}\longrightarrow\mathbb{C}\setminus\{0\}$ defined by $\pi(r,\theta)=re^{i\theta}$. We let $|(r,\theta)| := r$ be the **modulus** and $\arg(r,\theta):=\theta$ be the **argument** of (r,θ). We usually write $x=(|x|,\arg x)$ for an element of $\mathbb{L}$, and we identify the positive real half-line $(0,+\infty)$ with the set $\{x\in\mathbb{L}:\ \arg x = 0\}$.

To define an η-domain, we first introduce **real domains** as the sets of the form

$$\mathfrak{U}_h := \{x\in H_{\mathbb{L}}(a):\ |\arg x| < h(|x|)\},$$

where $a\ge 0$,

$$H_{\mathbb{L}}(a) := \{x\in\mathbb{L}:\ |x|>a\}$$

and $h:(a,+\infty)\longrightarrow(0,+\infty)$ is continuous. Identifying $\mathbb{L}$ with the set $(0,+\infty)\times\mathbb{R}$, a real domain $\mathfrak{U}_h$ is definable in $\mathbb{R}_{\text{an,exp}}$ if and only if h is a log-exp-analytic germ. Considering two real domains $\mathfrak{U}_{h_1}$ and $\mathfrak{U}_{h_2}$ equivalent if there exists $a>0$ such that $\mathfrak{U}_{h_1}\cap H_{\mathbb{L}}(a)=\mathfrak{U}_{h_2}\cap H_{\mathbb{L}}(a)$, and calling the corresponding equivalence classes of real domains **germs at ∞ of real domains**, we get a bijective map $h\mapsto\mathfrak{U}_h:\mathcal{H}^{>0}\longrightarrow\mathcal{G}^{\infty}_{\text{dfr}}(\mathbb{L})$, where $\mathcal{H}^{>0}$ is the set of all positive log-exp-analytic germs and $\mathcal{G}^{\infty}_{\text{dfr}}(\mathbb{L})$ denotes the set of all germs at ∞ of definable real domains. This bijection satisfies $h_1<h_2$ if and only if $\mathfrak{U}_{h_1}\subset\mathfrak{U}_{h_2}$ as germs at ∞; in particular, any measure of size on $\mathcal{H}^{>0}$, such as valuation or level, can be transferred to $\mathcal{G}^{\infty}_{\text{dfr}}(\mathbb{L})$.

To understand what measure of size on $\mathcal{H}^{>0}$ is appropriate for our purposes, we consider the analytic continuations on $\mathbb{L}$ of the elementary functions: scalar multiplication

$$m_r(x) := rx$$

and the power function

$$p_r(x) := x^r,$$

for $r > 0$ and $x > 0$, as well as exp and log. It is easy to see that m_r and p_r have definable, biholomorphic continuations $\mathfrak{m}_r, \mathfrak{p}_r : \mathbb{L} \longrightarrow \mathbb{L}$, respectively, while exp has a holomorphic continuation $\mathfrak{exp} : \mathbb{L} \longrightarrow \mathbb{L}$ that is neither definable nor injective. However, log has a definable, biholomorphic continuation $\mathfrak{log} : H_{\mathbb{L}}(1) \longrightarrow S_{\mathbb{L}}(\pi/2)$, where

$$S_{\mathbb{L}}(a) := \{x \in \mathbb{L} : \ |\arg x| < a\}$$

for $a > 0$. Indeed, one of our reasons for working with $\mathbb{L}$-analytic rather than complex analytic continuations is that the definable, biholomorphic restriction of $\mathfrak{exp}$ to $S_{\mathbb{L}}(\pi/2)$ has a larger domain than any definable, biholomorphic complex continuation of exp in the right complex half-plane.

It is not hard to show that each of these biholomorphic continuations maps definable real domains to definable real domains. But for any $f \in \mathcal{H}^{>0}$, if $g \in \mathcal{H}^{>0}$ is such that $\mathfrak{U}_g = \mathfrak{log}(\mathfrak{U}_f)$, then g is bounded, as $\mathfrak{U}_g$ is a subset of $S_{\mathbb{L}}(\pi/2)$ (as germs at ∞). This "big crunch" indicates that none of the measures of size mentioned earlier are quite right to describe the behaviour of $\mathfrak{log}$. We show that there is a decreasing map $\mathrm{al} : \mathcal{H}^{>0} \longrightarrow \mathbb{N} \cup \{-1\}$, called **angular level**, such that the following hold for $f \in \mathcal{H}^{>0}$ and $r > 0$:

(i) if $g \in \mathcal{H}^{>0}$ is such that $\mathfrak{U}_g = \mathfrak{m}_r(\mathfrak{U}_f)$, then $\mathrm{al}(g) = \mathrm{al}(f)$;
(ii) if $g \in \mathcal{H}^{>0}$ is such that $\mathfrak{U}_g = \mathfrak{p}_r(\mathfrak{U}_f)$, then $\mathrm{al}(g) = \mathrm{al}(f)$;
(iii) if $g \in \mathcal{H}^{>0}$ is such that $\mathfrak{U}_g = \mathfrak{log}(\mathfrak{U}_f)$, then $\mathrm{al}(g) = \mathrm{al}(f) + 1$;
(iv) if $f \le 1/\log$, then $\mathrm{al}(f) = \mathrm{level}(f) + 1$.

This angular level is related to the usual level, as point (iv) indicates, but takes into account the "big crunch" of $\mathfrak{log}$ mentioned earlier.

Finally, not all $f \in \mathcal{H}^{>0}$ are as well behaved as the elementary ones above: if $t_a(x) := x + a$ for $a > 0$ and $x > 0$, then t_a has an injective, holomorphic continuation $\mathfrak{t}_a : \mathbb{L} \longrightarrow \mathbb{L}$ that is periodic in $\arg x$, hence is not definable and, in general, maps definable real domains to domains that are neither definable nor real. However, it is easy to see that, given $f \in \mathcal{H}^{>0}$, there are $g_1, g_2 \in \mathcal{H}^{>0}$ such that $\mathrm{al}(f) = \mathrm{al}(g_1) = \mathrm{al}(g_2)$ and $\mathfrak{U}_{g_1} \subseteq \mathfrak{t}_a(\mathfrak{U}_f) \subseteq \mathfrak{U}_{g_2}$. This leads to our desired definition: given a domain $\mathfrak{U} \subseteq \mathbb{L}$ and $\eta \in \mathbb{N} \cup \{-1\}$, we call $\mathfrak{U}$ an **η-domain** if there exist $g_1, g_2 \in \mathcal{H}^{>0}$ such that $\mathrm{al}(g_1) = \mathrm{al}(g_2) = \eta$ and

$$\mathfrak{U}_{g_1} \subseteq \mathfrak{U} \subseteq \mathfrak{U}_{g_2}.$$

Every definable, real domain is an η-domain, for some appropriate η, and the maps $\mathfrak{m}_r$ and $\mathfrak{p}_r$, as well as $\mathfrak{t}_a$, map η-domains to η-domains, while $\mathfrak{log}$ maps η-domains to $(\eta+1)$-domains.

Denoting by $\mathcal{H}$ the set of all log-exp-analytic germs, we are now ready to state the Simplified Continuation Theorem, which describes biholomorphic continuations of all infinitely increasing log-exp-analytic germs in the spirit of those of the elementary germs described above. A map $\varphi : \mathfrak{U} \longrightarrow \mathbb{L}$, with $\mathfrak{U} \subseteq \mathbb{L}$, is called **angle-positive** if

$$\operatorname{sgn}(\arg\varphi(x)) = \operatorname{sgn}(\arg x)$$

for all $x \in \mathfrak{U}$.

Theorem 1 (Simplified Continuation). *Let $f \in \mathcal{H}$ be infinitely increasing, and set $\eta := \max\{0, \operatorname{eh}(f)\}$ and $\lambda := \operatorname{level}(f)$. Then there exist an $(\eta-1)$-domain $\mathfrak{U}$, an $(\eta-1-\lambda)$-domain $\mathfrak{V}$ and an angle-positive, biholomorphic continuation $\mathfrak{f} : \mathfrak{U} \longrightarrow \mathfrak{V}$ of f that maps k-domains to $(k-\lambda)$-domains, for $k \geq \eta - 1$.*

This theorem was what we wanted to prove originally, but we were unable to do so without making the statement considerably more precise. These are contained in the Continuation Theorem; we do not give its formulation here.

For arbitrary germs in $\mathcal{H}$, we have the following consequence of the Continuation Theorem:

Corollary 2 (Continuation). *Let $f \in \mathcal{H}$ and $\eta \in \mathbb{N}$. Then $\operatorname{eh}(f) \leq \eta$ if and only if there exist an $(\eta-1)$-domain $\mathfrak{U}$ and a half-bounded, analytic continuation $\mathfrak{f} : \mathfrak{U} \longrightarrow \mathbb{K}$ of f, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{L}$.*

The left-to-right implication of this corollary is a straightforward consequence of the Continuation Theorem. We obtain the right-to-left implication of the Continuation Corollary by combining the Continuation Theorem with a consequence of the Phragmén-Lindelöf principle found in Ilyashenko and Yakovenko [1, Lemma 24.37].

We now give two applications of the Continuation Theorem and its corollary. The first of these considers the following question: if f is infinitely increasing, and if f^{-1} denotes the compositional inverse of f, is there a bound on $\operatorname{eh}(f^{-1})$ described in terms of $\operatorname{eh}(f)$? (Note that this question with level in place of exponential height has the natural answer $\operatorname{level}(f^{-1}) = -\operatorname{level}(f)$, see [3].) What we find is the following more general statement:

Application 3. *Assume that f is infinitely increasing. Then*

$$\mathrm{eh}\left(g \circ f^{-1}\right) \leq \max\{\mathrm{eh}(g) + \mathrm{eh}(f) - 2\,\mathrm{level}(f), \mathrm{eh}(f) - \mathrm{level}(f)\},$$

where g is any log-exp-*analytic germ.*

We also call a log-exp-analytic germ **simple** if $\mathrm{eh}(f) = \mathrm{level}(f)$, and we establish several consequences of Application 3 for such germs.

The second application concerns the definability of the analytic continuations obtained in the Continuation Theorem and its corollary. Wilkie [9, Theorem 1.11] obtains complex definable continuations for the germs contained in the subset $\mathcal{R}_{\mathrm{poly}}$ of $\mathcal{H}$, in fact characterizing $\mathcal{R}_{\mathrm{poly}}$ as the set of all $f \in \mathcal{H}$ that have a definable, complex continuation $\mathbf{f}$ on some right translate of a sector properly containing a right half-plane of $\mathbb{C}$.

Building on the Continuation Theorem, we determine exactly which restrictions of our maximal $\mathbb{L}$-analytic continuations are definable: we call a set $\mathfrak{S} \subseteq \mathbb{L}$ **angle-bounded** if the set $\{|\arg x| : x \in \mathfrak{S}\}$ is bounded.

Theorem 4 (Definability). *Let $f \in \mathcal{H}$ be infinitely increasing, and set $\eta := \max\{0, \mathrm{eh}(f)\}$ and $\lambda := \mathrm{level}(f)$. Let $\mathfrak{f} : \mathfrak{U} \longrightarrow \mathfrak{V}$ be one of the biholomorphic continuations of f obtained from the Simplified Continuation Theorem, and let $\mathfrak{U}' \subseteq \mathfrak{U}$ be a definable domain. If $\mathfrak{f}(\mathfrak{U}')$ is angle-bounded, then $\mathfrak{f}\restriction_{\mathfrak{U}'}$ is definable.*

As a consequence, we obtain a similar (but not identical) characterization of membership in $\mathcal{R}_{\mathrm{poly}}$ that strengthens the complex continuation part of [9, Theorem 1.11]:

Application 5. *Let $f \in \mathcal{H}$. Then $f \in \mathcal{R}_{poly}$ if and only if there exist a (-1)-domain $\mathfrak{U}$ and a half-bounded, analytic continuation $\mathbf{f} : \mathfrak{U} \longrightarrow \mathbb{C}$ of f such that, for every angle-bounded, definable domain $\mathfrak{U}' \subseteq \mathfrak{U}$, the restriction $\mathbf{f}\restriction_{\mathfrak{U}'}$ is definable.*

We note that the continuations in Application 5 are complex-valued analytic continuations on domains in $\mathbb{L}$.

Finally, in the Complex Continuation Corollary, we give a description of *complex* analytic continuations of the germs in $\mathcal{H}$ implied by the Continuation Theorem and Corollary. For $a \geq 0$, we set

$$H(a) := \{z \in \mathbb{C} : \mathrm{Re}\, z > a\}.$$

We denote by arg the standard branch of the argument on $\mathbb{C} \setminus (-\infty, 0]$ and, for $\alpha \in (0, \pi]$, we set

$$S(\alpha) := \{z \in \mathbb{C} : |\arg z| < \alpha\}.$$

The following special case of the Complex Continuation Corollary is worth writing down, as it avoids all $\mathbb{L}$-related terminology introduced earlier:

Corollary 6 (Complex continuation). *Let $f \in \mathcal{H}$ be such that* $\mathrm{eh}(f) \leq 0$.

(1) *There are $a \geq 0$ and a half-bounded complex analytic continuation $\mathbf{f} : H(a) \longrightarrow \mathbb{C}$ of f.*

(2) *If f is infinitely increasing and $\lim_{x \to +\infty} \frac{f(x)}{x^2} = 0$, there are $a \geq 0$, a domain $V \subseteq \mathbb{C} \setminus (-\infty, 0]$ and a biholomorphic complex continuation $\mathbf{f} : H(a) \longrightarrow V$ of f such that*
   (a) $|\mathbf{f}(z)| \to \infty$ *as* $|z| \to \infty$, *for* $z \in H(a)$;
   (b) $\mathrm{sgn}(\arg \mathbf{f}(z)) = \mathrm{sgn}(\arg z) = \mathrm{sgn}(\mathrm{Im}\, z) = \mathrm{sgn}(\mathrm{Im}\, \mathbf{f}(z))$ *for* $z \in H(a)$;
   (c) *if* $\mathrm{eh}(f) < 0$ *then, for every* $\alpha > 0$, *there exists* $b \geq a$ *such that* $V \cap H(b) \subseteq S(\alpha)$. □

Indeed, in our forthcoming paper generalizing the construction of Ilyashenko algebras in [6], the only results we need from this paper are Corollary 6 and Application 3.

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SPECTRAL SPACES OF COUNTABLE ABELIAN LATTICE-ORDERED GROUPS (EXTENDED ABSTRACT)

FRIEDRICH WEHRUNG

ABSTRACT. The *ℓ-spectrum* of an Abelian ℓ-group is defined as the set of its prime ℓ-ideals, endowed with the standard hull-kernel topology. We state, without proof, a characterization of ℓ-spectra of countable Abelian ℓ-groups, as second countable, completely normal generalized spectral spaces.

1. INTRODUCTION

A *lattice-ordered group*, or *ℓ-group* for short, is a group G endowed with a translation-invariant lattice ordering. An *ℓ-ideal* of G is an order-convex, normal ℓ-subgroup I of G. We say that I is *prime* if $I \neq G$ and $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$, for all $x, y \in G$. We define the *ℓ-spectrum* of G as the set $\mathrm{Spec}_\ell\, G$ of all prime ℓ-ideals of G, endowed with the "hull-kernel" topology, whose closed sets are exactly the sets $\{P \in \mathrm{Spec}_\ell\, G \mid X \subseteq P\}$ for $X \subseteq G$. Characterizing the topological spaces $\mathrm{Spec}_\ell\, G$, for *Abelian* ℓ-groups G, is a long-standing open problem, which we shall call the *ℓ-spectrum problem.*

A (not necessarily Hausdorff) topological space X is *generalized spectral* if it is sober (i.e., every irreducible closed set is the closure of a unique singleton) and the collection $\overset{\circ}{\mathcal{K}}(X)$ of all compact open subsets of X forms a basis of the topology of X, closed under intersections of any two members. If, in addition, X is compact, we say that it is *spectral.* It is well known that the ℓ-spectrum of any Abelian ℓ-group is a generalized spectral space; in addition, this space is *completely normal*, that is, for any points x and y in the closure of a singleton $\{z\}$, either x is in the closure of $\{y\}$ or y is in the closure of $\{x\}$; this is a (properly) weaker concept than saying that every subspace of X is normal. Delzell and Madden constructed in 1994 an example of a completely normal spectral space which is not an ℓ-spectrum. However, their example is not second countable. The main result of our paper [1] is that there is no such counterexample in the second countable case:

Theorem. *A second countable topological space is the ℓ-spectrum of an Abelian ℓ-group iff it is completely normal generalized spectral.*

We also prove in [1] that the class of all Stone dual lattices of ℓ-spectra is neither closed under infinite products nor under homomorphic images, and that they have no $\mathscr{L}_{\infty,\omega}$-characterization.

We have omitted all proofs in this extended abstract. They can be found in the full version of the author's paper [1], together with references, further results, and discussion.

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2. Strategy of the proof

2.1. **Reduction to a lattice-theoretical problem.** We first reduce the problem about topological spaces to a problem about distributive lattices. This is done *via* the classical *Stone duality*, between distributive lattices with zero and zero-preserving lattice homomorphisms with cofinal[1] range on the one hand, generalized spectral spaces and spectral[2] maps on the other hand. The dual of a distributive lattice D with zero is the set $\operatorname{Spec} D$ of all its (proper) prime ideals, endowed with the usual hull-kernel topology. The dual of a generalized spectral space X is the lattice $\overset{\circ}{\mathcal{K}}(X)$ of all its compact open subsets, under set inclusion.

Characterizing all ℓ-spectra of Abelian ℓ-groups amounts to characterizing all their Stone duals, which are distributive lattices with zero. We call those lattices *ℓ-representable*. They are exactly the lattices isomorphic to the lattice $\operatorname{Id_c} G$ of all finitely generated (equivalently, principal) ℓ-ideals of some Abelian ℓ-group G. Hence, *a topological space X is homeomorphic to the ℓ-spectrum of an Abelian ℓ-group iff it is generalized spectral and the lattice $\overset{\circ}{\mathcal{K}}(X)$ is ℓ-representable.*

Monteiro proved in 1954 that a generalized spectral space X is completely normal iff for all $A, B \in \overset{\circ}{\mathcal{K}}(X)$ there are $U, V \in \overset{\circ}{\mathcal{K}}(X)$ such that $A \cup B = A \cup V = U \cup B$ and $U \cap V = \varnothing$. We then say that the lattice $\overset{\circ}{\mathcal{K}}(X)$ is completely normal. Hence, our main theorem can be stated lattice-theoretically as follows:

> *Every countable completely normal distributive lattice with zero is ℓ-representable.*

2.2. **Closed homomorphisms.** The following definition plays a crucial role in the proof.

Definition 2.1. A join-homomorphism $f\colon A \to B$, between join-semilattices A and B, is *closed* if whenever $a_0, a_1 \in A$ and $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there exists $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$.

For Abelian ℓ-groups G and H, every ℓ-homomorphism $f\colon G \to H$ induces, functorially, a closed zero-preserving lattice homomorphism $\operatorname{Id_c} f\colon \operatorname{Id_c} G \to \operatorname{Id_c} H$.

For every Abelian ℓ-group G and every distributive lattice L with zero, every surjective closed lattice homomorphism $f\colon \operatorname{Id_c} G \twoheadrightarrow L$ induces an isomorphism $\overline{f}\colon \operatorname{Id_c}(G/I) \to L$, for a suitable ℓ-ideal I of G; in particular, this implies the ℓ-representability of L. Hence, in order to prove that a given countable, completely normal distributive lattice L with zero is ℓ-representable, it is sufficient to construct a surjective closed lattice homomorphism $f\colon \operatorname{Id_c} \mathrm{F}_\ell(\omega) \twoheadrightarrow L$, where $\mathrm{F}_\ell(\omega)$ denotes the free Abelian ℓ-group on the first infinite ordinal $\omega \underset{\mathrm{def}}{=} \{0, 1, 2, \dots\}$.

2.3. **Elementary blocks: the lattices** $\mathrm{Op}^-(\mathcal{H})$**.** Our construction of a closed surjective lattice homomorphism $f\colon \operatorname{Id_c} \mathrm{F}_\ell(\omega) \twoheadrightarrow D$ is performed stepwise, by expressing $\operatorname{Id_c} \mathrm{F}_\ell(\omega)$ as a countable ascending union $\bigcup_{n<\omega} E_n$, for suitable finite sublattices E_n (the "elementary blocks" of the construction) and homomorphisms $f_n\colon E_n \to L$, then extending each f_n to f_{n+1}. Each step of the construction is one of the following:

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[1] A subset X in a poset P is *cofinal* if every element of P lies below some element of X.

[2] A map between generalized spectral spaces is *spectral* if the inverse image of any compact open set is compact open.

(1) extend the domain of f_n — in order to get the final map f defined on all of $\mathrm{Id_c}\,\mathrm{F}_\ell(\omega)$;
(2) correct "closure defects" of f_n (i.e., $f_n(a_0) \leq f_n(a_1) \vee b$ with no x such that $a_0 \leq a_1 \vee x$ and $f_n(x) \leq b$) — in order to get f closed;
(3) add elements to the range of f_n — in order to get f surjective.

It should be noted that not all the E_n can be taken completely normal. They are defined as sublattices, of the powerset lattice of a countably infinite-dimensional vector space $\mathbb{E}$, generated by the open half-spaces arising from finite collections $\mathcal{H}$ of hyperplanes. Those lattices are denoted in the form $\mathrm{Op}^-(\mathcal{H})$. The reduction to the context of topological vector spaces is made possible by the Baker-Beynon duality.

The construction process relies on a suitable sufficient condition, for a zero-preserving lattice homomorphism $f\colon D \to L$ between finite distributive lattices, to be extendable to a lattice homomorphism $g\colon E \to L$, for certain extensions E of D. Among the assumptions, a noteworthy item is that D should be a Heyting subalgebra of E. Our goals (more specifically Step (2) above) require the extension process from f to g be performed constructively.

On the topological vector space front, a crucial role is played by the observation that for every join-irreducible member P of $\mathrm{Op}^-(\mathcal{H})$, both P and $P \setminus P_*$ are convex, where P_* denotes the unique lower cover of P in $\mathrm{Op}^-(\mathcal{H})$.

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Fonctions définissables dans un corps valué C-minimal

Françoise Delon

La C-minimalité est un cadre naturel où étudier les corps valués algébriquement clos. Elle a été introduite dans les années 90 et joue le même rôle pour les C-relations que l'o-minimalité pour les ordres. Une C-relation est une relation ternaire généralisant légèrement la relation $v(x-y) = v(x-z) < v(y-z)$ d'un espace ultramétrique. La relation pour laquelle $C(x, y, z)$ équivaut à $x \neq y = z$ est une C-relation, appelée triviale. Dans un corps on exigera que la C-relation soit compatible avec l'addition et la multiplication par un élément non nul. Un corps C-minimal pour une C-relation non triviale est nécessairement un corps valué algébriquement clos, le lien entre la C-relation et la valuation étant comme ci-dessus. On pourrait espérer développer la théorie des corps valués C-minimaux sur le modèle des corps o-minimaux. La présence d'une topologie à base définissable et la maximalité algébrique des modèles sont des arguments en ce sens. Mais des différences notables apparaissent immédiatement. Ainsi la maximalité définissable est une conséquence de l'o-minimalité, non de la C-minimalité. Dans le texte qui suit (écrit en collaboration avec Pablo Cubides-Kovacsics), nous mettons en évidence une nouvelle différence : un corps C-minimal valué dans le groupe additif des rationnels est nécessairement polynomialement borné. Ce résultat s'applique en particulier à $\mathbb{C}_p$ (qui est la complétion de la clôture algébrique de $\mathbb{Q}_p$ muni de la valuation p-adique). Or $\mathbb{C}_p$ est à beaucoup de points de vue l'analogue p-adique de $\mathbb{R}$ et un résultat essentiel de l'o-minimalité est que le corps $\mathbb{R}$ muni de l'exponentielle est o-minimal. Un corps C-minimal est-il toujours polynomialement borné ? Nous laissons la question ouverte. Nous étudions également la question de la dérivabilité. La dérivabilité presque partout des fonctions en une seule variable est une conséquence de l'o-minimalité. Elle ne découle pas de la C-minimalité puisque celle-ci autorise une caractéristique positive, avec en conséquence la présence de l'automorphisme de Frobenius,

à dérivée partout nulle, et ayant donc un inverse nulle part dérivable. Nous montrons la dérivabilité presque partout en caractéristique nulle, modulo une hypothèse additionnelle sur le groupe de valuation. Que $(\mathbb{Q}, +, <)$ satisfasse cette hypothèse est une conjecture en o-minimalité.

DEFINABLE FUNCTIONS IN TAME EXPANSIONS OF ALGEBRAICALLY CLOSED VALUED FIELDS

PABLO CUBIDES KOVACSICS AND FRANÇOISE DELON

Abstract. In this article we study definable functions in tame expansions of algebraically closed valued fields. For a given definable function we have two types of results: of type (I), which hold at a neighborhood of infinity, and of type (II), which hold locally for all but finitely many points in the domain of the function. In the first part of the article, we show type (I) and (II) results concerning factorizations of definable functions over the value group. As an application, we show that tame expansions of algebraically closed valued fields having value group $\mathbb{Q}$ (like $\mathbb{C}_p$ and $\overline{\mathbb{F}_p}^{alg}((t^{\mathbb{Q}}))$) are polynomially bounded. In the second part, under an additional assumption on the asymptotic behavior of unary definable functions of the value group, we extend these factorizations over the residue multiplicative structure RV. In characteristic 0, we obtain as a corollary that the domain of a definable function $f\colon X \subseteq K \to K$ can be partitioned into sets $F \cup E \cup J$, where F is finite, $f|E$ is locally constant and $f|J$ satisfies locally the Jacobian property.

The o-minimality of the real field expanded by the exponential function, proved by Wilkie in [14], was one of the major achievements of model theory during the nineties. Its ubiquity and importance can also be seen in the light of Miller's growth dichotomy [11]: an o-minimal expansion of a real field is either polynomially bounded or the exponential function is definable in it.

During the same years, analogous minimality conditions were introduced as candidates to define tame expansions of different structures. In the case of expansions of algebraically closed valued fields, the corresponding notion was introduced by Haskell, Macpherson and Steinhorn in [7, 10] and is called *C-minimality*.

In this article we study definable functions in tame expansions of algebraically closed valued fields where C-minimality is taken as the tameness condition. The main theorems obtained concern the factorization of definable functions over the value group (Γ-factorization) and over the residue multiplicative structure (RV-factorization). The formal definition of what factorization means will be given in the next section. For both factorizations we have two types of results: of type (I), which hold at a neighborhood of infinity, that is, they hold outside some neighborhood of 0; and of type (II), which hold locally, for all but finitely many points of the domain.

As an application of type (I) Γ-factorization, we show a major difference concerning the behavior at infinity of definable functions with respect to the o-minimal context: C-minimal valued fields having $\mathbb{Q}$ as their value group are polynomially bounded. This yields, in particular, that C-minimal expansions of algebraically closed valued fields like $\mathbb{C}_p$ or $\overline{\mathbb{F}_p}^{alg}((t^{\mathbb{Q}}))$ are polynomially bounded, which radically restricts the class of definable functions in such expansions.

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RV-factorizations are proven under an additional hypothesis on the asymptotic behavior of unary definable functions of the value group. In characteristic 0, we deduce from type (II) RV-factorization that the domain of a definable function $f\colon X \subseteq K \to K$ can be partitioned into sets $F \cup E \cup J$, where F is finite, $f|E$ is locally constant and $f|J$ satisfies locally the Jacobian property (as defined in [1]). Let us now introduce these concepts and formally state the results of the article.

1. Main results

Let (K, v) be a valued field. All valued fields under consideration are non-trivially valued. We denote the value group by Γ_K, the valuation ring by $\mathcal{O}_K$, its maximal ideal by $\mathcal{M}_K$ and the residue field by K/v. For $a \in K$ and $\gamma \in \Gamma_K \cup \{+\infty\}$, we let $B(a, \gamma) := \{x \in K : v(x-a) \geq \gamma\}$ denote the closed ball centered at a of radius γ. Respectively, for $\gamma \in \Gamma_K \cup \{-\infty\}$, we let $B^\circ(a, \gamma) := \{x \in K : v(x-a) > \gamma\}$ denote the open ball centered at a of radius γ (thus K is treated as an open ball). By a ball we mean either a closed or an open ball. We let RV^* denote the quotient group

$$\mathrm{RV}^* := K^\times/(1 + \mathcal{M}_K)$$

and define $\mathrm{RV} := \mathrm{RV}^* \cup \{0\}$. The function $\mathrm{rv}\colon K \to \mathrm{RV}$ denotes the quotient map which in addition sends 0 to 0.

We study (K, v) as a first order structure using the language of valued fields $\mathcal{L}_{div} := (+, -, \cdot, 0, 1, \mathrm{div})$ where $\mathrm{div}(x, y)$ is a binary predicate interpreted in (K, v) by $v(x) \leq v(y)$. Given a language $\mathcal{L}$ extending $\mathcal{L}_{div}$, by an $\mathcal{L}$-definable set we mean a set defined in the language $\mathcal{L}$ with parameters. We will often omit the prefix $\mathcal{L}$ and talk about definable sets when $\mathcal{L}$ is clear from the context. Note that Γ_K, RV and K/v are all three $\mathcal{L}_{div}$-interpretable. Abusing of terminology, by a definable subset of Γ_K, K/v or RV we mean an interpretable subset. Let us recall the definition of C-minimality in this context:

Definition 1.1. An expansion $(K, \mathcal{L})$ of $(K, \mathcal{L}_{div})$ is *C-minimal* if for every elementary equivalent structure $(K', \mathcal{L})$, every $\mathcal{L}$-definable subset $X \subseteq K'$ is a boolean combination of balls.

Every algebraically closed valued field $(K, \mathcal{L}_{div})$ is C-minimal. Conversely, by a result of Haskell and Macpherson in [7], every C-minimal valued field is algebraically closed. Further examples of C-minimal valued fields include algebraically closed valued fields with analytic structure as studied by Lipshitz and Robinson in [9]. From now on we work over a C-minimal expansion $(K, \mathcal{L})$ of an algebraically closed valued field (K, v).

The first part of the paper (Sections 3 and 4) is devoted to study Γ-factorizations. Let us provide their formal definition.

Definition 1.2. Let $f\colon X \subseteq K \to K$ be a function.

(1) The function f *factorizes at infinity over* Γ if there is $h\colon \Gamma_K \to \Gamma_K$ and $\gamma_0 \in \Gamma_K$ such that $v(f(x)) = h(v(x))$ for all $x \in X \setminus B(0, \gamma_0)$. We say in this case that f factorizes at infinity over Γ *through* h or that h is a Γ*-factorization of* f *at infinity*.
(2) A function $f\colon X \subseteq K \to K$ *factorizes over* Γ if there is a function $h\colon \Gamma_K \to \Gamma_K$ such that $v(f(x) - f(y)) = h(v(x-y))$ for all distinct $x, y \in X$. In this case we say that f factorizes over Γ *through* h or that h is a Γ*-factorization of* f.
(3) We say that f *locally factorizes over* Γ if for every $x \in X$ there is an open ball $B_x \subseteq X$ containing x such that $f|B_x$ factorizes over Γ. For $h\colon Y \subseteq X \times \Gamma_K \to \Gamma_K$, we say that f locally factorizes over Γ *through* h if for every $x \in X$ there is an open ball $B_x \subseteq X$ containing x such that $f|B_x$ factorizes over Γ through h_x.

The following results correspond to type (I) and (II) Γ-factorization (later Theorems 3.1 and 3.3).

Theorem (Γ-factorization I)**.** *Let $(K,\mathcal{L})$ be a C-minimal valued field and let $f\colon X\subseteq K\to K^\times$ be a definable function defined at a neighborhood of infinity. Then f factorizes at infinity over Γ through a definable function h.*

Theorem (Γ-factorization II)**.** *Let $(K,\mathcal{L})$ be a C-minimal valued field and let $f\colon X\subseteq K\to K$ be a definable local C-isomorphism. Then there is a finite subset $F\subseteq X$ such that $f|(X\setminus F)$ locally factorizes over Γ through a definable function h.*

It is natural to restrict our study to definable local C-isomorphisms (Definition 2.3) in the second theorem above, since by a result of Haskell and Macpherson (see Theorem 2.4), every definable function $f\colon X\subseteq K\to K$, X definably decomposes into $X=F\cup E\cup J$ where F is a finite set, $f|_E$ is locally constant and $f|_J$ is a local C-isomorphism.

Both type (I) and type (II) Γ-factorizations also hold uniformly in definable families (see later Theorems 3.7 and 3.8).

From type (I) Γ-factorization we deduce the above mentioned result about polynomially bounded C-minimal valued fields. Let us recall what polynomially bounded means in this context. The structure $(K,\mathcal{L})$ is said to be *polynomially bounded* if for every definable function $f\colon X\subseteq K\to K$ there is $\gamma\in\Gamma_K$ and a non-zero integer n such that $v(f(x))>nv(x)$ for all $x\in X\setminus B(0,\gamma)$. We say it is *uniformly polynomially bounded*, if the analogous statement holds over definable families (see Definition 4.2 for the precise definition).

Theorem (later Theorem 4.8)**.** *Let $(K,\mathcal{L})$ be a C-minimal valued field with $\Gamma_K=\mathbb{Q}$. Then $(K,\mathcal{L})$ is uniformly polynomially bounded. In particular, any C-minimal expansion of $\mathbb{C}_p$ or $\overline{\mathbb{F}_p}^{acl}(\!(t^{\mathbb{Q}})\!)$ is polynomially bounded. More generally, any C-minimal valued field $(K,\mathcal{L})$ which is $\mathcal{L}$-elementary equivalent to a valued field having $\mathbb{Q}$ as its value group, is uniformly polynomially bounded.*

The previous theorem is obtained using a dichotomy for o-minimal expansions of ordered groups due to Miller and Starchenko [12], and the following result (which uses type (I) Γ-factorization).

Theorem (later Theorem 4.7)**.** *Let $(K,\mathcal{L})$ be a C-minimal valued field such that Γ_K is $\mathbb{Q}$-linearly bounded. Then $(K,\mathcal{L})$ is uniformly polynomially bounded.*

In the second part of the paper (Sections 5 and 6), we extend Γ-factorizations to RV-factorizations under additional hypotheses concerning definable functions of the value group. In particular, we derive RV-factorization under the hypothesis that such functions are *eventually* $\mathbb{Q}$*-linear* (see later Definition 5.1). For simplicity, we will now state both type (I) and (II) RV-factorization only for definable unary functions, although we will later prove these results in families (see later Theorems 5.4 and 5.5).

Let p denote the *characteristic exponent* of (K,v), that is, $p=1$ if the characteristic of K is 0 and otherwise p equals the characteristic of K.

Theorem (RV-factorization I)**.** *Let $(K,\mathcal{L})$ be a C-minimal valued field, $f\colon X\subseteq K\to K^\times$ be a definable function defined at a neighborhood of infinity. Let $h\colon Y\subseteq\Gamma_K\to\Gamma_K$ be a Γ-factorization of f at infinity. If h is eventually $\mathbb{Q}$-linear, then there are integers $m,n\in\mathbb{Z}$ and $c\in RV$ such that, in a neighborhood of infinity,*

$$rv(f(x))=rv(x)^{n/p^m}c.$$

If in addition $(K,\mathcal{L})$ is definably complete and all definable unary Γ-functions are eventually linear, the limit

$$a := \lim_{x\to\infty} \frac{f(x)}{x^{n/p^m}}$$

exists in K and $c = rv(a)$.

Theorem (RV-factorization II)**.** *Let $(K,\mathcal{L})$ be a C-minimal valued field and $f\colon X \subseteq K \to K$ be a definable local C-isomorphism. Suppose $h\colon Y \subseteq X\times\Gamma_K \to \Gamma_K$ is a local Γ-factorization of f which is eventually $\mathbb{Q}$-linear. Then there are a finite definable partition $X = X_1 \cup \cdots \cup X_\ell$, integers $n_1,\ldots,n_\ell$ and definable functions $\delta\colon X \to \Gamma_K$ and $c\colon X \to RV$ such that for all $x \in X_i$, $B^\circ(x,\delta(x)) \subseteq X_i$ and for all distinct $y,z \in B^\circ(x,\delta(x))$,*

$$rv((f(y) - f(z)) = rv((y - z))^{p^{n_i}} c(x).$$

In particular, when $p = 1$ we may assume that $\ell = 1$. If in addition $(K,\mathcal{L})$ is definably complete and all definable unary Γ-functions are eventually linear, the limit

$$a(x) := \lim_{y\to x} \frac{f(x) - f(y)}{(x - y)^{p^{n_i}}}$$

exists in K and $c(x) = rv(a(x))$.

Recall that an expansion $(K,\mathcal{L})$ is *definable complete* if every definable family of nested balls whose set of radii tends to $+\infty$ has non-empty intersection. Clearly, every expansion of a complete valued field is definably complete. Note moreover that definable completeness is a first order property. Therefore, since every algebraically closed valued field $(K,\mathcal{L}_{div})$ in the language of valued fields has a complete elementary extension, they are all definably complete. Unfortunately, there are C-minimal expansions of valued fields which are not definably complete (see for example [5, Theorem 5.4]). Nonetheless, since most algebraically closed valued fields of interest are complete (for example $\mathbb{C}_p$), any of their C-minimal expansions will satisfy this assumption.

In characteristic 0, type (II) RV-factorization implies the following result for definably complete C-minimal valued fields.

Theorem (later Theorem 6.2 in families)**.** *Let $(K,\mathcal{L})$ be a definably complete C-minimal valued field of characteristic 0 in which all definable unary Γ-functions are eventually $\mathbb{Q}$-linear. Let $f\colon X \subseteq K \to K$ be a definable local C-isomorphism. Then there is a finite set F such that $f|(X \setminus F)$ has locally the Jacobian property.*

The Jacobian property is taken from [1] and will be later recalled (Definition 6.1). A weaker version of Theorem 6.2 was already obtained by the second author for valued fields of equicharacteristic zero in [5]. Here we generalize and extend the result to mixed characteristic. We also obtain the following corollaries.

Corollary (later Corollary 6.3 in families)**.** *Let $(K,\mathcal{L})$ be as in the previous theorem and $f\colon X \subseteq K \to K$ be a definable function. Then there is a definable partition of X into sets $X = F \cup E \cup J$ where F is finite, $f|_E$ is locally constant and $f|_J$ has locally the Jacobian property.*

Corollary. *Let $(K,\mathcal{L})$ and f be as in the previous corollary. Let D be the definable set $D := \{x \in X : f'(x) = 0\}$. Then D can be decomposed into sets $F \cup L$ such that F is finite and $f|L$ is locally constant. In particular, f is C^1 on a cofinite subset of X.*

Note that the previous corollary cannot be extended to characteristic $p > 0$. Indeed, in an algebraically closed valued field of characteristic $p > 0$ the function $x \mapsto x^p$ is injective and has null derivative.

It is worthy to note that over complete algebraically closed valued fields, C-minimal expansions by analytic functions as studied by Lipschitz and Robinson [9] do satisfy all hypothesis of Theorem 6.2, but our theorem is not giving anything new. Indeed, much stronger results follow (like local analyticity of definable functions) from [9] and [1]. On the other hand, type (II) Γ-factorization and RV-factorization generalize results of Hrushovski-Kazhdan in [8, Section 5]. Their results hold in algebraically closed valued fields of residue characteristic 0 under the stronger assumption of V-minimality, which implies both definable completeness and that the induced structure on RV is exactly the induced structure by $\mathcal{L}_{div}$. As far as we know, type (I) factorizations are new in the literature.

We would also like to point out that, modulo the following conjecture about o-minimal expansions of $(\mathbb{Q}, <, +, 0)$, all definable functions would have RV-factorizations in C-minimal valued fields $(K, \mathcal{L})$ for which $\Gamma_K = \mathbb{Q}$. In particular, if $(K, \mathcal{L})$ is definably complete of characteristic 0, the previous corollary implies that every definable function $f\colon X \subseteq K \to K$ is C^1, for all but finitely many points in X.

Conjecture 1. *Every definable function $f\colon \mathbb{Q} \to \mathbb{Q}$ in an o-minimal expansion of $(\mathbb{Q}, <, +, 0)$ is eventually $\mathbb{Q}$-linear.*

The article is organized as follows. In Section 2, we provide the needed background on C-minimality. Section 3 is devoted to show the results about Γ-factorization. Polynomially bounded C-minimal valued fields are studied in Section 4. The results on RV-factorizations are presented in Section 5. Finally, all results related to the Jacobian property are shown in Section 6.

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2. Preliminaries and auxiliary results

Hereafter, (K, v) will denote an algebraically closed valued field, $(K, \mathcal{L})$ a C-minimal expansion of $(K, \mathcal{L}_{div})$. We define a ternary relation on K, called the *C-relation*, by

$$C(x, y, z) \Leftrightarrow v(x - y) = v(x - z) < v(y - z).$$

Boolean combinations of balls correspond precisely to quantifier free formulas in the language which only contains a predicate for the C-relation, which partly explains the analogy between o-minimality and C-minimality.

We use the following conventions regarding definable families of sets and functions. Given a definable set W, a definable family of sets in K parametrized by W is a definable set $X \subseteq W \times K$ such that the projection of X onto the coordinates of W is equal to the set W. We will often omit the mention "in K parametrized by W" and simply write $X \subseteq W \times K$ when no confusion arises. Given a definable family $X \subseteq W \times K$, the fiber at $w \in W$ corresponds to $X_w := \{x \in K : (w, x) \in X\}$. Similarly, a definable family of functions is a definable function $f\colon X \subseteq W \times K \to K$, and we use the notation $f_w\colon X_w \to K$ for the function given by $f_w(x) := f(w, x)$.

Remark 2.1. By C-minimality, every definable non-empty subset $X \subseteq K$ is a finite disjoint union of "Swiss cheeses". A Swiss cheese is a set of the form $B \setminus (\bigcup_{i=1}^{m} B_i)$ where B is a ball and each B_i is a ball strictly contained in B such that $B_i \cap B_j = \emptyset$ for $i \neq j$. Furthermore, this also holds in families: if $X \subseteq W \times K$ is a definable family, then there is a finite partition $W_1, \ldots, W_n$ of W and integers k_i, m_i with $i \in \{1, \ldots, n\}$ such that for all $w \in W_i$, the fiber X_w is a disjoint union of k_i Swiss cheeses of the form $B(w) \setminus (\bigcup_{i=1}^{m_i} B_i(w))$.

The reader interested in the general study of C-minimal structures is referred both to the original articles [7, 10] and to [3, 4, 5].

The following two theorems due to Haskell and Macpherson in [7] will be extensively used.

Theorem 2.2 (Haskell-Macpherson). *Let $(K, \mathcal{L})$ be C-minimal.*

(1) The induced structure on Γ_K is o-minimal, that is, every definable set $Y \subseteq \Gamma_K$ is a finite union of intervals and points.
(2) Any definable function $g\colon X \subseteq K \to \Gamma_K$ is locally constant on a cofinite subset of X.
(3) The induced structure on K/v is strongly-minimal. In particular, every definable set $Y \subseteq K/v$ is either finite or cofinite, and no infinite subset of K/v can be linearly ordered by a definable relation.
(4) Let W be a definable set and $X \subseteq W \times K$ be a definable family such that each fiber X_w is finite. Then there is $n \in \mathbb{N}$ such that each fiber has cardinality less than n.

To state their second theorem we need to introduce the notion of C-isomorphism.

Definition 2.3. Given an open ball $B \subseteq K$, a function $f\colon B \to K$ is a *C-isomorphism* if $f(B)$ is an open ball and f preserves both the C-relation and its negation, that is, if for all $x, y, z \in B$

$$C(x, y, z) \Leftrightarrow C(f(x), f(y), f(z)).$$

Note that a C-isomorphism must be injective. The original definition of C-isomorphism in [7] does not include the condition "$f(B)$ is an open ball", but for our purposes such assumption is harmless. Indeed, for $f\colon X \subseteq K \to K$ a definable local C-isomorphism as defined in [7] (i.e., without the assumption "$f(B)$ is an open ball"), by C-minimality we have that the set

$$\{x \in X : \text{there is no ball } B \subseteq X \text{ containing } x \text{ such that f(B) is an open ball}\},$$

is finite (one can use Fact 1 in [7], Lemma 1.9 in [3]).

Theorem 2.4 (Haskell-Macpherson). *Let $(K, \mathcal{L})$ be C-minimal and $f\colon X \subseteq K \to K$ be a definable function. Then there is a definable partition of X into sets $F \cup E \cup I$ such that F is finite, $f|(X \setminus F)$ is continuous, $f|E$ is locally constant and $f|I$ is a local C-isomorphism.*

Using part (4) of Theorem 2.2, it is not difficult to see that Theorem 2.4 holds also for definable families of functions as every condition is uniformly definable. For a definable set W and a definable family of functions $f\colon X \subseteq W \times K \to K$, we say that f is a family of local C-isomorphisms if for each $w \in W$ the fiber f_w is a local C-isomorphism. The following is the family version of Theorem 2.4.

Corollary 2.5. *Let $(K, \mathcal{L})$ be C-minimal, W be a definable set and $f\colon X \subseteq W \times K \to K$ be a definable family of functions. Then there is a definable partition of X into sets $F \cup E \cup I$ such that for all $w \in W$, F_w is finite, $f|(X_w \setminus F_w)$ is continuous, $f_w|E_w$ is locally constant and $f_w|I_w$ is a local C-isomorphism.*

Note that for all $x, y \in K^*$

$$\mathrm{rv}(x) = \mathrm{rv}(y) \Leftrightarrow v(x) = v(y) < v(x - y),$$

and therefore the condition $C(0, x, y)$ recovers the equivalence relation defined by $rv(x) = rv(y)$. We will often denote this relation, for notation convenience, by $x \sim y$. We will later need the following lemma about RV-classes.

Lemma 2.6. *Suppose that $a, b, c \in K^\times$ are elements lying in different RV-classes. If $a \sim a', b \sim b'$ and $c \sim c'$ then $C(a, b, c) \Leftrightarrow C(a', b', c')$.*

Proof. We show the following two equivalences from which the result follows:

(1) $C(a, b, c) \Leftrightarrow C(a', b, c)$;
(2) $C(a, b, c) \Leftrightarrow C(a, b', c)$.

By assumption on a, b, c and a' we have that

$$\begin{aligned} C(a,b,c) &\Leftrightarrow v(a) < \min(v(b), v(c)) \\ &\Leftrightarrow v(a) = v(a') < \min(v(b), v(c)) \\ &\Leftrightarrow C(a', b, c), \end{aligned}$$

which shows (1). For (2),

$$\begin{aligned} C(a,b,c) &\Leftrightarrow v(a) < \min(v(b), v(c)) \\ &\Leftrightarrow v(a) < \min(v(b'), v(c)) \\ &\Leftrightarrow C(a, b', c). \end{aligned}$$

□

Let $(M, \mathcal{L})$ be a structure and S be a $\emptyset$-definable ($\emptyset$-interpretable) set. The induced structure by $\mathcal{L}$ on S will be denoted by $(S, \mathcal{L}_{ind})$. It consists of all $\emptyset$-$\mathcal{L}$-definable (resp. $\emptyset$-$\mathcal{L}$-interpretable) subsets of cartesian powers of S. A $\emptyset$-definable ($\emptyset$-interpretable) set S is *stably embedded* if every $\mathcal{L}$-definable subset $X \subseteq S^n$ is already $\mathcal{L}_{ind}$-definable with parameters in S. The following result is a particular case of Corollary 1.10 from [2]. It essentially follows from [13, Theorem 1.4].

Proposition 2.7. *Let K be a C-minimal valued field. Then Γ_K and K/v are stably embedded.*

In our setting there are morally three infinities. We have $-\infty$ and $+\infty$ as additional elements of Γ_K which are respectively smaller and bigger than any element in Γ_K. Furthermore we have an infinity element "∞" (without sign) which we treat as an additional point of K which satisfies the following: a set $X \subseteq K$ contains ∞ if and only there is $\gamma \in \Gamma_K$ such that $K \setminus B(0, \gamma) \subseteq X$. Thus, by a *neighborhood of infinity* we mean a set of the form $K \setminus B(0, \gamma)$ for some $\gamma \in \Gamma_K$. We say that a family of functions $f : X \subseteq W \times K \to K$ is *defined at a neighborhood of infinity* if for all $w \in W$, the function f_w is defined at a neighborhood of infinity.

3. Γ-factorization

In what follows we use the following notation: for $\delta \in \Gamma_K \cup \{-\infty\}$, let $\Gamma_K^{>\delta}$ denote the set $\Gamma_K^{>\delta} := \{\gamma \in \Gamma_K : \gamma > \delta\}$ and analogously for $\Gamma_K^{<\delta}$ with $\delta \in \Gamma_K \cup \{+\infty\}$.

Theorem 3.1 (Γ-factorization I)**.** *Let $(K, \mathcal{L})$ be C-minimal and $f : X \subseteq K \to K^\times$ be a definable function defined at a neighborhood of infinity. Then f factorizes at infinity over Γ through a definable function h.*

Proof. Consider the function $g\colon \Gamma_K \to \Gamma_K \cup \{-\infty\}$ given by

$$h(\gamma) := \inf\{v(f(x)) : v(x) = \gamma\}.$$

By o-minimality (part (1) of Theorem 2.2), the function h is well-defined.

Claim 3.2. *There is $\delta_1 \in \Gamma_K$ such that $h(\gamma) \in \Gamma_K$ for all $\gamma < \delta_1$.*

For $\gamma, \mu \in \Gamma_K$, define

$$D_{\gamma,\mu} := f^{-1}(v^{-1}(]-\infty, \mu])) \cap (v^{-1}(\gamma)),$$

and let $Y := \{\gamma \in \Gamma_K : \forall \mu \in \Gamma_K\ (D_{\gamma,\mu} \neq \emptyset)\}$. Fix $\gamma \in Y$ (if $Y \neq \emptyset$). For each $\mu \in \Gamma_K$, the set $D_{\gamma,\mu}$ is contained in $B(0,\gamma) \setminus B^\circ(0,\gamma)$. By part (1) of Theorem 2.2 and the fact that definable families of subsets of K are families of Swiss cheeses (see Remark 2.1), there is an integer n such that, coinitially in Γ_K, only one of the following happens

(1) $D_{\gamma,\mu}$ contains a set of the form $B(0,\gamma) \setminus \bigcup_{i=1}^n B_{i,\mu}$, or
(2) $D_{\gamma,\mu}$ is contained in $\bigcup_{i=1}^n B_{i,\mu}$,

where the balls $B_{1,\mu}, \ldots, B_{n,\mu}$ are of the form $B_{i,\mu} = B^\circ(a_{i,\mu}, \gamma)$ with $v(a_{i,\mu}) = \gamma$.

Let us first show that for all $\gamma \in Y$, (1) cannot hold coinitially in Γ_K. For suppose for a contradiction that there is $\gamma \in Y$ such that (1) holds coinitially in Γ_K. Since $D_{\gamma,\mu'} \subseteq D_{\gamma,\mu}$ for all $\mu' \leqslant \mu$, by part (1) of Theorem 2.2, the balls $B_{i,\mu}$ can be taken equal coinitiallly in Γ_K. But this implies that the intersection $\bigcap_{\mu\in\Gamma_K} D_{\gamma,\mu} \neq \emptyset$, which is a contradiction since for every $x \in B(0,\gamma)$ there is μ such that $x \notin D_{\gamma,\mu}$.

Therefore, for all $\gamma \in Y$, we must have that (2) holds coinitially in Γ_K. So for every $\gamma \in Y$, there is an element $\mu_\gamma \in \Gamma_K$ such that for all $\mu < \mu_\gamma$, the set $D_{\gamma,\mu}$ is contained in a union of n open balls $B^\circ(a,\gamma)$ with $v(a) = \gamma$. Consider the definable set

$$\bigcup_{\gamma\in Y, \mu<\mu_\gamma} D_{\gamma,\mu}.$$

If Y is infinite, we contradict C-minimality, as such set is not a finite disjoint union of Swiss cheeses. Therefore, Y is finite. Let δ_1 be the minimal element of Y if $Y \neq \emptyset$ or $\delta_1 = 0$ if $Y = \emptyset$. By construction, δ_1 satisfies the claim.

For $\gamma \in \Gamma_K$, define

$$A_\gamma := \{x \in K : v(x) = \gamma \wedge v(f(x)) = h(\gamma)\}.$$

By Claim 3.2, the union $\bigcup_{\gamma\in\Gamma_K} A_\gamma$ contains elements of arbitrarily small valuation. Therefore, by C-minimality, it contains a set of the form $K \setminus B(0,\gamma_0)$ with $\gamma_0 \in \Gamma_K$. Let δ_2 be the maximal such γ_0, if existing, or $\delta_2 = 0$ otherwise. To complete the proof, we show that $v(f(x)) = h(v(x))$ for all $x \in K \setminus B(0,\delta_2)$. Indeed, if $x \in K \setminus B(0,\delta_2)$, then $x \in A_\gamma$ for some $\gamma \in \Gamma_K$. But by the definition of A_γ this only holds if $v(x) = \gamma$ and $v(f(x)) = h(x)$. □

Theorem 3.3 (Γ-factorization II)**.** *Let $(K, \mathcal{L})$ be a C-minimal valued field and let $f\colon X \subseteq K \to K$ be a definable local C-isomorphism. Then there is a finite subset $F \subseteq X$ such that $f|(X \setminus F)$ locally factorizes over Γ through a definable function h.*

Proof. Let $x \in X$ and $B_x = B^\circ(x, \gamma_0(x))$ be maximal such that $f_{|B_x}$ is a C-isomorphism. Note that $\gamma_0\colon X \to \Gamma_K \cup \{-\infty\}$ is definable. The definition of C-isomorphism implies that the function $h_x\colon \Gamma_K^{>\gamma_0(x)} \to \Gamma_K$ defined by

$$h_x(\gamma) = v(f(x) - f(y)) \text{ for some (all) } y \text{ such that } v(x - y) = \gamma,$$

is well defined. For $\square \in \{=,<,>\}$, we write $[h_x]\square[h_y]$ to express the following property

$$[h_x]\square[h_y] \Leftrightarrow (\exists\delta \in \Gamma)(\forall\gamma > \delta)(h_x(\gamma)\square h_y(\gamma)).$$

Thus, $[h_x] = [h_y]$ means that the functions h_x and h_y are eventually equal.

Claim 3.4. *There is a definable function $\gamma_1\colon X \to \Gamma_K \cup \{-\infty\}$ such that for all $x \in X$, $\gamma_1(x) \geqslant \gamma_0(x)$ and for all $y \in B^\circ(x, \gamma_1(x)) \setminus \{x\}$, either $[h_x] = [h_y]$, $[h_x] > [h_y]$ or $[h_x] < [h_y]$.*

Consider the definable sets $A_\square(x) := \{y \in B_x : [h_x]\square[h_y]\}$ where $\square \in \{=,<,>\}$. These sets are disjoint and, by o-minimality, they cover B_x. Therefore, by C-minimality, there is $\gamma_1(x) \geqslant \gamma_0(x)$ minimal such that $B^\circ(x,\gamma_1(x)) \setminus \{x\}$ is contained in one of them, which shows the claim.

In what follows, set $B_x := B^\circ(x, \gamma_1(x))$.

Claim 3.5. *For all but finitely many $x \in X$, for all $y \in B_x$, $[h_x] = [h_y]$.*

Suppose not. Then, by Claim 3.4, there are infinitely many $x \in X$ for which for all $y \in B_x \setminus \{x\}$, either $[h_x] < [h_y]$ or $[h_x] > [h_y]$. Suppose that there are infinitely many $x \in X$ for which $[h_x] < [h_y]$ for all $y \in B_x \setminus \{x\}$ (the other case is analogous). By C-minimality, there is an open ball $B \subseteq X$ such that for all $x \in B$ for all $y \in B_x \setminus \{x\}$, $[h_x] < [h_y]$. But this implies a contradiction, since for $x, y \in B$ such that $x \neq y$ we have that $[h_x] < [h_y] < [h_x]$. This shows the claim.

Let $F \subseteq X$ be the finite set given by Claim 3.5 and let $x \in X \setminus F$. Let $g_x\colon B_x \to \Gamma_K \cup \{-\infty\}$ be the function sending y to the smallest $\delta \in \Gamma_K \cup \{-\infty\}$ such that $(\forall\gamma > \delta)(h_x(\gamma) = h_y(\gamma))$, which exists since $[h_x] = [h_y]$. By part (2) of Theorem 2.2, the function g_x is locally constant on $B_x \setminus W_x$, with W_x a finite set. Let $\gamma_3\colon X \setminus F \to \Gamma_K \cup \{\infty\}$ be the definable function sending x to the minimal γ such that $B^\circ(x,\gamma) \subseteq B_x \setminus W_x$. Reset $B_x := B^\circ(x, \gamma_3(x))$. Consider the set

$$A := \{x \in X \setminus F : \neg(\exists\gamma \in \Gamma_K)(\gamma \geqslant \gamma_3(x) \wedge (\forall y, z \in B^\circ(x,\gamma) \setminus \{x\})(g_x(y) = g_x(z))\},$$

consisting of elements $x \in X \setminus F$ for which there is no open ball B around x such that g_x restricted to $B \setminus \{x\}$ is constant.

Claim 3.6. *The set A is finite.*

Suppose not. Since A is definable, by C-minimality, there is an open ball $B \subseteq A$. Pick $x \in B$ and $y \in B \cap B_x$. Then, since $y \in B_x$, there is an open ball $B' \subseteq B \cap B_x$ containing y and such that g_x is constant on B'. We claim that g_y is defined and constant on $B' \setminus \{y\}$, which contradicts that $y \in A$. It suffices to show that there is some δ such that for all $z \in B' \setminus \{y\}$, $h_y(\gamma) = h_z(\gamma)$ for all $\gamma > \delta$. Indeed, if such δ exists, then there is a minimal such element which would be equal to $g_y(z)$ for all $z \in B' \setminus \{y\}$. We take $\delta = g_x(z) = g_x(y)$, which is well defined since by assumption g_x is constant on B'. By definition of g_x we have that for all $\gamma > g_x(z)$

$$h_z(\gamma) = h_x(\gamma) = h_y(\gamma),$$

which completes the claim.

Let $x \in X \setminus (A \cup F)$. Since $x \notin A$, let $\gamma_4\colon X \setminus (A \cup F) \to \Gamma_K \cup \{-\infty\}$ be the definable function sending x to the minimal $\gamma \geqslant \gamma_3(x)$ such that g_x is constant on $B^\circ(x,\gamma) \setminus \{x\}$. Finally, reset B_x to now denote the maximal open ball $B^\circ(x, \gamma_5(x))$, where

$$\gamma_5(x) = \max\{\gamma_4(x), g_x(z)\} \text{ for some (all) } z \in B^\circ(x, \gamma_4(x)) \setminus \{x\}.$$

Then, $f_{|B_x}$ factorizes over Γ through h_x. □

Uniform results. The reader can check that the proofs of both Theorem 3.1 and 3.3 are uniform in parameters. We provide the exact statements of what we mean by 'uniform in parameters' for the reader's convenience.

Theorem 3.7. *Let $(K,\mathcal{L})$ be C-minimal and $f\colon X\subseteq W\times K\to K^\times$ be a definable family of functions defined at a neighborhood of infinity. Then there is a definable family of functions $h\colon Y\subseteq W\times\Gamma_K\to\Gamma_K$ such that for all $w\in W$, f_w factorizes at infinity over Γ through h_w.*

Theorem 3.8. *Let $(K,\mathcal{L})$ be C-minimal and $f\colon X\subseteq W\times K\to K$ be a definable family of local C-isomorphisms. Then there are a definable family of functions $h\colon Y\subseteq X\times\Gamma_K\to\Gamma_K$ and a definable subset $F\subseteq X$ such that, for all $w\in W$ the set F_w is finite, and $f_w|(X_w\setminus F_w)$ locally factorizes over Γ through h_w.*

4. Polynomially bounded C-minimal valued fields

Let us start by defining the main concepts of this section:

Definition 4.1. Let $(R,<,+)$ be a divisible ordered abelian group definable inside some structure $(M,\mathcal{L})$. The structure $(R,\mathcal{L}_{ind})$ is said to be *linearly bounded* if for every $\mathcal{L}_{ind}$-definable function $f\colon R\to R$ there are a definable endomorphism $\lambda\in End(R,<,+)$ and $a\in R$ such that $|f(x)|<|\lambda(x)|$ for all x such that $|x|>|a|$. It is said to be *$\mathcal{L}$-uniformly linearly bounded* if for every $\mathcal{L}$-definable set W and every $\mathcal{L}$-definable family of functions $f\colon W\times R\to R$, there are an $\mathcal{L}$-definable $\lambda\in End(R,<,+)$ and an $\mathcal{L}$-definable function $a\colon W\to R$ such that, for all $w\in W$, $|f_w(x)|<|\lambda(x)|$ for all $x>a(w)$. It is $\mathbb{Q}$-linearly bounded (resp. $\mathcal{L}$-uniformly $\mathbb{Q}$-linearly bounded) if in addition λ can be chosen to be $\lambda(x)=nx$ for some integer n.

Definition 4.2. An expansion $(K,\mathcal{L})$ of $(K,\mathcal{L}_{div})$ is said to be *uniformly polynomially bounded*, if for every definable set W and definable family of functions $f\colon X\subseteq W\times K\to K$, there is an integer n and a definable function $a\colon W\to\Gamma_K$ such that for each $w\in W$, $v(f_w(x))>nv(x)$ for all $x\in X$ such that $v(x)<a(w)$.

We will need the following dichotomy due to Miller and Starchenko in o-minimal expansions of ordered groups.

Theorem 4.3 (Miller-Starchenko [12, Theorems A and B])**.** *Suppose that $(R,\mathcal{L})$ is an o-minimal expansion of an ordered group $(R,<,+)$. Then exactly one of the following holds:*

(a) $(R,\mathcal{L})$ defines a binary operation $\cdot$ such that $(R,<,+,\cdot)$ is an ordered real closed field or
(b) for every definable $\alpha\colon R\to R$ there exist $c\in R$ and a definable $\lambda\in\{0\}\cup Aut(R,+)$ with $\lim_{x\to+\infty}[\alpha(x)-\lambda(x)]=c$.

If (b) holds and there is a distinguished $\emptyset$-definable element $1>0$ then every definable endomorphism of $(R,+)$ is $\emptyset$-definable.

Remark 4.4. Note that condition (b) implies that the structure $(R,\mathcal{L})$ is linearly bounded. Indeed one may always suppose without loss of generality that there is a distinguished $\emptyset$-definable element $1>0$.

Corollary 4.5. *Every o-minimal expansion of $(\mathbb{Q},<,+,0,1)$ is $\mathbb{Q}$-linearly bounded.*

Proof. Since multiplication is not definable in any o-minimal expansion of $(\mathbb{Q},<,+,0,1)$, by Theorem 4.3 (and the previous remark), for any definable $\alpha\colon\mathbb{Q}\to\mathbb{Q}$, there exist a $\emptyset$-definable $\lambda\in\{0\}\cup Aut(\mathbb{Q},+)$ and $a\in\mathbb{Q}$ satisfying that $|\alpha(x)|<|\lambda(x)|$ for all x such that $|x|>|a|$. The result follows from the fact that every automorphism of $(\mathbb{Q},+)$ is of the form qx for $q\in\mathbb{Q}$. □

Theorem 4.6. *Let R be a definable (or interpretable) and stably embedded set in a structure $(M, \mathcal{L})$. Suppose moreover that $(R, \mathcal{L}_{ind})$ is an o-minimal expansion of an ordered divisible group $(R, <, +, 0, 1)$. Then $(R, \mathcal{L}_{ind})$ is linearly bounded if and only if is $\mathcal{L}$-uniformly linearly bounded. Moreover, it is $\mathbb{Q}$-linearly bounded if and only if it is $\mathcal{L}$-uniformly $\mathbb{Q}$-linearly bounded.*

Proof. One direction is trivial, so suppose $(R, \mathcal{L}_{ind})$ is linearly bounded and let $f\colon W \times R \to R$ be an $\mathcal{L}$-definable family of functions. By Theorem 4.3, the two following conditions are equivalent

(1) $(R, \mathcal{L}_{ind})$ is linearly bounded;
(2) no field expanding $(R, +)$ is $\mathcal{L}_{ind}$-definable.

Therefore, the assumption implies that no field expanding $(R, +)$ is $\mathcal{L}_{ind}$-definable. Since this last condition is expressible by an $\mathcal{L}_{ind}$-axiom scheme, Theorem 4.3 implies that every $\mathcal{L}_{ind}$-elementary equivalent structure is linearly bounded. Furthermore, by Theorem 4.3, every $\mathcal{L}_{ind}$-definable endomorphism of $(R, +)$ is $\emptyset$-definable. Let Λ be the ring of all $\mathcal{L}_{ind}$-definable endomorphisms of $(R, <, +)$. By stable embeddedness, for every $w \in W$, the function $f_w\colon R \to R$ is $\mathcal{L}_{ind}$-definable. Our assumption implies there are a $\emptyset$-definable endomorphism $\lambda_w \in \Lambda$ and $a_w \in R$ such that $|f_w(x)| < |\lambda(x)|$ for all x such that $|x| > |a_w|$ (see Remark 4.4). Since Λ has bounded cardinality, by compactness and the fact that every $\mathcal{L}_{ind}$-elementary equivalent structure is linearly bounded, there is $\lambda \in \Lambda$ such that

$$(\forall w \in W)(\exists a_w \in R)(\forall x)(|x| > |a_w| \to |f_w(x)| < |\lambda(x)|,$$

which shows that $(R, \mathcal{L}_{ind})$ is $\mathcal{L}$-uniformly bounded.

If furthermore Γ_K is $\mathbb{Q}$-linearly bounded, then every $\lambda \in \Lambda$ is bounded by a function of the form $\lambda(x) = nx$ for $n \in \mathbb{N}$. Therefore $(R, \mathcal{L}_{ind})$ is $\mathcal{L}$-uniformly $\mathbb{Q}$-linearly bounded. □

Theorem 4.7. *Suppose $(K, \mathcal{L})$ is a C-minimal valued field such that $(\Gamma_K, \mathcal{L}_{ind})$ is $\mathbb{Q}$-linearly bounded. Then $(K, \mathcal{L})$ is uniformly polynomially bounded.*

Proof. Let $f\colon X \subseteq W \times K \to K^{\times}$ be a definable family of functions defined at a neighborhood of infinity. By Theorem 3.7, let $h\colon Y \subseteq W \times \Gamma_K \to \Gamma_K$ be a definable Γ-factorization of f at infinity, that is, for every $w \in W$, f_w factorizes at infinity over Γ through h_w. It suffices to show that h is $\mathcal{L}$-uniformly $\mathbb{Q}$-linearly bounded at infinity. Since C-minimality is preserved by addition of constants, we may assume there is a $\emptyset$-definable element $1 > 0$ in Γ_K. The result now follows by Theorem 4.6 taking $(M, \mathcal{L}) = (K, \mathcal{L})$ and $R = \Gamma_K$.

□

Using Corollary 4.5 and Theorem 4.7, we obtain the following:

Theorem 4.8. *Let $(K, \mathcal{L})$ be a C-minimal valued field with $\Gamma_K = \mathbb{Q}$. Then $(K, \mathcal{L})$ is uniformly polynomially bounded. In particular, any C-minimal expansion of $\mathbb{C}_p$ or $\overline{\mathbb{F}_p}^{acl}((t^{\mathbb{Q}}))$ is polynomially bounded. More generally, any C-minimal valued field $(K, \mathcal{L})$ which is $\mathcal{L}$-elementary equivalent to a valued field having as value group $\mathbb{Q}$, is uniformly polynomially bounded.*

We end up this section by giving an example of a polynomially bounded C-minimal valued field in which definable Γ-functions are not $\mathbb{Q}$-linearly bounded.

Example 4.9. Consider the algebraically closed valued field $K = \mathbb{C}((t^{\mathbb{R}}))$ having value group $\mathbb{R}$. Consider the two-sorted structure $(\mathcal{K}, \mathcal{L})$

$$\mathcal{K} := \begin{cases} (K, \mathcal{L}_{\text{ring}}) \\ (\mathbb{R}, \mathcal{L}_{or}) \\ v\colon K \to \mathbb{R} \cup \{\infty\} \end{cases}$$

where $\mathcal{L}_{or} := \mathcal{L}_{\text{ring}} \cup \{<\}$. The theory of $\mathcal{K}$ has elimination of quantifiers (one follows the same argument as for the reduct in which the value group has only the ordered abelian group structure, see [6]). Since $(\mathbb{R}, \mathcal{L}_{or})$ is o-minimal, it is easy to see that definable sets in one valued field variable can only be finite boolean combinations of balls (in any model of $Th(\mathcal{K})$). Consider the one sorted language $\mathcal{L}^*$ containing $\mathcal{L}_{div}$ and a relation symbol for every $\mathcal{L}$-definable subset of K^n (for all n). Interpreting the language in the obvious way, we have that $(K, \mathcal{L}^*)$ is C-minimal. Clearly, there are definable unary Γ-functions which are not linearly bounded since we have multiplication in the value group. Nevertheless, it is worthy to note that $(K, \mathcal{L}^*)$ is polynomially bounded. Indeed, by quantifier elimination, every $\mathcal{L}^*$-definable function $f\colon K \to K$ is already $\mathcal{L}_{div}$-definable. Therefore, although the fact of having all definable Γ-functions to be $\mathbb{Q}$-linearly bounded is a sufficient condition for a C-minimal valued field to be polynomially bounded, the example shows it is not a necessary condition. We finish this section with the following natural question:

Question 1. Is every C-minimal valued field polynomially bounded?

5. RV-factorization

Let us start by defining what eventually linear means. By a unary Γ-function in K we simply mean a function $g\colon \Gamma_K \to \Gamma_K$.

Definition 5.1. Let $g\colon \Gamma_K \to \Gamma_K$ be a unary Γ-function.

(1) We say g is *eventually linear* if there are $\delta, \alpha \in \Gamma_K$ and an endomorphism λ of $(\Gamma_K, +)$ such that $g(\gamma) = \lambda(\gamma) + \alpha$ for all $\gamma \in \Gamma_K^{>\delta}$.
(2) We say g is *eventually* $\mathbb{Q}$*-linear* if in the previous definition, λ can be taken of the form $\lambda(\gamma) = r\gamma$ for $r \in \mathbb{Q}$.
(3) Given a definable set of parameters W, a definable family of functions $f\colon W \times \Gamma_K \to \Gamma_K$ is *eventually* $\mathbb{Q}$*-linear* if there is a finite subset $Z_f \subseteq \mathbb{Q}$ such that for every $w \in W$, the function f_w is eventually $\mathbb{Q}$-linear with slope $r \in Z_f$.

In algebraically closed valued fields, every $\mathcal{L}_{div}$-definable Γ-function is already definable in the language of ordered groups $\mathcal{L}_{og} := \{<, +, 0\}$ (see for instance [6, Theorem 2.1.1]). It follows, using quantifier elimination of divisible ordered abelian groups, that unary $\mathcal{L}_{div}$-definable Γ-functions are eventually $\mathbb{Q}$-linear. The same holds for C-minimal expansions with analytic structure as studied in [9], as the induced structure on Γ_K is again the pure structure of an ordered abelian group.

Remark 5.2. Let $g\colon \Gamma_K \to \Gamma_K$ be a definable function. It follows from the assumption that unary definable Γ-functions are eventually linear, that there are $\delta', \alpha' \in \Gamma_K$ and λ in $End(\Gamma_K, +)$ such that for all $\gamma \in \Gamma_K^{<\delta'}$

$$g(\gamma) = \lambda(\gamma) + \alpha'.$$

Given a function $f\colon X \subseteq K \to K$, we adopt the convention concerning limits in $K \cup \{\infty\}$:

- for $a \in K$, $\lim_{x\to a} f(x) = \infty$ holds if for every $\gamma \in \Gamma_K$, there is an open ball B containing a such that $v(f(B)) < \gamma$;
- for $b \in K$, $\lim_{x\to\infty} f(x) = b$ holds if for every $\gamma \in \Gamma_K$, there is $\delta \in \Gamma_K$ such that $v(f(x) - b) > \gamma$ for all x for which $v(x) < \delta$;
- $\lim_{x\to\infty} f(x) = \infty$ holds if for every $\gamma \in \Gamma_K$, there is $\delta \in \Gamma_K$ such that $v(f(x)) < \gamma$ for all x for which $v(x) < \delta$;

- for $a, b \in K$, $\lim_{x\to a} f(x) = b$ is the usual definition.

The following Lemma follows from the proof of [5, Proposition 4.5] with minor modifications. It ensures the existence of limits in definably complete C-minimal valued fields for which all definable unary Γ-functions are eventually linear.

Lemma 5.3. *Let $(K, \mathcal{L})$ be a definably complete C-minimal valued field in which definable unary Γ-functions are eventually linear. Let $f\colon X \subseteq K \to K$ be a definable function. Then, for any $a \in \overline{X}$ (the topological closure of X in $K \cup \{\infty\}$), the limit $\lim_{x\to a} f(x)$ exists in $K \cup \{\infty\}$.*

We have all the elements to state and prove our RV-factorization results. Recall that p denotes the characteristic exponent of K (that is, $p = \mathrm{char}(K)$ if $\mathrm{char}(K) > 0$ and $p = 1$ otherwise). We will directly show them in families.

Theorem 5.4 (RV-factorization I)**.** *Let $(K, \mathcal{L})$ be a C-minimal valued field and $f\colon X \subseteq W \times K \to K^\times$ be a definable family of functions defined at a neighborhood of infinity. Let $h\colon Y \subseteq W \times \Gamma_K \to \Gamma_K$ be a Γ factorization of f at infinity, namely, for every $w \in W$, f_w factorizes at infinity over Γ through h_w. If h is eventually $\mathbb{Q}$-linear, then there are a finite definable partition of W into sets $W_1 \cup \cdots \cup W_\ell$, integers $n_1, \ldots, n_\ell, m_1, \ldots, m_\ell \in \mathbb{Z}$ and a definable function $c\colon W \to RV$, such that for all $w \in W_i$, in a neighborhood of infinity,*

$$rv(f_w(x)) = rv(x)^{n_i/p^{m_i}} c(w).$$

If in addition $(K, \mathcal{L})$ is definably complete and all definable unary Γ-functions are eventually linear, for all $w \in W_i$, the limit

$$a(w) := \lim_{x\to\infty} \frac{f_w(x)}{x^{n_i/p^{m_i}}}$$

exists in K and $c(w) = rv(a(w))$.

Proof. By assumption, there are rational numbers $r_1, \ldots, r_\ell$ and definable functions $\alpha\colon W \to \Gamma_K$ and $\delta\colon W \to \Gamma_K \cup \{+\infty\}$ such that for each $w \in W$ there is $i \in \{1, \ldots, \ell\}$ satisfying that for all $\gamma < \delta(w)$

$$v(f_w(x)) = h_w(\gamma) = r_i\gamma + \alpha(w) \text{ for all } x \text{ such that } v(x) = \gamma.$$

This induces naturally a definable partition of $W = W_1 \cup \cdots \cup W_\ell$ depending on the slopes r_i. From now on we work over $W = W_i$ and let r be r_i. Without loss of generality, suppose that $r \geqslant 0$. Let $r = st^{-1}$, with s and t coprime positive integers or $s = 0$ and $t = 1$. Fix $w \in W$ and let $b \in K$ be such that $v(b) = \alpha(w)$. Thus, we have that

$$v(f_w(x)^t) = v(x^s b^t) \text{ for all } x \text{ such that } v(x) < \delta(w).$$

Consider the definable family of sets $A^{w,b} \subseteq \Gamma_K^{<\delta(w)} \times (K/v)^\times$ defined by having fibers of the form

$$A^{w,b}_\gamma := \{[f_w(x)^t x^{-s} b^{-t}]/v : \gamma < \delta(w), v(x) = \gamma\}.$$

Let $T^{w,b}\colon K/v \to \Gamma_K \cup \{-\infty\}$ be the definable partial function given by

$$c/v \mapsto \gamma_c := \inf\{\gamma \in \Gamma_K^{<\delta(w)} \cup \{-\infty\} : (c/v) \in A^{w,b}_\gamma\}.$$

By o-minimality, $T^{w,b}$ is well-defined and by Theorem 2.2 (part (3)), $T^{w,b}$ has finite image. Moreover, the image of $T^{w,b}$ is coinitial in Γ_K, so it must attain the value $-\infty$. Therefore, again by o-minimality, there are $c \in K$ and $\delta'(w) \leqslant \delta(w)$ such that $(c/v) \in A^{w,b}_\gamma$ for all

$\gamma < \delta'(w)$. By quantifying over b and c, we can suppose that $\delta'\colon W \to \Gamma_K \cup \{+\infty\}$ is a definable function: we define $\delta'(w)$ to be be the minimal element $\gamma_1 \in \Gamma_K \cup \{+\infty\}$ such that

$$\text{(E1)} \qquad \gamma_1 \geqslant \delta(w) \wedge (\exists b \in K)(\exists c \in \mathcal{O}_K)(v(b) = \alpha(w) \wedge (\forall \gamma < \gamma_1)((c/v) \in A_\gamma^{w,b})).$$

For any $b, c \in K^\times$ satisfying (E1), we have that

$$\text{(E2)} \qquad \mathrm{rv}(f_w(x))^t = \mathrm{rv}(x)^s \mathrm{rv}(b^t c),$$

for all $x \in K \setminus B(0, \delta'(w))$. Let us show that $t = p^m$ for some integer $m \in \mathbb{N}$ (so we assume that $s \neq 0$). Since s and t are positive, by C-minimality we have that $f_w(K \setminus B(0, \delta'(w))$ contains a neighborhood of infinity. Let $t = t' p^{v_p(t)}$ and $s = s' p^{v_p(s)}$, where v_p denotes the p-adic valuation when $p > 1$, and v_1 denotes the constant 0 function. Now, for x in a neighborhood of infinity consider the following sets:

- $A_x := \{\mathrm{rv}(y) : \mathrm{rv}(f_w(x)) = \mathrm{rv}(f_w(y)) \wedge v(x) = v(y)\}$,
- $B_x := \{\mathrm{rv}(y) : \mathrm{rv}(f_w(x))^t = \mathrm{rv}(f_w(y))^t \wedge v(x) = v(y)\}$ and
- $C_x := \{\mathrm{rv}(y) : \mathrm{rv}(x)^s = \mathrm{rv}(y)^s\}$.

By C-minimality, there is $d \in \mathbb{N}^* \cup \{+\infty\}$ such that, for all x in a neighborhood of infinity, $|A_x| = d$. Similarly, for all x in a neighborhood of infinity, we have that $|B_x| = dt'$ and $|C_x| = s'$. By (E2), we have that $B_x = C_x$, and therefore $d \in \mathbb{N}^*$ and $dt' = s'$. This shows that t' divides s'. Since s and t are coprime, we must have that $t' = 1$, which shows what we wanted.

Define $c(w) := \mathrm{rv}(bc^{1/p^m})$, for any $b, c \in K$ satisfying (E1). We have thus that $\mathrm{rv}(f_w(x)) = \mathrm{rv}(x)^{s/p^m} c(w)$.

To show the last statement, by Lemma 5.3 there is $a(w) \in K \cup \{\infty\}$ such that

$$a(w) := \lim_{x \to \infty} \frac{f_w(x)}{x^r}.$$

Since in a neighborhood of infinity all values of $f_w(x)x^{-r}$ lie in a single ball of radius 0, we must have that $a(w) \in K$. Clearly, $\mathrm{rv}(a(w)) = c(w)$. □

In what follows p will still denote the characteristic exponent of K and $q \geqslant 0$ will denote the characteristic of the residue field K/v.

Theorem 5.5 (RV-factorization II). *Let $(K, \mathcal{L})$ be a C-minimal valued field and $f\colon X \subseteq W \times K \to K$ be a definable family of local C-isomorphisms. Suppose $h\colon Y \subseteq X \times \Gamma_K \to \Gamma_K$ is a local Γ-factorization of f (namely, for all $w \in W$, f_w locally factorizes over Γ through h_w) which is eventually $\mathbb{Q}$-linear. Then, there is a finite definable partition of X into sets $X_1 \cup \cdots \cup X_\ell$, integers $n_1, \ldots, n_\ell$ and definable functions $\delta\colon X \to \Gamma_K$ and $c\colon X \to RV$ such that for all $(w, x) \in X_i$, $B^\circ(x, \delta(w, x)) \subseteq (X_i)_w$ and for all distinct $y, z \in B^\circ(x, \delta(w, x))$,*

$$rv((f_w(y) - f_w(z)) = rv((y - z))^{p^{n_i}} c(w, x).$$

In particular, when $p = 1$ we may assume that $\ell = 1$. If in addition $(K, \mathcal{L})$ is definably complete and all definable unary Γ-functions are eventually linear, the limit

$$a(w, x) := \lim_{y \to x} \frac{f_w(x) - f_w(y)}{(x - y)^{p^{n_i}}}$$

exists in K and $c(w, x) = rv(a(w, x))$.

Proof. Let $\theta\colon X \to \Gamma_K \cup \{-\infty\}$ be the definable function sending $(w,x) \in X$ to the minimal γ such that $f_w|B^\circ(x,\gamma)$ is a C-isomorphism and $f_w|B^\circ(x,\gamma)$ factorizes over Γ through $h_{w,x}$. Therefore, for every $(w,x) \in X$, all distinct $y,z \in B^\circ(x,\theta(w,x))$ and all $\gamma > \theta(w,x)$ we have that

$$h_{w,x}(\gamma) = v(f_w(z) - f_w(y)) \text{ if and only if } v(y-z) = \gamma.$$

Since h is eventually $\mathbb{Q}$-linear, there is a finite set $Z_h \subseteq \mathbb{Q}$ satisfying that for every $(w,x) \in X$, there is some $r \in Z_h$ such that for all $\gamma > \theta(w,x)$ (possibly replacing θ by a definable function taking higher values)

$$h_{w,x}(\gamma) = r\gamma + \alpha(w,x) \tag{E3}$$

where $\alpha\colon X \to \Gamma_K$ is a definable function. By Theorem 3.8 we have moreover that for all $y \in B^\circ(x,\theta(w,x))$ and all $\gamma > \theta(w,x)$

$$h_{w,y}(\gamma) = r\gamma + \alpha(w,x). \tag{E4}$$

Suppose that $Z_h = \{r_1,\dots,r_\ell\}$. We partition X into sets $X_1,\dots,X_\ell$ where X_i is given by

$$\{(w,x) \in X : (\forall\gamma > \theta(w,x))(h_{w,x}(\gamma) = r_i\gamma + \alpha(w,x))\}.$$

From now on we suppose that X is some X_i and drop the indices (so r will denote the slope r_i).

Fix $(w,x) \in X$. Since $f_w|B^\circ(x,\theta(w,x))$ is a C-isomorphism we must have that $r > 0$. Write $r = st^{-1}$ with s and t coprime positive integers and let $b \in K$ be such that $v(b) = \alpha(w,x)$. Unraveling definitions, we thus obtain that for all distinct $u,z \in B^\circ(x,\theta(w,x))$

$$v((f_w(u) - f_w(z))^t) = v((u-z)^s \cdot b^t).$$

Consider the definable family of sets $A^{w,x,b} \subseteq (\Gamma_K^{>\theta(w,x)}) \times (K/v)^\times$ defined by having fibers

$$A_\gamma^{w,x,b} := \{[(f_w(u) - f_w(z))^t(u-z)^{-s}b^{-t}]/v : u,z \in B^\circ(x,\theta(w,x)), v(u-z) = \gamma\},$$

and the definable partial function $T^{w,x,b}\colon K/v \to \Gamma_K \cup \{+\infty\}$

$$d \mapsto \gamma_d := \sup\{\gamma \in \Gamma_K^{>\theta(w,x)} \cup \{+\infty\} : d \in A_\gamma^{w,x,b}\}.$$

By o-minimality of Γ_K, $T^{w,x,b}$ is well-defined. By Theorem 2.2 (part (3)),the image of $T^{w,x,b}$ has finite image. Therefore, since its image is cofinal in $\Gamma_K^{>\theta(w,x)} \cup \{+\infty\}$, $T^{w,x,b}$ must attain the value $+\infty$. Again by o-minimality, this implies that there are $c \in K$ and $\gamma_1 \geqslant \theta(w,x)$ such that $(c/v) \in A_\gamma^{w,x,b}$ for all $\gamma > \gamma_1$. By quantifying over b and c, we can make γ_1 definable over (w,x): we let $\delta(w,x)$ be the minimal element $\gamma_1 \in \Gamma_K \cup \{-\infty\}$ such that

$$\gamma_1 \geqslant \theta(w,x) \wedge (\exists b \in K)(\exists c \in K)(v(b) = \alpha(w,x) \wedge (\forall\gamma > \gamma_1)((c/v) \in A_\gamma^{w,x,b})). \tag{E5}$$

Fix $b,c \in K$ satisfying (E5). We have that for all distinct $u,z \in B^\circ(x,\delta(w,x))$

$$(f_w(u) - f_w(z))^t \sim (u-z)^s b^t c. \tag{E6}$$

Recall that $q \geq 0$ denotes the characteristic of the residue field.

Claim 5.6. *Either $s = t = 1$, or for some $k > 0$ either $s = q^k$ and $t = 1$, or $s = 1$ and $t = q^k$.*

Pick $d \in K$ such that $v(d) > \delta(w,x)$, $e \in K$ a t-root of $d^{-s}b^{-t}c$ and consider the function defined on $\mathcal{O}_K$ given by

$$g(z) := e(f_w(dz + x) - f_w(x)).$$

Note that $g(0) = 0$, g is a C-isomorphism and moreover, for all $u, z \in \mathcal{O}_K$ we have that $(g(u) - g(z))^t \sim (u - z)^s$. Indeed, for $u, z \in \mathcal{O}_K$ we have that:

$$\begin{aligned}(g(u) - g(z))^t &= e^t(f_w(du + x) - f_w(dz + x))^t \\ &\sim e^t[(du + x) - (dz + x)]^s b^t c^{-1} && \text{by (E6)} \\ &= e^t d^s b^t c^{-1}(u - z)^s = (u - z)^s.\end{aligned}$$

This shows in particular that $v(g(u)) = st^{-1}v(u)$ and that $g(u + \mathcal{M}) \subseteq g(u) + \mathcal{M}$ for any $u \in \mathcal{O}_K$. Hence g induces a residue map $\bar{g} \colon K/v \to K/v$, which satisfies for all $u, z \in \mathcal{O}_K$

$$(\bar{g}(u/v) - \bar{g}(z/v))^t = (u/v - z/v)^s.$$

We now work in K/v (we will use $u, z \in K/v$ instead of u/v, z/v for $u, z \in \mathcal{O}_K$).

Since $(\bar{g}(u) - \bar{g}(z))^t = (u - z)^s$, we have both that $\bar{g}(u)^t = u^s$ and that $\bar{g}$ is injective. This implies, by Theorem 2.2, that there is a finite subset $F \subseteq K/v$ such that $\bar{g}(K/v) = (K/v) \setminus F$. If $q \neq 0$, we let v_q denote the q-adic valuation on $\mathbb{Z}$. We split the argument in three cases:

Case 1 ($q \nmid s, q \nmid t$): The equality $\bar{g}(u)^t = u^s$ implies that for all $z \in K/v$, u is an s-root of z if and only if $\bar{g}(u)$ is a t-root of z. It is enough then to find some $z \neq 0$ such that all t-roots of z are in $\bar{g}(K/v)$. For if this is true, $\bar{g}$ will be a bijection between all s-roots of z and all t-roots of z which implies $s = t$. From the assumption $gcd(s, t) = 1$, we can conclude that $s = t = 1$. To show such z exists, consider the set $E = \{z \in K/v : \exists a \in F, a^t = z\}$. Given that F is finite, so is E. Any $z \in (K/v) \setminus E$ satisfies the required property.

Case 2 ($q \neq 0, q \mid s, q \nmid t$): Let $k := v_q(s)$ and $m \in \mathbb{N}$ such that $s = q^k m$. This implies that $gcd(m, t) = 1$ and our equality becomes $\bar{g}(u)^t = (u^{q^k})^m$. Thus for all $z \in K/v$, u is an m-root of z^{q^k} if and only if $\bar{g}(u)$ is a t-root of z. As in the previous case it is enough to find $z \neq 0$ such that all t-roots of z are in $\bar{g}(K/v)$ to get a bijection from m-roots to t-roots, which implies $m = t = 1$. The existence argument is exactly the same as in the previous case.

Case 3 ($q \neq 0, q \nmid s, q \mid t$): As in Case 2, let $k := v_q(t)$ and set $m \in \mathbb{N}$ such that $t = q^k m$. Thus, $gcd(m, s) = 1$ and our equality becomes $(\bar{g}(u)^{q^k})^m = u^s$. Therefore for all $z \in K/v$, u is an s-root of z if and only if $\bar{g}(u)^{q^k}$ is an m-root of z. Let $\sigma(u) = u^{q^k}$. Here we need $z \neq 0$ such that all s-roots of z are in $(\sigma \circ \bar{g})(K/v)$. Given that σ is a bijection, a similar argument as in Case 1 shows the existence of such $z \in k$. Thus, $m = s = 1$. This completes the claim.

In all cases we will define the function $c \colon X \to \mathrm{RV}$ by $c(w, x) := rv(bc^{1/t})$ for any b, c satisfying (E5). Note that Claim 5.6 already implies the theorem for equicharacteristic valued fields. Moreover, if $p = 1$ (recall p is the characteristic exponent), then the only possible case is to have $s = t = 1$, and therefore $Z_h = \{1\}$ (so assertion (2) of the theorem holds). For valued fields of mixed characteristic, it remains to show that the only possible case in the disjunction of Claim 5.6 is $s = t = 1$. Since in Case 1 we already obtained that $s = t = 1$, we may assume we are in either Case 2 or Case 3. We work again in K and we assume that $1 = p$ and $0 < q$:

(i) Suppose we are in Case 2 above. Then there is some integer $k > 0$ such that $s = q^k$, $t = 1$ and therefore $g(u) \sim u^{q^k}$ for all $u \in \mathcal{O}_K$. Let $a \in \mathcal{O}_K \setminus \{0\}$ and $z \in K \setminus \{1\}$ such that

$z^{q^k} = 1$. This implies that $v(z) = 0$ and thus that $az \in \mathcal{O}_K$ and $a \neq az$. Let $a_0 \in \mathcal{O}_K \setminus \{0\}$ be such that

(1) $a_0^{q^k} \neq a^{q^k}$
(2) $v(a - a_0) = v(az - a_0) = v(az - a)$.

Such an element a_0 always exists. Indeed, since K/v is infinite, there are infinitely many elements a_0 satisfying (2), but at most q^k roots of a^{q^k}. Now, on the one hand, the second condition on a_0 implies $\neg C(a_0, a, az)$. Since g is a C-isomorphism, this implies $\neg C(g(a_0), g(a), g(az))$. On the other hand $C(a_0^{q^k}, a^{q^k}, a^{q^k})$ always holds, and since $a^{q^k} = (az)^{q^k}$, we also have that $C(a_0^{q^k}, a^{q^k}, (az)^{q^k})$. But $g(u) \sim u^{q^k}$ for all $u \in \mathcal{O}_K$, which contradicts Lemma 2.6.

(ii) Suppose we are in Case 3 above. Then, for some integer $k > 0$, $s = 1$, $t = q^k$ and therefore $g(u)^{q^k} \sim u$ for all $u \in \mathcal{O}_K$. Since g is a C-isomorphism and given that $g(0) = 0$, $g(\mathcal{O}_K)$ is a closed ball centered at 0. Given $u \in \mathcal{O}_K$ and $z = g(u)$, we have that

$$g(u)^{q^k} \sim u \Leftrightarrow z^{q^k} \sim g^{-1}(z)$$

and get again the contradiction from case (i) for the C-isomorphism g^{-1}. This completes the proof of the first part.

For the last assertion, suppose that $(K, \mathcal{L})$ is definably complete and all definable unary Γ-functions are eventually linear. Define $a(w, x)$ as

$$a(w, x) := \lim_{y \to x} \frac{f_w(x) - f_w(y)}{(x - y)^r},$$

which exists by Lemma 5.3 (note that it cannot be the value ∞). Clearly, we have that $c(w, x) = \mathrm{rv}(a(w, x))$. □

6. The Jacobian Property

Let $(K, \mathcal{L})$ be a definably complete C-minimal expansion of an algebraically closed valued field of characteristic 0 in which all definable unary Γ-functions are eventually $\mathbb{Q}$-linear. Note that by Lemma 5.3, we have in this setting a well-defined notion of derivative for definable functions since limits exist. Let us recall the definition of the Jacobian property from [1].

Definition 6.1. (Jacobian Property) Let (K, v) be a valued field of characteristic 0. Let $B \subseteq K$ be an open ball and $f \colon B \to K$ be a function. We say that f has the *Jacobian Property* if

(i) f is injective and $f(B)$ is an open ball;
(ii) f is differentiable and f' is nonvanishing;
(iii) $rv(f'(x))$ is constant on B, say equal to c;
(iv) for all $x, y \in B$, $rv(f(x) - f(y)) = rv(x - y)c$.

Theorem 6.2. *Let $(K, \mathcal{L})$ be a definably complete C-minimal valued field of characteristic 0 in which definable unary Γ-functions are eventually $\mathbb{Q}$-linear. Let $f \colon X \subseteq W \times K \to K$ be a definable family of local C-isomorphisms. Then there is a set $F \subseteq W$ such that for all $w \in W$, F_w is finite and $f_w|(X_w \setminus F_w)$ has locally the Jacobian property.*

Proof. By Theorem 5.5, there is a set $F \subseteq X$ such that, for each $w \in W$

(1) F_w is finite and

(2) there are definable functions $\delta\colon W \to \Gamma_K$ and $c\colon X \to K$ such that for all $w \in W$, all $x \in X_w \setminus F_w$ and all distinct $y, z \in B^\circ(x, \delta(w, x))$,

$$rv(f_w(y) - f_w(z)) = rv(y - z)c(w, x).$$

We left to the reader to check that $f_w|B^\circ(x, \delta(w, x))$ satisfies the Jacobian property: since $f_w|X_w$ is a C-isomorphism, condition (i) of Definition 6.1 is already satisfied; conditions (ii)-(iv) easily follow from point (2) above. □

Corollary 6.3. *Let $(K, \mathcal{L})$ be a definably complete C-minimal valued field of characteristic 0 in which definable unary Γ-functions are eventually $\mathbb{Q}$-linear. Let $f\colon X \subseteq W \times K \to K$ be a definable family of functions. Then X decomposes into definable sets $X = F \cup E \cup J$ such that for each $w \in W$:*

(1) F_w is a finite set;
(2) $f_w|_{E_w}$ is a locally constant function;
(3) $f_w|_{J_w}$ has locally the Jacobian property.

Proof. Let $f\colon X \subseteq W \times K \to K$ be a definable family of functions. By Corollary 2.5, there is a definable partition of X into sets $F \cup C \cup I$ where for all $w \in W$, F_w is finite, $f_w|C_w$ is locally constant and $f_w|I_w$ is a local C-isomorphism. The result follows by applying Theorem 6.2 to the family $f|I$. □

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COMPONENT-CLOSED EXPANSIONS OF THE REAL LINE (PRELIMINARY REPORT)

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Abstract. We investigate structures on the set of real numbers having the property that connected components of definable sets are definable.

This is a very preliminary report on some recent work. The original motivation lies in real-analytic geometry, but we quickly realized that some fundamental issues in the model theory of expansions of the real line are involved. We thank Pantelis Eleftheriou, Alex Savatovsky and Charles Steinhorn for useful discussions.

Throughout, we abbreviate "connected component" by "component". We always regard $\mathbb{R}^n$ as equipped with its usual topology. We usually identify interdefinable structures; when syntactic notions are important, we shall make it clear. The reader is assumed to be familiar with very basic definability theory and o-minimality (see, e.g., van den Dries [2]).

Definition. A structure with underlying set $\mathbb{R}$ is **component closed** if, for every definable set E (of any arity), every component of E is definable.

The basic question. What can be said about component-closed structures on $\mathbb{R}$?

1. Before attacking this seriously, we should attempt to ascertain which structures might be involved. Here are some examples and counterexamples:

(1) Trivially, the expansion of $\mathbb{R}$ by (predicates for) each subset of each $\mathbb{R}^k$ ($k \geq 1$) is component closed.
(2) $(\mathbb{R}, <)$ is component closed (by quantifier elimination).
(3) $(\mathbb{R}, +, \cdot, \mathbb{Z})$ is component closed. (It defines every closed set in any $\mathbb{R}^n$—see, e.g., [2, (2.6)]—and in any topological space, every component of a set is closed in the set.)
(4) Every o-minimal expansion of $(\mathbb{R}, <)$ is component closed (see exercises 5 and 7 of [2, (2.19)]).
(5) $(\mathbb{R})$, as a structure in the empty language, is not component closed. (The set $\{ (x, y) \in \mathbb{R}^2 : x \neq y \}$ is definable, but by quantifier elimination, $<$ is not definable.)
(6) We will prove below (Proposition 7 and the paragraph subsequent to the proof) that if $E \subseteq \mathbb{R}$ is closed, discrete and infinite, then $(\mathbb{R}, <, E)$ is not component closed.
(7) We will prove below (Theorem 12) that if $E \subseteq \mathbb{R}$ is closed and discrete, then the expansion of $(\mathbb{R}, <)$ by each subset of each E^k ($k \geq 1$) is component closed. (Of course, this reduces to (2) if E is finite.)

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As a corollary of the proof of (5),

2. **Proposition.** *Every component-closed structure on* $\mathbb{R}$ *defines* $<$.

The existence of structures that are not component closed motivates the following

Definition. Given a structure $\mathfrak{R}$ on the set $\mathbb{R}$, the **component closure**, $\mathfrak{R}^{\mathrm{cc}}$, of $\mathfrak{R}$ is the intersection of all component-closed expansions (in the sense of definability) of $\mathfrak{R}$.

Evidently, $\mathfrak{R}^{\mathrm{cc}}$ is component closed, and can be defined instead by induction:

— $\mathfrak{R}_0 = \mathfrak{R}$;
— for $m \in \mathbb{N}$, $\mathfrak{R}_{m+1}$ is the expansion of $\mathfrak{R}_m$ by all components of sets definable in $\mathfrak{R}_m$;
— $\mathfrak{R}^{\mathrm{cc}} = \bigcup_{m\in\mathbb{N}} \mathfrak{R}_m$.

By 1.4 and Proposition 2,

3. **Proposition.** *If* $\mathfrak{R}$ *is a structure on* $\mathbb{R}$ *such that* $(\mathfrak{R}, <)$ *is o-minimal, then* $\mathfrak{R}^{\mathrm{cc}} = (\mathfrak{R}, <)$.

Hence, we make a

Reduction. From now on, we are concerned only with expansions of the real line $(\mathbb{R}, <)$ that are not o-minimal. As every component of any subset of $\mathbb{R}$ is definable in $(\mathbb{R}, <)$, we care only about components of definable sets of arity at least 2.

Currently, we do not have any documented examples of component-closed expansions of $(\mathbb{R}, +, \cdot)$ that neither are o-minimal nor define $\mathbb{Z}$ (though we do have some candidates). Moreover, working over the field structure is rather daunting at this stage, so we first attempt to understand some toy models. Some preliminary considerations are in order.

Let X be a topological space. We take the position that any general investigation of components of subsets of X should begin by considering the open sets, which leads immediately to considering also boundaries of open sets, then closed sets, and more generally, the **constructible** (boolean combinations of open) sets. Note that components of constructible sets are constructible.

4. **Fact** (Dougherty and Miller [1]). *If* $A \subseteq \mathbb{R}^n$ *is constructible, then* A *is a boolean combination of open sets that are each* $\emptyset$*-definable in* $(\mathbb{R}, <, A)$.

Hence, the study of expansions of $(\mathbb{R}, <)$ by constructible sets is exactly the same as the study of expansions of $(\mathbb{R}, <)$ by open—or closed—sets.

5. **Fact.** *If* $A \subseteq \mathbb{R}$ *is unbounded and discrete, then* $(\mathbb{R}, <, +, A)$ *defines a closed infinite discrete subset of* $\mathbb{R}$.

Proof. By replacing A with its set of negatives if needed, we reduce to the case that A is unbounded above. Put $S = \{ (t, a) \in \mathbb{R}^{>0} \times A : \mathrm{d}(a, A \setminus \{a\}) \geq t \}$ where d indicates distance with respect to the max norm. If some fiber S_t $(= \{ a \in A : (t, a) \in S \})$ of S is infinite, then we are done, so assume otherwise. As A is discrete, for each $a \in A$ there exists $t(a) > 0$ such that $a \in S_t$ for all $0 < t \leq t(a)$. As A is nonempty, there exists $t_0 > 0$ such that: (i) if $0 < t \leq t_0$, then $\max S_t$ exists; (ii) for all $0 < t < t' \leq t_0$, if $\max S_t \neq \max S_{t'}$, then $\max S_t \geq t + \max S_{t'}$; and (iii) $\lim_{t\downarrow 0} \max S_t = +\infty$. Hence, $\{ \max S_t : 0 < t \leq t_0 \}$ is closed, infinite, discrete and definable in $(\mathbb{R}, <, +, A)$. ☐

Remark: some history. The above (essentially) was communicated in personal conversation to author Miller by Michael Tychonievich, while the latter was a PhD student of the former, in response to an issue raised in Miller and Speissegger [8]). A more abstract (and more difficult) version was announced independently and shortly thereafter by Fornasiero [4, Remark 4.16], soon followed by another variant due to Hieronymi [6] (the earliest published account).

6. **Fact.** *Let $\mathfrak{R}$ be a non-o-minimal expansion of $(\mathbb{R},<)$ by constructible sets.*

(1) *$\mathfrak{R}$ defines a countably infinite subset of $\mathbb{R}$.*
(2) *$(\mathfrak{R},+)$ defines an infinite discrete subset of $\mathbb{R}$.*
(3) *If f is a bijection between a bounded interval and an unbounded interval, then $(\mathfrak{R},+,f)$ defines a closed infinite discrete subset of $\mathbb{R}$.*
(4) *$(\mathfrak{R},+,\cdot)$ defines a closed infinite discrete subset of $\mathbb{R}$.*

Proof. By Fact 4, it suffices to consider the case that $\mathfrak{R}$ is an expansion of $(\mathbb{R},<)$ by open sets; then (1) and (2) are by [8]. Item (3) is essentially immediate from (2) and Fact 5, and (4) is immediate from (3). □

Thus, in order to understand component-closed expansions of the real field by constructible sets, we must understand at the very least structures of the form $(\mathbb{R},+,\cdot,E)$ where $E\subseteq\mathbb{R}$ is infinite, closed and discrete; as part of this, we would need to understand the structure induced on E in $(\mathbb{R},+,\cdot,E)$. This suggests that we should first try to understand $(\mathbb{R},<,E)^{\mathrm{cc}}$. Up to isomorphism, there are only three cases: $E=\mathbb{N}$; $E=-\mathbb{N}$; and $E=\mathbb{Z}$. Evidently, $(\mathbb{R},<,\mathbb{N})$ and $(\mathbb{R},<,-\mathbb{N})$ are interchangeable via $x\mapsto -x$ and $(\mathbb{R},<,\mathbb{Z})$ is interdefinable with $(\mathbb{R},<,\mathbb{N},-\mathbb{N})$, so we first attempt to understand $(\mathbb{R},<,\mathbb{N})^{\mathrm{cc}}$.

For $r\in\mathbb{R}$, put $r\mathbb{N}=\{\,rk : k\in\mathbb{N}\,\}$.

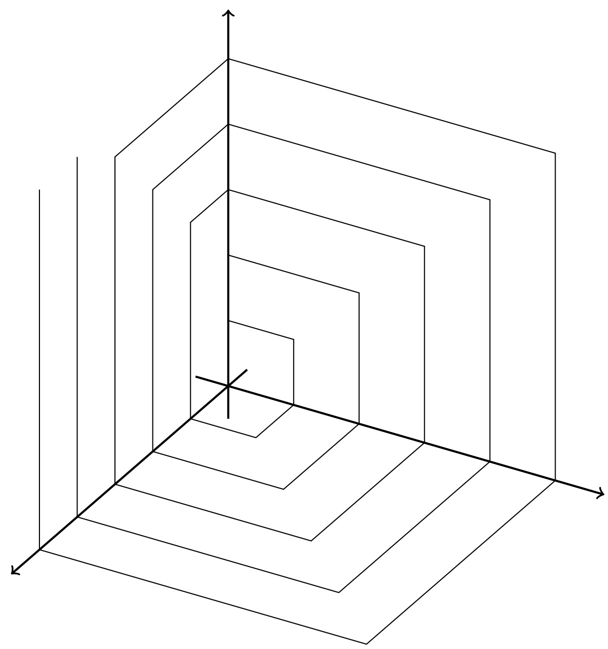
7. **Proposition.** *If $d\in\mathbb{N}$, then $(\mathbb{R},<,\mathbb{N})^{\mathrm{cc}}$ defines $d\mathbb{N}$.*

Proof. Let S be the union of the following subsets of $\mathbb{R}^3$:

$$\{\,(x,y,0) : x\in\mathbb{N}\ \&\ 0\le y\le x\,\}$$
$$\{\,(x,y,0) : y\in\mathbb{N}\ \&\ 0\le x\le y\,\}$$
$$\{\,(0,y,z) : y\in\mathbb{N}\ \&\ 0\le z\le y\,\}$$
$$\{\,(0,y,z) : z\in\mathbb{N}\ \&\ 0\le y\le z\,\}$$
$$\{\,(x,0,z) : x\in\mathbb{N}\ \&\ 0\le z\le x+d\,\}$$
$$\{\,(x,0,z+d) : z\in\mathbb{N}\ \&\ 0\le x\le z\,\}.$$

(See the figure below for the idea behind the case $d=2$.) As $t\mapsto t+d\colon\mathbb{N}\to\mathbb{N}$ is definable in $(\mathbb{R},<,\mathbb{N})$, so is S. Let C be the component of S that contains the point $(0,0,d)$; then the intersection of C with the positive z-axis is the set $\{\,(0,0,dn) : 0<n\in\mathbb{N}\,\}$. Hence, $(\mathbb{R},<,\mathbb{N})^{\mathrm{cc}}$ defines $d\mathbb{N}$. □

It is an exercise, via quantifier elimination in an appropriate extension by definitions, to see that no $d\mathbb{N}$ with $1<d\in\mathbb{N}$ is definable in $(\mathbb{R},<,\mathbb{N})$. Hence, by Proposition 7, $(\mathbb{R},<,\mathbb{N})$ is not component closed. Similarly, neither $(\mathbb{R},<,-\mathbb{N})$ nor $(\mathbb{R},<,\mathbb{Z})$ is component closed. (See [3] for some related material). We do not yet know whether $(\mathbb{R},<,(d\mathbb{N})_{d\in\mathbb{N}})$ is the component closure of $(\mathbb{R},<,\mathbb{N})$. (This should not be construed as an open question; we just have not yet found the time to deal with it seriously.) But we can show that every bounded set definable in $(\mathbb{R},<,\mathbb{N})^{\mathrm{cc}}$



is definable in $(\mathbb{R},<)$, and so $(\mathbb{R},<,\mathbb{N})^{\mathrm{cc}}$ does not define even the restriction of $x \mapsto x+1\colon \mathbb{R} \to \mathbb{R}$ to any nonempty open interval; in particular, $(\mathbb{R},<,\mathbb{N})^{\mathrm{cc}}$ is considerably less complicated than the obvious upper bound of $(\mathbb{R},+,\cdot,\mathbb{Z})$. We shall obtain this as a special case of a more technical result that we now begin to describe.

8. **Lemma.** *Let $n \in \mathbb{N}$. There is a countable decomposition, $\mathcal{D}_n$, of $\mathbb{R}^n$ into $\mathbb{N}$-definable cells of $(\mathbb{R},<)$ such that $\mathcal{D}_n$ is compatible with every subset of $\mathbb{R}^n$ that is $\mathbb{N}$-definable in $(\mathbb{R},<)$.*

(Note the order of the quantifiers. Countable cell decompositions are defined just like ordinary cell decompositions except that they are allowed to be countably infinite; see [7, §4] for more detail.)

Proof. The proof is an exercise in induction, essentially, but it is worth considering the case $n=2$ in some detail. We shall simply list the elements of $\mathcal{D}_2$. (Recommended: Draw the picture.) For each $p,m \in \mathbb{N}$ with $p \neq m$, the open box $(p,p+1)\times(m,m+1)$ is in $\mathcal{D}_2$, as are its edges

$$(p,p+1)\times\{m\},\ (p,p+1)\times\{m+1\},\ \{p\}\times(m,m+1),\ \{p+1\}\times(m,m+1)$$

and vertices. For each $m \in \mathbb{N}$, we need the sets

$$\{\,(x,y) : m<x<y<m+1\,\},\ \{\,(x,y) : m<y<x<m+1\,\},$$
$$\{\,(t,t) : t \in (m,m+1)\,\},$$
$$(-\infty,0)\times\{m\},\quad (-\infty,0)\times(m,m+1),$$
$$\{m\}\times(-\infty,0),\quad (m,m+1)\times(-\infty,0).$$

Finally, we need also

$$\{\,(t,t) : t<0\,\},\quad \{\,(x,y) : x<y<0)\,\},\quad \{\,(x,y) : y<x<0)\,\}. \qquad \square$$

Let us continue to focus on $n = 2$ a bit longer. Up to permutation of coordinates and translation by points in $\mathbb{N}^2$, there are only finitely many elements of $\mathcal{D}_2$, say:

$$\{(0,0)\}, \quad (0,1)\times\{0\}, \quad \{\,(t,t) : 0 < t < 1\,\},$$
$$(1,2)\times(0,1), \quad \{\,(x,y) : 0 < y < x < 1\,\},$$
$$(-\infty,0)\times\{0\}, \quad (-\infty,0)\times(0,1),$$
$$\{\,(t,t) : t < 0\,\}, \quad \{\,(x,y) : x < y < 0\,\}.$$

Hence, it is an exercise to see that there exists $k \in \mathbb{N}$ (depending only on $n = 2$) such that if $E \subseteq \mathbb{R}^2$ is a union of elements of $\mathcal{D}_2$, then there are $V_1, \dots, V_k \subseteq \mathbb{N}^2$ such that E is $\emptyset$-definable in $(\mathbb{R}, <, V_1 \times \cdots \times V_k)$. In other words, E can be $\emptyset$-definably encoded over $(\mathbb{R}, <)$ by some subset of $\mathbb{N}^N$, where N depends only on $n = 2$. Evidently, if E is also bounded, then it is a finite union of elements of $\mathcal{D}_2$, and thus is $\mathbb{N}$-definable in $(\mathbb{R}, <)$. More generally:

9. **Lemma.** *Let $n \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that if $E \subseteq \mathbb{R}^n$ is a union of elements of $\mathcal{D}_n$, then there exists $V \subseteq \mathbb{N}^N$ such that E is $\emptyset$-definable in $(\mathbb{R}, <, V)$. If moreover E is bounded, then E is $\mathbb{N}$-definable in $(\mathbb{R}, <)$.*

We have a sort of converse:

10. **Lemma.** *Let $\mathfrak{R}$ be the expansion of $(\mathbb{R}, <)$ by each subset of each $\mathbb{N}^k$ $(k \geq 1)$. If $E \subseteq \mathbb{R}^n$ is $\emptyset$-definable in $\mathfrak{R}$, then E is a union of elements of $\mathcal{D}_n$.*

The proof is a routine induction on complexity of formulas.

Remark. Of course, $\mathrm{Th}(\mathfrak{R})$ is horrible—it interprets every countable theory—but this is not our concern at the moment.

11. **Proposition.** *Let $\mathfrak{R}$ be the expansion of $(\mathbb{R}, <)$ by each subset of each $\mathbb{N}^k$ $(k \geq 1)$. Then $\mathfrak{R}$ is component closed and its bounded definable sets are definable in $(\mathbb{R}, <)$.*

Proof. Let $E \subseteq \mathbb{R}^n$ be definable in $\mathfrak{R}$ and C be a component of E.

First, suppose that E is $\emptyset$-definable. By Lemma 10, E is a union of elements of $\mathcal{D}_n$. As cells are connected and C is maximally connected, C is a union of elements of $\mathcal{D}_n$. By Lemma 9, C is $\emptyset$-definable in $\mathfrak{R}$, and E is $\mathbb{N}$-definable in $(\mathbb{R}, <)$ if it is bounded.

Now suppose that E is A-definable for some finite $A \subseteq \mathbb{R}$. Lemmas 8, 9 and 10 hold with $\mathbb{N}$ replaced uniformly by $A \cup \mathbb{N}$. Argue as in the preceding paragraph to finish. □

Mutatis mutandis, the result holds with $\mathbb{N}$ replaced by either $-\mathbb{N}$ or $\mathbb{Z}$ in the specification of $\mathfrak{R}$. Recall that neither $(\mathbb{R}, <, -\mathbb{N})$ nor $(\mathbb{R}, <, \mathbb{Z})$ is component closed. Hence, as promised earlier:

12. **Theorem.** *If $E \subseteq \mathbb{R}$ is closed and discrete, then:*

— *$(\mathbb{R}, <, E)$ is component closed if and only if E is finite.*
— *The expansion of $(\mathbb{R}, <)$ by each subset of each E^k $(k \geq 1)$ is component closed, and its bounded definable sets are definable in $(\mathbb{R}, <)$.*

Indeed, with minor modifications, the following are component closed:

$$(\mathfrak{R}, -x), \quad (\mathfrak{R}, x+1), \quad (\mathfrak{R}, -x, x+1)$$

as well as each of these with $\mathbb{N}$ replaced by either $-\mathbb{N}$ or $\mathbb{Z}$ in the specification of $\mathfrak{R}$. Moreover, in each case, the bounded definable sets are definable in the underlying o-minimal structure (either $(\mathbb{R},<)$, $(\mathbb{R},<,-x)$, $(\mathbb{R},<,x+1)$ or $(\mathbb{R},<,-x,x+1)$, as appropriate).

This is as far as we are at this time with results that can be considered positive (in the sense of identifying component closures, that are neither o-minimal nor equal to $(\mathbb{R},+,\cdot,\mathbb{Z})$, of expansions of $(\mathbb{R},<)$ by constructible sets). We currently suspect that the following are component closed:

— the expansion of $(\mathbb{R},<)$ by the graphs of addition and multiplication restricted to $\mathbb{N}^2$ (or $\mathbb{Z}^2$);
— the expansion of $(\mathbb{R},<,+)$ by each subset of each $\mathbb{N}^k$ $(k \geq 1)$;
— the expansion of $(\mathbb{R},<,+)$ by the graphs of addition and multiplication restricted to $\mathbb{N}^2$;
— the expansion of $(\mathbb{R},+,\cdot\,)$ by each subset of each E^k $(k \geq 1)$, where E is a fast sequence for $(\mathbb{R},+,\cdot\,)$ as defined in Friedman and Miller [5].

We close with an intriguing issue: What is the model-theoretic content of our basic question? Suppose, for example, that we try to work instead with structures on the set of real algebraic numbers. Then components (with respect to the order topology) of definable sets are simply points, so all such structures are trivially component closed. Yes, we could redefine "component" to mean "trace of a component of the closure in the appropriate $\mathbb{R}^n$", but then this still is tied to the Dedekind completeness of the real line, and thus is not model theory per se. The notion of definable connectedness (see, e.g., [2]) comes naturally to mind, but this also degenerates, because a "definably-connected component" would wind up being definable by definition.

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EXTENDED ABSTRACT

ON NON-COMMUTATIVE QUANTIFIER ELIMINATION IN REAL ALGEBRA

TIM NETZER

1. Commutative Quantifier Elimination

We will start by briefly describing quantifier elimination in standard (= commutative) real algebra. We first need the notion of a semialgebraic set:

Definition 1.1. A *semialgebraic set* in $\mathbb{R}^n$ is a Boolean combination of sets of the form

$$\{a \in \mathbb{R}^n \mid p(a) \geq 0\}$$

where $p \in \mathbb{R}[x_1, \ldots, x_n]$ is a polynomial.

The first step towards quantifier elimination, which is already the hardest, is the Projection Theorem:

Theorem 1.2 (Projection Theorem)**.** *Projections of semialgebraic sets are again semialgebraic.*

Proofs and remarks on the Projection Theorem can be found in [1, 4, 5]. Note that it constitutes a non-trivial result, far deeper than for example the Fourier-Motzkin-Elimination for polytopes. It immediately generalizes to arbitrary polynomial images of semialgebraic sets, and can be applied to show that for examples closures, interiors, convex hulls etc. of semialgebraic sets are again semialgebraic.

The Projection Theorem admits constructive proofs; they are uniform in the input data (the polynomials that describe the semialgebraic set), and hold over arbitrary real closed fields. Since a projection corresponds to an existential quantifier in a formula, the following Quantifier Elimination is almost immediate:

Theorem 1.3 (Quantifier Elimination)**.** *For every first-order formula in the language of ordered rings, there is a quantifier-free such formula, which is equivalent over each real closed field.*

This now directly implies the strong Transfer Principle of Tarski & Seidenberg. It yields decidability of the first-order theory of real closed fields, and lies at the core of almost every Positivstellensatz in real algebra.

Theorem 1.4 (Transfer Principle)**.** *Any two real closed fields are elementary equivalent, i.e. they fulfill the same first-order formulas in the language of ordered rings.*

2. Non-commutative real algebra

In applications from control theory, electrical engineering and quantum physics, one considers Hermitian matrices instead of real numbers. Often the size of the matrices is unbounded a priori. Matrix multiplication is non-commutative. To adapt the notion of semialgebraic sets to this context, one thus has to pass to a non-commutative algebraic setup first. So let $\mathbb{R}\langle x_1,\ldots,x_n\rangle$ denote the ring of real polynomials in non-commuting variables. A polynomial is thus a real linear combination of words in the variables $x_1,\ldots,x_n$, where the order of variables *does matter!* For example, the two expressions x_1x_2 and x_2x_1 constitute different polynomials.

Into a non-commutative polynomial $p\in\mathbb{R}\langle x_1,\ldots,x_n\rangle$ one can plug in a (real or complex) matrix-tuple $(M_1,\ldots,M_n)$; the result $p(M_1,\ldots,M_n)$ is a matrix of the same size as the M_i. Since positivity is usually only defined for Hermitian matrices, one has to keep track of this property as well. So we define an involution $*$ on $\mathbb{R}\langle x_1,\ldots,x_n\rangle$, which leaves coefficients and variables invariant, und simply reverses the order of words, e.g.

$$(7x_2x_1+x_1^2)^*=7x_1x_2+x_1^2.$$

Let $\mathbb{R}\langle x_1,\ldots,x_n\rangle_h$ denote the set of Hermitian elements, i.e. fixed-points of the involution. Whenever $p\in\mathbb{R}\langle x_1,\ldots,x_n\rangle_h$ and $(M_1,\ldots,M_n)$ are all Hermitian, so is $p(M_1,\ldots,M_n)$. Thus it makes sense to define

$$\mathcal{W}_s(p):=\{(M_1,\ldots,M_n)\in \mathrm{Her}_s(\mathbb{C})^n \mid p(M_1,\ldots,M_n)\succcurlyeq 0\}$$

where $\succcurlyeq 0$ means that the matrix is positive semidefinite. The whole collection

$$\mathcal{W}(p)=\bigcup_{s\geq 1}\mathcal{W}_s(p)$$

might then be called a *basic closed non-commutative semialgebraic set*. It is a direct generalization of the same notion from the commutative context, that takes into account matrices of all sizes simultaneously. This is precisely what matters in the above mentioned applications.

So far, a more general notion of a semialgebraic set has not yet been developed. To obtain results as the Projection Theorem, one will surely have to go beyond such basic closed sets, by for example allowing for Boolean combinations. The most suitable notion will hopefully develop in context with growing insight in the near future.

3. Non-commutative quantifier elimination

Let us talk about a possible Projection Theorem in the non-commutative setup first. A projection can be defined for matrix tuples of fixed size in the obvious way:

$$\begin{aligned} p_s\colon \mathrm{Her}_s(\mathbb{C})^n &\to \mathrm{Her}_s(\mathbb{C})^{n-1}\\ (M_1,\ldots,M_n)&\mapsto (M_2,\ldots,M_n).\end{aligned}$$

It extends to

$$p\colon \mathrm{Her}(\mathbb{C})^n:=\bigcup_{s\geq 1}\mathrm{Her}_s(\mathbb{C})^n\to \mathrm{Her}(\mathbb{C})^{n-1}$$

by applying p_s on matrices of size s. Now one can clearly ask whether an analog of the Projection Theorem holds, or how non-commutative semialgebraic sets have to be defined to make it hold. Although we have no clear answer to this question, it seems like a positive answer should not be expected. Some first results and examples in that spirit appear in [3, 6]. We will in the following explain the most important results from [2].

Example 3.1. [2] With basic closed non-commutative semialgebraic sets, Boolean combinations and projections, one can define the set $W = \bigcup_{s\geq 1} W_s \subseteq \mathrm{Her}(\mathbb{C})$ with

$$W_s = \begin{cases} \mathrm{Her}_s(\mathbb{C}) & : s \text{ prime} \\ \emptyset & : \text{ else} \end{cases}$$

This set is surely not a Boolean combination of basic closed sets, since for example whether the zero matrix belongs to such a combination is independent of the size s. It is probably not semialgebraic under any suitable notion. So a possible Projection Theorem will require stronger assumptions or will provide weaker results.

As explained above, the classical quantifier elimination implies decidability of the theory of real closed fields. Although our below result on undecidability is not directly comparable (since different matrix sizes are used instead of different real closed fields), it still imposes some severe restrictions on possible elimination of quantifiers in a non-commutative setup. First note that non-commutative formulas are defined similarly as in the classical setup. Atomic formulas are of the form $p \geqslant 0$ for some $p \in \mathbb{R}\langle x_1, \ldots, x_n\rangle_h$, general formulas arise by combining atomic formulas with $\wedge, \neg, \forall$. A formula is *closed*, if it does not contain free variables.

Theorem 3.2. [2] *The question whether a non-commutative closed formula holds for at least one size of matrices is undecidable.*

But let us finally also mention some positive result (in its most basic form).

Theorem 3.3. [2] *Let* $p_1, \ldots, p_r \in \mathbb{R}\langle x_1, \ldots, x_n\rangle_h$ *and* $b_1, \ldots, b_r \in \mathbb{R}$. *Then for any* $s \geq 1$ *and* $M_1, \ldots, M_n \in \mathrm{Her}_s(\mathbb{C})$, *the following are equivalent:*

(i) $\exists N \in \mathrm{Her}_s(\mathbb{C}) : p_i(M_1, \ldots, M_n) + b_i N \geqslant 0$ *for all* $i = 1, \ldots, r$.

(ii) $\forall W_j \in \mathrm{Mat}_{r,s}(\mathbb{C})$ *with* $\sum_j W_j^* \mathrm{diag}(b_1, \ldots, b_r) W_j = 0$ *we have*

$$\sum_j W_j^* \mathrm{diag}(p_1(M_1, \ldots, M_n), \ldots, p_r(M_1, \ldots, M_n)) W_j \geqslant 0.$$

Let us finish with some remarks on this result. It is obviously deficient in many ways. We only remove an existential quantifier that quantifies over a separated linear variable. The description we obtain contains infinitely many inequalities. One might even argue that (ii) still contains a quantifier. On the other hand, the universal quantifier in (ii) acts on a different level; the description in (ii) depends only on the input $p_1, \ldots, p_r, b_1, \ldots, b_r$, but not on the M_i! The dependence on s is very uniform, much better than what a naive application of the classical Projection Theorem for fixed matrix size (when all entries of the matrices are commutative variables) could provide. Also, the infinitely many inequalities are parametrized in a good way (semialgebraic in the classical sense). This might provide for example an approximation of the set of all $(M_1, \ldots, M_n)$ that fulfill (i) for optimization purposes.

To sum up, we see that a Projection Theorem/Quantifier Elimination can probably not be expected in full generality in the non-commutative setup. There are even problems that are undecidable, which constitutes a severe obstacle to any such result. One will have to accept stronger assumptions or weaker results. The last theorem is a very first step in this direction.

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