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F. Delon, M. A. Dickmann, D. Gondard

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Delon – Dickmann – Gondard

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17.11.92 – **M. Messmer** (Freiburg, Allemagne).
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1.12.92 – **Y.L. Ershov** (Novosibirsk, Russie).
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16.02.93 – **A. Macintyre** (Oxford, Grande–Bretagne).
p–adic exponentiation ; the state of the art.

23.02.93 – **F. Miraglia** (Sào Paulo, Brésil).
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2.03.93 – **R. Cignoli** (Barcelone, Espagne et Buenos–Aires, Argentine).
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16.03.93 – **F. Lucas** (Angers).
ℓ–groupes divisibles et produits de Hahn.

23.03.93 – **A. Khélif** (Paris VII)
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4.05.93 – **R. Villemaire** (UQAM de Montréal, Canada).
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11.05.93 – **C. Stheinhorn** (Vassar, Etats–Unis et Oxford, Grande–Bretagne).
On the existence of an o–minimal expansion of the ordered rational group.

25.05.93 – **C.J. Mulvey** (Sussex, Grande–Bretagne).
Quantales : C*–algebras and a Quantisation of the Calculus of Relations.

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22.06.93 – **T. Krick** (Limoges et Buenos–Aires, Argentine)
Complexité de la construction des chemins semi–algébriques et linéaires par morceaux dans $\mathbb{R}^n$.

29.06.93 – **F. Lucas et D. Gluschankof** (Angers).
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6.07.93 – **F. Delon** (Paris VII) et **D. Gondard** (Paris VI).
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Liste des contributions.

M. Messmer (Freibourg, Allemagne).
Groups and fields interpretable in separably closed fields.

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Quelques résultats sur la théorie des modèles des groupes cycliquement ordonnés.

F. Lucas (Angers).
Quelques précisions sur la théorie des modèles des groupes cycliquement ordonnés abéliens divisibles.

L. Bröcker (Münster, Allemagne).
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R. Cignoli (Buenos Aires, Argentine)
The algebras of Lukasiewicz many–valued logics.

A. Khélif (Paris VII).
There are $2^{\aleph_0}$ existentially closed non–elementarily equivalent countable groups.

R. Villemaire (Québec, Montréal, Canada).
Joining k– and ℓ– recognizable sets of natural numbers.

H. Carrieu (Toulouse).
Problème de James Ax.

D. Gluschankof et **F. Lucas** (Angers).
The representation of ℓ–groups over Priestley spaces.

F. Delon (Paris VII).
Corps presque réels clos ou "ARC".

F. Delon (Paris VII) et **R. Farré** (Barcelone, Espagne).
Some model theory for almost real closed fields.

E. Becker, R. Berr (Dortmund, Allemagne) et **D. Gondard** (Paris VI).
On henselian fields with real–closed residue fields.

Groups and Fields Interpretable in Separably Closed Fields

by Margit Messmer

The following two theorems are part of the author's Ph.D. thesis, written under the supervision of Prof.D.Marker at the University of Illinois at Chicago, 1992, and will appear in the Transactions of the American Society ([9]).

Theorem A *Let F be a separably closed field of finite Eršov-invariant (degree of imperfection), and let G be an infinite group interpretable in F. Then G is definably isomorphic to an F-algebraic group.*

Theorem B *Let F be a separably closed field, and let K be an infinite field interpretable in F. Then K is separably closed, $char(K) = char(F)$ and K has the same Eršov-invariant as F. Moreover, if F has finite Eršov-invariant, then K is definably isomorphic to a finite (purely inseparable) extension of F.*

Both theorems generalize analogous results for algebraically closed fields, see [8, 4, 12, 2, 13]. The proofs make use of model theoretic results like stability, quantifier elimination and elimination of imaginaries, developed in [6, 14, 3, 11, 5]. Furthermore, techniques from stable group theory (see [10]) and linear algebraic group theory (see [1, 7]) are needed.

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QUELQUES RESULTATS SUR LA THEORIE DES MODELES DES GROUPES CYCLIQUEMENT ORDONNES. (Résumé).

M.Giraudet

Université du Maine
BP535 72017 Le Mans cedex

F.Lucas

Université d'Angers
2 bd Lavoisier 49045 Angers cedex

Equipe de Logique Mathématique
CNRS URA 753
2 Place Jussieu, 75251 Paris cedex

Abstract:
First, following the representation theorem of Rieger we prove that if a cyclically ordered group C is given by $C=G/\langle z\rangle$ where G is a totally ordered group and z a central and cofinal element, the first order theory of C is determined by the first order theory of the ordered group G with distinguished element z.
After that we look at orderability and give an axiom system for the elementary class of cyclically orderable groups.
Finally we remark that there are different universal theories of abelian cyclically ordered groups and prove that all the abelian cyclically ordered groups containing elements of order n for each n, have the same universal theory.

I Introduction

Rappellons les définitions introduites dans [R] et [F]:
Si A est un ensemble et R une relation ternaire sur A, R est un *ordre cyclique* sur A ssi

1) $\forall x,y,z\ (R(x,y,z) \Rightarrow x\neq y\neq z\neq x)$
2) $\forall x,y,z,\ (x\neq y\neq z\neq x \Rightarrow (R(x,y,z) \text{ or } R(z,y,x))$
3) $\forall x,y,z,\ (R(x,y,z) \Rightarrow R(y,z,x))$
4) $\forall x,y,z,u,\ ((R(x,y,z)\ \&\ R(y,u,z)) \Rightarrow R(x,u,z))$

Si maintenant G est un groupe et R une relation ternaire sur G, (G,R) est un *groupe cycliquement ordonné* si R est un ordre cyclique compatible avec la loi de groupe:

5) $\forall x,y,z,u,v,\ (R(x,y,z) \Rightarrow R(uxv,uyv,uzv))$.

On peut déjà remarquer que la théorie des groupes cycliquement ordonnés est finiment axiomatisable et universelle si l'on choisit de manière adéquate le langage de groupe.

Exemples:

1) Un groupe totalement ordonné est cycliquement ordonné par la relation :

$R(x,y,z)$ ssi ($x<y<z$ or $y<z<x$ or $z<x<y$)

2) Le groupe des nombres complexes de module 1: $\mathbb{K}=\{x\in\mathbb{C}/\ |x|=1\}=\{e^{2i\pi\theta}, 0\leq\theta<2\pi\}$ muni de la relation $R(e^{2i\pi\theta},e^{2i\pi\theta'},e^{2i\pi\theta''})$ ssi $\theta<\theta'<\theta''$, est un groupe cycliquement ordonné qui a des eléments de torsion et des éléments d'ordre infini. On note $\mathbb{U}$ les éléments de torsion de $\mathbb{K}$: ce sont les racines de l'unité de $\mathbb{C}$.

3) *Produit lexicographique*. Un des moyens de construire ou de décrire des groupes cycliquement ordonnés est le suivant: Si (C,R) est un groupe cycliquement ordonné et $(L,\leq)$ un groupe totalement ordonné, on définit un ordre cyclique R' sur $L\times C$ par

$R'((c,r),(c',r'),(c''r''))$ ssi $(c\neq c'\neq c''\neq c$ & $R(c,c',c''))$ ou $(c=c'\neq c''$ & $r<r')$ ou $(c\neq c'=c''$ & $r'<r'')$ ou $(c=c''\neq c'$ & $r<r'')$ ou $(c=c'=c''$ & $r<r'<r'')$.

On note $L\vec{\times}C$ le groupe cycliquement ordonné obtenu,on l'appelle produit lexicographique de L et de C.

4) *Enroulé de Rieger*: Si $(G,\leq)$ est un groupe cycliquement ordonné et z un de ses éléments positif, central et cofinal, la relation R définie ci-dessous fait de $G/\langle z\rangle$ un groupe cycliquement ordonné.

$R(\bar{g},\bar{h},\bar{k})$ ssi $\exists\ g',h',k'/\ \bar{g}=\bar{g}',\ \bar{h}=\bar{h}',\ \bar{k}=\bar{k}'$ &
$(e\leq g'<h'<k'<z$ ou $e\leq h'<k'<g'<z$ ou $e\leq k'<g'<h'<z)$.

Rieger a montré que (Theorème de Rieger) que tout groupe cycliquement ordonné peut être obtenu de cette manière, la démonstration donne la construction de ce que nous appellerons le *déroulé de Rieger* (uw(G)) du groupe cycliquement ordonné (G,R): uw(G) a pour ensemble de base le produit cartesien $\mathbb{Z}\times G$, pour ordre total l'ordre lexicographique $\mathbb{Z}\vec{\times}G$, la loi de groupe étant donnée par les formules suivantes: (On utilise ici la notation multiplicative aussi l'élément m de $\mathbb{Z}$ est ici représenté par z^m).

$(z^k,e).(z^m,h)=(z^{k+m},h)$, $(z^k,g).(z^m,e)=(z^{k+m},g)$ (e est l'élément neutre de G)

if $R(e,g,gh)$: $(z^k,g).(z^m,h)=(z^{k+m},gh)$

if $g\neq e$ and $gh=e$ then $(z^k,g).(z^m,h)=(z^{k+m+1},e)$

if $R(e,gh,g)$: $(z^k,g).(z^m,h)=(z^{k+m+1},gh)$.

On note z_G l'élément (z,e), il est positif, central et cofinal. On montre que $uw(G)/\langle z_G\rangle\simeq G$

Un sous groupe H d'un groupe cycliquement ordonné (g.c.o.) (G,R) est dit *c-convexe* si $\forall h\in H,g\in G\ (h^2\neq e$ & $R(h^{-1},e,h)$ & $R(e,g,h))\Rightarrow g\in H$.

Remarque I.1: Un sous groupe c-convexe n'est pas toujours pur dans G: Dans le sous groupe G de $(\mathbb{Z}/3\mathbb{Z})\vec{\times}\mathbb{Z}$ engendre par (1,1), le sous groupe H engendre par (0,3) est c-convexe, mais l'élement (0,3) n'est pas divisible par 3 dans H et il est divisible par 3 dans G: $3\times(1,1)=(0,3)$.

Un g.c.o. G est *c-archimédien* si $\forall g\neq e\in G,\forall h\neq e\in G,\ \exists n\in\mathbb{N}\ (\neg R\ (e,g^n,h))$.

G est archimédien ssi il n'a pas de sous groupe convexe propre; si G n'est pas totalement ordonné, G est archimedien ssi son déroulé de Rieger l'est.

II Equivalence élémentaire

On prouve ici un théorème de transfert de l'équivalence élémentaire entre un g.c.o. et son déroulé.

Theorem 2.1: Soient G et G' deux g.c.o.: $G\equiv G'$ ssi $(uw(G),z_G)\equiv(uw(G'),z_{G'})$
(on a aussi le transfert de l'inclusion élémentaire).

La preuve est donnée par les lemmes suivant où l'on utilise trois sortes

de structures:
-(G,R) dans le langage de g.c.o.,loi de groupe et ordre cyclique.
-(uw(G),$<z_G>$) dans le langage des paires de groupe totalement ordonné, avec un prédicat P pour le petit de la paire.
-$\mathbb{Z}\times G$ avec la loi de groupe,deux prédicats, l'un pour $\mathbb{Z}$, l'autre pour G, l'ordre sur $\mathbb{Z}$ et l'ordre cyclique sur G.

Lemme 2.2: On peut interpréter (G,R) dans (uw(G),<z>) et (uw(G),$<z_G>$) dans $\mathbb{Z}\times G$.
Lemme 2.3: G≡G' iff (uw(G),$<z_G>$)≡(uw(G'),$<z_{G'}>$), et la même chose pour l'inclusion élémentaire.
Lemme 2.4: (uw(G),z_G)≡(uw(G'),$z_{G'}$) ssi (uw(G),$<z_G>$)≡(uw(G'),$<z_{G'}>$), (idem pour l'inclusion élémentaire).
(L'idée de la preuve est de prendre deux structures saturées isomorphes de (uw(G),z_G) et (uw(G'),$z_{G'}$), dans ces structures z_G et $z_{G'}$ ne sont plus des éléments cofinaux, mais on peut se restreindre aux sous-groupes convexes engendrés.

En utilisant ceci on peut montrer un résultat sur le produit lexicographique:
Corollaire 2.5: Si G et G' sont deux c.o.g., et H et H' deux groupes cycliquement ordonnés:
(G≡G' & H≡H')⇒$G\vec{\times}H \equiv G'\vec{\times}H'$ et la même chose pour l'inclusion élémentaire.

III Groupes cycliquement ordonnables.

On utilise ici et par la suite le théorème de plongement de Swirczkowski:

Théorème (Swirczkowski) [Sw]: Pour chaque g.c.o. G il existe un groupe totalement ordonné (g.t.o.) H et un plongement f de G dans le produit lexicographique $\mathbb{K}\vec{\times}H$, (on dit que f est une représentation de G).
Jakubik [JP] a montré que la première projection est indépendante (à isomorphisme près) de la représentation, on la note K(G).

En utilisant ce résultat Zheleva a caractérisé les groupes cycliquement ordonnables:

Théorème (Zheleva) [Z]: Un groupe G admet un ordre cyclique ssi le sous groupe de ses torsions U(G) est central, locallement cyclique et G/U(G) est ordonnable.
(Dans le cas abélien ce résultat est à relier à celui de G.Sabbagh [S] qui donne la même caractérisation pour les groupes qui se plongent dans le groupe multiplicatif d'un corps).
La preuve utilise le fait que toute extension centrale d'un groupe totalement ordonné par un groupe cycliquement ordonné peut être muni d'un ordre cyclique.

On peut par cette même méthode montrer:
Théorème 3.1: G est cycliquement ordonnable ssi son centre Z(G) est cycliquement ordonnable et le quotient G/Z(G) est ordonnable.
(Résultat bien sûr à relier avec celui de Kokorin et Kopitov [KK]: un groupe G est ordonnable ssi son centre Z(G) et le quotient G/Z(G) sont ordonnables).

Nous donnons enfin un système d'axiomes pour la théorie des groupes cycliquement ordonnables en utilisant la caractérisation des groupes ordonnables donnée par Onishi et Los [O], [L]:

Un groupe G est ordonnable ssi pour tout n-uple d'éléments non nuls $x_1,\ldots,x_n$ de G il existe $\varepsilon\in\{1,-1\}^n$ telque e n'appartient pas au demi-groupe engendré par les conjugués de $x_1^{\varepsilon(1)},\ldots,x_n^{\varepsilon(n)}$.

Theorem 3.2: G est cycliquement ordonnable ssi il satisfait les axiomes suivant:
-pour chaque n: $\forall x\ (x^n=e \Rightarrow \forall y\ xy=yx)$
-pour chaque n:
$$\exists x_1,\ldots,x_n((\bigwedge_{(i,j)\in\{1,\ldots,n\}} x_i\neq x_j \wedge x_i^n=e)\wedge\forall x(x^n=e \Rightarrow \bigvee_{i\in\{1,\ldots,n\}} x=x_i)$$
-pour chaque n,k, chaque mot à kn variables $m(t_{i,j}\ 1\leq i\leq n,\ 1\leq j\leq k)$ et chaque $\varepsilon\in\{-1,1\}^n$:
$$\forall x,x_1,..,x_i,..,x_n,y_{1,1},.,y_{i,k}..,y_{n,n}\ ((\bigwedge_{i=1,..,n}x_i\neq e \wedge x^n=e) \Rightarrow (\bigvee_{\varepsilon\in\{-1,1\}^n} m(y_{i,k}x_i^{\varepsilon(i)}y_{i,k}^{-1})\neq x)).$$

IV Quelques remarques sur le cas abélien en particulier sur les théories universelles de g.c.o. abéliens .

Un résultat utile pour l'étude des groupes abéliens totalement ordonnés est que pour tout sous groupe convexe C, G est élémentairement équivalent à $(G/C)\vec{\times}C$; la preuve de ce résultat est basée sur les fait suivants: tout sous groupe convexe est pur et si G est ω_1-saturé $G\simeq(G/C)\vec{\times}C$.
Dans le cas des g.c.o. un sous groupe c-convexe n'est pas toujours pur comme on l'a vu en I.

Nous avons prouvé les résultats suivants:
Lemme 4.1: Si $\mathbb{U}\subseteq G$, tout sous groupe c-convexe de G est pur dans G.
Lemme 4.2: Si $\mathbb{U}\subseteq G$ et G est ω_1-saturé alors $G/G_0=\mathbb{K}$.
Théorème 4.3: Si C est un sous groupe c-convexe et pur de G alors $G\equiv(G/C)\vec{\times}C$.

Gurevich et Kokorin ([GK], [Gl]) ont montré que tous les groupes abéliens totalement ordonnés ont la même theorie universelle.
Il est clair que le résultat n'est pas valable pour les g.c.o. abéliens en effet l'existence d'éléments d'ordres finis différents donne des théories universelles differentes. Même dans le cas de g.c.o. abeliens sans torsions on peut avoir des theories universelles différentes, par exemple la formule
$\exists x\ R(x,4x,2x,0,3x)$ (abréviation pour une conjonction) est satisfaite dans le sous groupe de $\mathbb{K}$ engendré par un élément $e^{2i\pi\theta}$ avec θ irrationnel tel que $\pi/3<2\pi\theta<\pi$ $(1/3<\theta<1/2)$, mais n'est pas satisfaite dans l'exemple de la remarque 1.1.

En toute généralité on peut démontrer un résultat du type du théorème de transfert 2.1:
Theorème 4.5: Si G et G' sont des g.c.o.:
$G\equiv_\forall G'$ ssi $(uw(G),z_G)\equiv_\forall(uw(G'),z_{G'})$.

Dans le cas abélien nous montrons:
Théorème 4.6: Tous les g.c.o. abéliens qui contiennent $\mathbb{U}$ ont la même théorie universelle.
Théorème 4.7: Soient G et G' deux g.c.o. abéliens tels que uw(G) et uw(G') sont des g.t.o. abéliens réguliers (i.e. élémentairement équivalent à des archimédiens) et $G\cap\mathbb{U}=G'\cap\mathbb{U}$, alors $G\equiv_\forall G'$.

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QUELQUES PRECISIONS SUR LA THEORIE DES MODELES DES GROUPES CYCLIQUEMENT ORDONNES ABELIENS DIVISIBLES. (Résumé).

F.Lucas

Abstract: We use a transfert result between cyclically ordered groups and associated structures of ordered groups, proved by M.Giraudet and the author [GL], and model theory of totally ordered abelian groups as developped by A.Robinson and E.Zakon [RZ], M.I.Kargapolov [K], Y.Gurevich [G] and P.Smitt [Sc]. We remark in 1) that the theory of divisible cyclically ordered groups is not complete and study its completions. One of these completions, we call the theory of super-divisible is proved in 2) to be the model completion of cyclically ordered groups. Following an idea of D.Macpherson [M] we prove in 3) that the abelian super-divisible are exactly the circle-minimal models (i.e. those in which the only parametricaly definable subsets are the finite unions of intervals). In 4) we give a description of the different cyclically ordering of the additive group of the rationals $\mathbb{Q}$.

Les définitions et notions de base concernant les groupes cycliquement ordonnés ont été rappelés dans un résumé précédent (M.Giraudet & F.lucas Quelques resultats sur la théorie des modèles des groupes cycliquement ordonnés).

1) **Theories de groupes cycliquement ordonnés abélien divisibles.**(g.c.o.a.d.).

On remarque tout d'abord que la théorie des g.c.o.a.d. n'est pas complète, un tel groupe peut être totalement ordonné ou non, il peut avoir des éléments d'ordre fini ou ne pas en avoir...On peut donc développer une classification de ces théories, on en donne ici quelques éléments. On utilisera le théorème de transfert présenté dans [GL]:

Théorème: [GL] Si G et G'sont deux g.c.o. on a $G \equiv G'$ ssi $(uw(G), z_G) \equiv (uw(G'), z_{G'})$.

Parmi les différences entre le cas totalement ordonné et le cas cycliquement ordonné, on peut dès le départ noter:

1) Le fait que G soit divisible n'implique pas que uw(G) le soit.(Si $G=\mathbb{Q}(\pi)/\langle\pi\rangle$ et $G'=\mathbb{Q}[\pi]/\langle\pi\rangle$, G,G' et uw(G) sont divisibles mais uw(G') ne l'est pas.

2) un g.c.o.a.d. n'est pas toujours dense (soit $G=\mathbb{Q}\vec{\times}\mathbb{Z}/\langle(\alpha,1)\rangle$, le sous groupe engendré dans G par (0,1) est c-convexe et discret, et chaque élément de G est divisible puisque $(0,1)=(\alpha,1)-(\alpha,0)$).

On montre que cependant si un g.c.o.a.d. G a un élément d'ordre fini il est dense.

Dans le cadre des abéliens totalement ordonnés, la théorie des divisibles joue un rôle fondamental, aussi est-il naturel de définir:

G est *super-divisible* si uw(G) est divisible.

Lemme: G is super-divisible ssi il est divisible et pour tout n il existe g telque ng=0 ou encore il est divisible et contient $\mathbb{U}$.

La théorie des super-divisibles sera étudiée dans le paragraphe 2).

En utilisant le théorème de transfert cité ci-dessus on sait que la théorie de G est

donnée par celle de uw(G) et le type sur le vide de z_G et on peut utiliser les résultats de théorie des modèles sur les totalement ordonnés. On va commencer par le cas régulier avant de donner une idée du cas général:
Suivant [RZ], si n est un nombre premier on dira que G est n-regular si $\forall x_1,\ldots,x_n(0<x_1<\ldots<x_n \Rightarrow \exists y(\ x_1<ny\leq x_n))$.
cette condition équivaut au fait que tout quotient par un sous-groupe convexe propre est n-divisible.
On dit que G est régulier si il est n-régulier pour tout premier n. Si G est divisible ou archimédien il est régulier.

Si maintenant G est un g.c.o.a. on dira qu'il est *n-régulier* si il est totalement ordonné et régulier ou si uw(G) is n-regular; on dira qu'il est *régulier* si il est n-régulier pour tout premier n.
Une différence est qu'un g.c.o.a.divisible n'est pas toujours régulier; par exemple $G=(\mathbb{Q}(\pi)\vec{\times}\mathbb{Q})/<\pi>$, est divisible mais $uw(G)=\mathbb{Q}(\pi)\vec{\times}\mathbb{Q}$ n'est pas régulier.

Dans le cas dense la propriété d'être n-régulier s'exprime par les formules suivantes
1) $\exists x\neq 0/nx=0$
2) $\forall y(R(-y,0,y)\Rightarrow \exists xR((x,2x,\ldots,(n-1)x,0,nx)\&R(0,nx,y)))$.
et on a le résultat suivant:

Théorème: Deux g.c.o.a.divisibles denses et réguliers sont élémentairement équivalents ssi ils le sont dans le langage des groupes c'est à dire s'ils ont les mêmes éléments de torsion.

Dans le cas ordonné non régulier on utilise les résultats de [G] et [Sc]
si $g\in G$, B(g) est le plus petit sous groupe convexe contenant g, $A_n(g)$ le plus grand sous groupe convexe tel que $B(g)/A_n(g)$ soit n-regular, and $B_n(g)$ le plus grand sous groupe convexe tel que $B_n(g)/A_n(g)$ soit n-regular, $F_n(g)$ le plus grand sous groupe convexe dont aucun élément n'est congru à g modulo n. $S_n(G)=\{A_n(g),\ F_n(g),\ g\in G\}$. La famille des chaînes $S_n(G)$ colorées de prédicats (précisant par exemple la théorie de $B_n(g)/A_n(g)$) détermine la théorie de G.

On a ici:
Lemme: Si G est divisible, pour tout premier p, uw(G)/puw(G) est un espace vectoriel sur $\mathbb{Z}/p\mathbb{Z}$ de dimension 0 ou 1.

Théorème: Si G est divisible, pour tout premier p, $S_p(uw(G))$ a au plus deux éléments.

2) **Super-divisibles**:

On a décidé dans le paragraphe 1) d'appeler super-divisible un g.c.o.a.d. G tel que uw(G) est divisible. On a vu que ceci était équivalent au fait d'être divisible et de contenir $\mathbb{U}$.
On peut encore remarquer que si G est super-divisible il est isomorphe à $K(G)\times G_0$.

Comme un corollaire de 1.1 on obtient:
Théorème 2.1: La théorie des super-divisibles est complète et modèle complète.

On montre que:
Théorème 2.2: La théorie des super-divisibles a la propriété d' amalgamation.

On en déduit:
Corollaire 2.3: La théorie des super-divisibles a l'élimination des quantificateurs.

Théorème 2.4: La théorie des super-divisibles a un modèle premier: $\mathbb{U}$.

Enfin:
Théorème 2.5: Tou g.c.o.a. admet une clôture super-divisible.

Donc:
Corollaire 2.6: La théorie des super-divisibles est la modèle-complétion de la théorie des groupes cycliquement ordonnés abéliens.

3) **Groupes cycliquement ordonnés K-minimaux.**

On sait que dans le corps $\mathbb{R}$ tout sous ensemble définissable est réunion finie d'intervalles, c'est à dire est définissable par une formule sans quantificateurs du langage d'ordre. L'étude générale des structures algébriques totalement ordonnées ayant une telle propriété a été menée par A.Pillay et C.Steinhorn puis D.Macpherson, M.Dickmann etc. sous le nom de structures o-minimales.
Je remercie D.Macpherson de m'avoir de m'avoir parlé de ses premiers résultats, obtenus dans un contexte très général [MS], concernant une notion analogue adaptée au cas des structures cycliquement ordonnées.

Soit (G,+,R) un g.c.o. et I un de ses sous ensembles: I est un *intervalle* si I est réduit à un point ou si $I=\{i\in G/R(g,i,g')\}$ pour des $g,g'\in G$; I est *convexe* si il est réduit à un point ou si pour tout $g\in G$ et tous $i,i'\in I$ si $R(i,g,i')$ alors $g\in I$.

On dira que G est:
-*K-minimal* si il est infini et que tout sous ensemble définissable est union finie d'intervalles.
-*fortement-K-minimal* si pour tout G' élémentairement équivalent à G, G' est *K*-minimal.
-*faiblement-K-minimal* si il est infini et que tout sous ensemble définissable est union finie de convexes.

Remarques:
1) Ces trois notions correspondent respectivement aux notions de o-minimal, fortement-o-minimal et faiblement-o-minimal dans le cas ordonné.
2) Dans le cas des groupes abéliens ordonnés ces trois notions coïncident et sont réalisées exactement par les abéliens divisibles.
3) Clairement si G est fortement-K-minimal il est K-minimal et si il est K-minimal il est faiblement-K-minimal.

On va caractériser ici les faiblement-K-minimaux et montrer que G est K-minimal ssi il est fortement-K-minimaux ssi il est super-divisible.

Si H est un sous groupe de G soit C(h,H) le plus grand sous ensemble convexe de G contenant h et contenu dans H; on l'appellera composante convexe en h de H.
On remarque que:
1) C(0,H) est symétrique: si $h\in C(0,H)$ alors $-h\in C(0,H)$.
2) $C(0,H)=\{h\in H\ \forall g\in G\ (R(0,g,h) \text{or } R(0,g,-h))\Rightarrow g\in H\}$.

Lemme: Si G est faiblement-K-minimal et H un sous groupe infini définissable:
1) C(0,H) est définissable dans G et H n'a qu'un nombre fini de composantes convexes.
2) si $H\leq G_0$ alors H est un sous groupe convexe de G_0.

Lemme: Si G est faiblement-K-minimal:
1) G_0 est divisible.
2) G est abélien.
3) si G n'est pas divisible $K(G)\leq\mathbb{U}$.

Lemme: Soit G faiblement-K-minimal:
- ou bien G est divisible et il est alors super-divisible,
- ou bien il n'est pas divisible et il est de la forme $(\mathbb{Z}/n\mathbb{Z})\vec{\times}G_0$ avec G_0 divisible.

Lemme: Si G est K-minimal il est super-divisible.

Lemme: 1) Pour tout D divisible totalement ordonné $G=(\mathbb{Z}/n\mathbb{Z})\vec{\times}D$ est faiblement-K-minimal.
2) Tout super-divisible est K-minimal.

On en déduit:
Théorème: Soit G un g.c.o.:
1) G est K-minimal ssi il est abélien et super-divisible.
2) G est faiblement-K-minimal ssi il est abélien super-divisible , ou bien de la forme $(\mathbb{Z}/n\mathbb{Z})\vec{\times}D$ avec D abélien divisible.

Remarque: si G' est élémentairement équivalent à G et G est faiblement-K-minimal G' l'est aussi.

4) Ordres cycliques sur $\mathbb{Q}$.

On montre d'abord que deux sous groupes cycliquement ordonnés de $\mathbb{K}$, isomorphes sont égaux.

Soit $G=(\mathbb{Q},R)$ un ordre cyclique sur $\mathbb{Q}$. $\mathbb{Q}$ muni de cet ordre peut se plonger dans un produit lexicographique $K(G)\vec{\times}L$ avec L totalement ordonné divisible, et il y est engendré comme $\mathbb{Q}$-espace vectoriel par un élément que l'on notera (a,b).

a) Si $a=0$ on a un groupe totalement ordonné, et l'on sait qu'il existe deux manières de munir $\mathbb{Q}$ d'un ordre total.

b) Si $b=0$, G est contenu dans $\mathbb{K}$:
on a alors à décrire les sous $\mathbb{Q}$-espaces vectoriels de $\mathbb{K}$ de dimension 1.
En utilisant la notation multiplicative complexe dans $\mathbb{K}$, on remarque d'abord que pour tout $g=e^{i\theta}\in G$, $\theta/2\pi$ est irrationnel. Soient Θ une base de $\mathbb{R}$ sur $\mathbb{Q}$, $(p_i)_{i\in\mathbb{N}}$ la suite des premiers, q_i et q'_i les suites définies par $q_i=\prod_{j\leq i}(p_j)^i$, $q'_i=(\prod_{j\leq i}p_j)(p_{i+1})^{i+1}$ et enfin $F=\{f\in \mathbb{N}^{\mathbb{N}} / \forall i\ 1\leq f(i)\leq(\prod_{j\leq i}p_j)(p_{i+1})^{i+1}\}$.
Pour chaque $\theta\in\Theta$ et chaque $f\in F$ on définit un $\mathbb{Q}$-espace vectoriel de dimension 1 engendré par $g=e^{i\theta}$ de la manière suivante: (il suffit de définir g/q_i pour tout $i\in\mathbb{N}$) on définit g/q_j par recurrence: si $g/q_j=e^{i\theta'}$ alors $g/q_{j+1}=e^{i(\theta+2\pi f(i))/q'_i}$.

c) if $a\neq0$ and $b\neq0$:
- si a n'est pas un élément de torsion on a un isomorphisme de groupe cycliquement ordonné entre le $\mathbb{Q}$-espace vectoriel engendré par (a,b) et le $\mathbb{Q}$-espace vectoriel engendré par $(a,0)$.
- si $na=0$, alors $n(a,b)=(0,nb)$, et le sous groupe engendré par $(0,nb)$ et les $(0,nb/m)$ pour $m\notin n\mathbb{N}$ est c-convexe et divisible par m pour chaque $m\notin n\mathbb{N}$. On peut alors déterminer les diviseurs manquant d'une manière analogue à celle du b).

On voit par là qu'il y a $2^{\aleph_0}$ ordres cyclique compatibles non isomorphes sur $(\mathbb{Q},=)$, certains d'entre eux n'étant pas archimédiens.

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Abstract Real Spectra

by Ludwig Bröcker, Münster

In this abstract we are going to present an axiomatic approach for the problem of the describing real constructible sets by few inequalities. In all situations, as semialgebraic geometry, semianalytic geometry, real spectra of rings, for the description of constructible sets one does not need the precise values of the defining functions but just their signs. This leads immediately to the very general notion, which we introduce by the following

Definition 1.1. A real space (X, G) consists of a set X and a multiplicative monoid G of maps $G : X \to \mathbb{F}_3$ such that the following conditions hold:
S_1: G contains the unit map $1 : X \to \mathbb{F}_3$; $x \mapsto +1$.
S_2: G contains the map $-1 : X \mapsto \mathbb{F}_3$; $x \mapsto -1$.
S_3: G separates points, that is, for any two points $x, y \in X$ there is an $f \in G$ with $f(x) \neq f(y)$.
Clearly, not very much can be done in this general setting. However, one has already a lot of notions, which can be fixed in the usual way: basic open sets, basic closed sets, constructible sets, Harrison topology on X and also constructible topology and Zariski topology on X. One may even imitate weak equivalence classes of quadratic forms. For this consider n-tuples $(a_1, \ldots, a_n)$, $a_i \in G$ and write $(a_1, \ldots, a_n) \equiv (b_1, \ldots, b_n)$ if

$$\sum_{i=1}^{n} a_i(x) = \sum_{i=1}^{n} b_i(x) \quad \text{for all} \quad x \in X.$$

Here, of course, the summation is done in $\mathbb{Z}$. Then a form ϱ over X is just an equivalence class under the relation $\equiv$. Also, write $\varrho(x) := \sum_{i=1}^{n} a_i(x)$ for $x \in X$. In the sequel, we write, for $a \in G$, $\{a > 0\}$ instead of $\{x \in X | a(x) = 1\}$. Similarly, we write $\{a = 0\}$, $\{a < 0\}$ and also $\{a > 0, b > 0\}$ for $\{a > 0\} \cap \{b > 0\}$ and so on.

2. Axioms for abstract real spectra

Definition 2.1. Let (X, G) be a real space. If X is compact with respect to the constructible topology, (X, G) is called an abstract real prespectrum.

Now let (X,G) be an abstract real prespectrum. We are going to define abstract real spectra by imposing three supplementary axioms. The notion we get generalizes real spectra of rings in [BCR, Chap. 7] in a similar way as Marshall's spaces of orderings [M] generalize real spectra of fields. Now the axioms:

PE: For any two $a,b \in G$ there exists $c \in G$ such that $\{a=0\,,\ b=0\} = \{c=0\}$.
HL: Let $C \subset X$ be constructible and closed. Let $f,g \in G$ such that $C \cap \{f=0\} \subset \{g=0\}$. Then there exists $f' \in G$ such that
a) $f' = f$ on C, $f' = g$ on $\{f=0\}$.
b) $< f,g > = < f'.f'fg >$.
MM: Let $\varrho = < a_1,\ldots,a_n >$, $\tau = < b_1,\ldots,b_m >$ be forms over X, and let $h \in G$ such that $\varrho + \tau = < h, c_2,\ldots,c_{n+m} >$, with $c_2,\ldots c_{n+m} \in G$. Let $x \in X$ be a point such that

$$h(x) \neq 0\,,\ a_i(x) \neq 0\,,\ b_j(x) \neq 0\,,\ c_k(x) \neq 0$$

for all i,j,k. then there exist forms $\varrho' = < f, a'_2,\ldots,a'_n >$ and $\tau' = < g, b'_2,\ldots,b'_m >$, an element $c \in G$ and a Zariski neighbourhood U of x such that, for all $y \in U$:

i) $a'_i(y) \neq 0$, $b'_j(y) \neq 0$, $f(y) \neq 0$, $g(y) \neq 0$, $c(y) \neq 0$ for all i,j.

ii) $\varrho(y) = \varrho'(y)$, $\tau(y) = \tau'(y)$.

iii) $< f,g > (y) = < h,c > (y)$.

Definition 2.2. An abstract real prespectrum (X,G) is called abstract real spectrum, if **PE** and **HL** hold for (X,G) and **MM** holds for any Zariski closed subset of (X,G).

Remark 2.3. Let (X,G) be an abstract real spectrum. If G is a group, then (X,G) is a space of orderings (see [M]).
Finally , we have

Proposition 2.3. Let A be a commutative ring with unit. Let $X := Spec_r(A)$ be the real spectrum of A and set $G := \{sign f : X \mapsto \mathbb{F}_3 \mid f \in A\}$. Then (X,G) is an abstract real spectrum.
Proof: [ABR, III 5]

3. Statement of a main result

The theory of abstract real spectra is surprisingly rich. We are able to develop, in this abstract setting, the whole theory of describing real constructible sets by few inequalities [B] and related problems, as far as it was known for real spectra of rings and beyond that.
Before we state a result we need

Definition 3.1. Let (X, G) be an abstract real spectrum, and let $Y \subset X$ be a set of the form

$$Y = (\cap_{f \in A}\{f = 0\}) \cap (\cap_{g \in B}\{g > 0\}) \quad \text{with} \quad A, B \subset G.$$

Then $(Y, G|Y)$ is called a subspace of (X, G).

Proposition 3.2. A subspace of an abstract real spectrum is again an abstract real spectrum.
Proof: [ABR, III 4].
Next consider an $\mathbb{F}_2$-vectorspace H of dimension $n+1$ which we write multiplicatively, and fix an element $-1 \in H$ with $-1 \neq 1$. Let $\hat{H}$ be the dual group of H and $F = \{x \in \hat{H} \mid x(-1) = -1\}$. Then H is a group of maps $h : F \mapsto \{1, -1\} \subset \mathbb{F}_3$; $x \mapsto h(x) := x(h)$. Then (F, H) is a space of orderings, where $\#(F) = 2^n$. Such a space is called a fan and denoted by F_n.
Now let (X, G) be an abstract real spectrum and let $(Y, G|Y)$ be a subspace. In general, $G|Y$ will contain the zero-function 0. However, it may happen, that $(G|Y)\backslash\{0\}$ is a group and even, that $(Y, (G|Y)\backslash\{0\})$ is a fan. Then we say, that Y is a fan in X.
Now we are able to state one of the central results for the theory of real spectra in our abstract setting.

Theorem 3.3. (generation formula). Let (X, G) be an abstract real spectrum and let $C \subset X$ be constructible such that C does not intersect the Zariski closure of its boundary. If for all fans F in X one has

$$\#(F) \equiv 0 \bmod \#(C \cap F)$$

and

$$2^k \#(C \cap F) \bmod \#(F),$$

then there are $g_1, \ldots, g_k \in G$ such that $C = \{g_1 > 0, \ldots, g_k > 0\}$.
Proof: [ABR, Th. 3.1.4].

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Mathematisches Institut
Einsteinstr. 62
48149 Münster

The algebras of Łukasiewicz many-valued logics

Roberto Cignoli

MV-algebras were introduced by Chang in 1958 as the algebraic counterparts of the infinite-valued Lukasiewicz logic. These algebras have appeared in the litterature under different names and presentations. Recently it was discovered that they are naturally related to the Murray-von Neumann order of projections in C*-algebras and that MV-algebras are also useful for the study of Ulam's searching games with lies.

The aim of this paper is to present a brief account of the theory of these algebras and their relations with Lukasiewicz many-valued logics. The forthcoming monograph [16] contains a rather detailed and self-contained account of the theory of MV-algebras.

For the applications of MV-algebras to AF C*-algebras and Ulam's games, we refer the reader to Mundici's papers [32, 35, 33]. For the connections with Moisil's Lukasiewicz algebras we refer to the survey article [11] and to the book [5]

1 Łukasiewicz many-valued systems of propositional calculi.

As in the classical case, the *propositional formulas* of Lukasiewicz propositional calculi are obtained from a denumerable set of *propositional variables*, **Var** $= \{p_0, p_1, \ldots\}$, by means of the connectives of *negation* $\neg$ and of *implication* $\rightarrow$, and the parentheses.

More precisely, the set **Form** of propositional formulas is given inductively as follows:

F1) Each variable is a formula.

F2) If P is a formula, then $\neg P$ is a formula.

F3) If P and Q are formulas, then $(P \rightarrow Q)$ is a formula.

The "truth tables" of the connectives $\rightarrow$ and $\neg$ are given by the following two functions defined on the segment $[0,1] = \{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$, where $\mathbf{R}$ denotes the set of real numbers:

(1) $\neg x =_{def} 1 - x$

and

(2) $x \to y =_{def} \min\{1, 1 - x + y\}$

A *Lukasiewicz subalgebra* of $[0,1]$ is a subset S of $[0,1]$ such that $1 \in S$ and S is closed under the operations $\neg$ and $\to$.
For each Lukasiewicz subalgebra S define an *S-valuation* as a function ν which assigns to each proposition P a truth-value $\nu(P) \in S$ satisfying the conditions:

(3) $\nu(\neg P) = \neg\nu(P)$

and

(4) $\nu(P \to Q) = \nu(P) \to \nu(Q)$

An S-tautology is a propositional formula P such that $\nu(P) = 1$ for each S-valuation ν.

The next result is obtained as in the case of classical propositional calculus:

Theorem 1.1 *Let S be a Lukasiewicz subalgebra of* $[0,1]$. *For each function f :* $\mathbf{Var} \longrightarrow S$ *there is a unique valuation* $\nu_f : \mathbf{Form} \longrightarrow S$ *such that* $\nu_f(p_n) = f(p_n)$ *for* $n = 0, 1, \ldots$.

For each $n \geq 2$, the following n-element sets are Lukasiewicz subalgebras of $[0,1]$:

$$\mathbf{L}_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$$

Note that $\mathbf{L}_2 = \{0,1\}$, and that the $\mathbf{L}_2$- tautologies are the classical tautologies, provided we identify, as usual, 0 with *false* and 1 with *true*.

If $\mathbf{A}$ and $\mathbf{Q}$ denote the set of algebraic and rational numbers respectively, then $\mathbf{A} \cap [0,1]$ and $\mathbf{Q} \cap [0,1]$ are examples of denumerable Lukasiewicz subalgebras of $[0,1]$.

If for each $n \geq 2$, $\mathbf{Taut}_n$ denotes the set of $\mathbf{L}_n$-tautogies, then it was proved by Lukasiewicz and Lindenbaum (see [28])that:

(5) $\mathbf{Taut}_n \subseteq \mathbf{Taut}_m$ *if and only if* $m - 1$ *divides* $n - 1$

It follows from (5) that $\mathbf{Taut}_m \neq \mathbf{Taut}_n$ for $m \neq n$. The set $\mathbf{Taut}_n$ is called in [28] *the n-valued system of propositional calculus*, for each $n \geq 2$.

On the other hand, Lindenbaum [28] proved that all the infinite Lukasiewicz subalgebras of $[0,1]$ have the same tautologies. The set of these tautologies, that we denote by $\mathbf{Taut}_{\aleph_0}$, is called in [28] *the $\aleph_0$-valued system of propositional calculus.*

The following relations among the n-valued systems of propositional calculi, for $n = 2, 3, \ldots, \aleph_0$ are given in [28]:

Theorem 1.2 *For each* $n \geq 3$, $\mathbf{Taut}_{\aleph_0} \subset \mathbf{Taut}_n \subset \mathbf{Taut}_2$

and

Theorem 1.3 (Tarski) *If $2 \leq n_1 < n_2 < \dots$ is an increasing sequence of natural numbers, then* $\mathbf{Taut}_{\aleph_0} = \bigcap_{k=1}^{\infty} \mathbf{Taut}_{n_k}$

It follows from (2) that for every x and y in $[0,1]$:

(6) $x \leq y$ *if and only if* $x \to y = 1$

Moreover

(7) $x \vee y =_{def} \max\{x, y\} = (x \to y) \to y$

and then, from (1):

(8) $x \wedge y =_{def} \min\{x, y\} = \neg(\neg x \vee \neg y)$

Thus the order structure of the segment $[0,1]$ can be recovered from the Lukasiewicz operations $\neg$ and $\to$. If we restrict these operations to $\mathbf{L}_2$, (7) gives the following well known relation between material implication and conjunction in classical logic:

$(P$ *or* $Q)$ *is equivalent to* $((P \to Q) \to Q)$

while (8) gives the De Morgan law. Therefore the operations $\vee$ and $\wedge$ can be regarded as many-valued generalizations of the classical truth tables of disjunction and conjunction, respectively.

In classical propositional calculus we also have that

$(\neg P \to Q)$ *is equivalent to* $(P$ *or* $Q)$

but from (1) and (2) we obtain:

(9) $x \oplus y =_{def} \min\{1, x + y\} = \neg x \to y$

It is not hard to see that $\mathbf{L}_2$ is the only Lukasiewicz subalgebra of $[0,1]$ on which the equation $x \vee y = x \oplus y$ holds. Hence we can consider the binary operation $\oplus$ as another many-valued generalization of the classical truth table for disjunction. Accordingly, the operation $\odot$ defined in the next formula, generalizes the truth table for conjunction:

(10) $x \odot y =_{def} \neg(\neg x \oplus \neg y) = \max\{0, x + y - 1\}$

Moreover, we have that for each x, y in $[0,1]$:

$$x \to y = \neg x \oplus y$$

This shows that, as in the classical case, the operation $\to$ can be recovered from the operations $\oplus$ and $\neg$. On the other hand, if $S \subseteq [0,1]$ is a Lukasiewicz subalgebra and $S \neq \mathbf{L}_2$, then there are proper subsets of S which contain 0 and 1 and are closed under the operations $\vee$ and $\neg$. Therefore the operation $\rightarrow$ on S cannot be defined in terms of $\vee$, $\neg$ and the constants 0 and 1.

It is well known that Boolean algebras are the algebraic counterparts of the classical propositional calculus. They are often defined in terms of operations that correspond to the logical connectives of conjuction, disjunction and negation. Accordingly, Chang [9] introduced the M(any)V(alued)-algebras as the algebraic counterparts of Lukasiewicz propositional calculi by taking the operations $\oplus$, $\odot$ and $\neg$ as primitive.

Before given the formal definition of MV-algebras, we want to say a few words about the sintactical aspects of Lukasiewicz calculi.

Lukasiewicz conjectured that all $L_{\aleph_0}$-tautologies could be deduced from all instances of the schemes L1)–L4) listed below, by means of the rule of detachment (or modus ponens): From $P \rightarrow Q$ and P, infere Q.

L1) $P \rightarrow (Q \rightarrow P)$

L2) $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

L3) $((P \rightarrow Q) \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow P)$

L4) $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$

Actually, Lukasiewicz considered one more scheme, bu it can be derived from L1)–L4), as was proved independently by Chang [8] and Meredith [31].

It is easy to check that the formulas obtained by the schemes L1)–L4) are S-tautologies for each Lukasiewicz subalgebra of $[0,1]$, and that the rule of modus ponens preserve tautologies. Therefore all the propositional formulas that can be derived from the schemes L1)–L4) by modus ponens are S-tautologies. Therefore, Lukasiewicz conjecture asserts that *all* $L_{\aleph_0}$*-tautologies are derivable from the schemes L1)–L4) by the rule of modus ponens.*

In an article of 1935, Wajsberg [45, p.240] announced that he had verified Lukasiewicz conjecture, but his proof was never published. A proof of this conjecture was published in 1958 by Rose and Rosser [40], and in 1959, Chang [10] published another proof, based on the properties of MV-algebras.

2 MV-algebras.

As we already mentioned, Chang [9] defined MV-algebras by axiomatizing the operations $\oplus$, $\odot$ and $\neg$ on $[0,1]$ considered in the previous section. The definition that we are given below, simpler than the original, is essentially due to Mangani [29] (see [34, 17, 18].

Definition 2.1 *An* MV-algebra *is an algebra* $(A, \oplus, \neg, 0)$ *with a binary operation* $\oplus$, *a unary operation* $\neg$ *and a constant* 0 *fulfilling the following equations:*

MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2) $x \oplus y = y \oplus x$

MV3) $x \oplus 0 = x$

MV4) $\neg\neg x = x$

MV5) $x \oplus \neg 0 = \neg 0$

MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$

As usual, we are going to denote an MV-algebra $(A, \oplus, \neg, 0)$ by its universe A, and the MV-algebra whose universe is the singleton $\{0\}$ is said to be *trivial.*

Note that axioms MV1)–MV3) imply that $(A, \oplus, 0)$ is an *abelian monoid.*

It is not hard to check that $([0,1], \oplus, \neg, 0)$, where the operations $\neg$ and $\oplus$ are defined by (1) and (9) respectively, is an MV-algebra.

This example can be generalized as follows.

Let G be a lattice-ordered abelian group (l-group for short). For each $u \in G$, $u > 0$, set $[0, u] =_{def} \{x \in G \mid 0 \leq x \leq u\}$, and for each x, y in $[0, 1]$, define

$$x \oplus y =_{def} u \wedge (x + y)$$

and

$$\neg x =_{def} u - x$$

It is not hard to see that $([0, u], \oplus, \neg, 0)$ is an MV-algebra (see [26, 32]), which will be denoted by $\Gamma(G, u)$.

Observe that the Lukasiewicz subalgebras of $[0, 1]$ considered in the previous section are subalgebras of the MV-algebra $\Gamma(\mathbf{R}, 1)$, and consequently, are MV-algebras.

On each MV-algebra A we define the constant 1 and the binary operations $\odot$, $\ominus$, $\vee$ and $\wedge$ as follows:

$$1 =_{def} \neg 0$$

$$x \odot y =_{def} \neg(\neg x \oplus \neg y)$$

$$x \ominus y =_{def} x \odot \neg y$$

$$x \vee y =_{def} \neg(\neg x \oplus y) \oplus y = (x \ominus y) \oplus y$$

$$x \wedge y =_{def} \neg(\neg x \odot y) \odot y = (x \oplus \neg y) \oplus y = \neg(\neg x \vee \neg y)$$

With these operations, the axioms MV5) and MV6) can be written as:

MV5') $x \oplus 1 = 1$

MV6') $x \vee y = y \vee x$

Note that in the MV-algebras $\mathbf{\Gamma}(G, u)$ we have that:

$1 = u$

$x \odot y = (x + y - u) \vee u$

$x \ominus y = (x - y) \vee 0$

For each MV-algebra A, $\mathbf{L}(A) = (A, \vee, \wedge, 0, 1)$ is a distributive lattice with smallest element 0 and greatest element 1. The corresponding order relation, which we call the *natural order* of A, is given by $x \leq y$ if and only if $\neg x \oplus y = 1$ (or equivalently, $x \ominus y = 0$), and the following relations hold in A:

(11) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$

An MV-algebra such that its natural order is total, is called an *MV-chain*.

Note that $\mathbf{L}(\mathbf{\Gamma}(G, u))$ coincides with the underlying lattice of the l- group G.

Since $x \wedge y = \neg(\neg x \vee \neg y)$; the system $< A, \vee, \wedge, \neg, 0, 1 >$ is a *De Morgan algebra*. As a matter of fact, it is a *Kleene algebra*, i.e. it satisfies the condition $x \wedge \neg x \leq y \vee \neg y$ (see [1] for details on De Morgan and Kleene algebras).

Let A be an MV-algebra. The Boolean algebra formed by the complemented elements of the lattice $\mathbf{L}(A)$ will be denoted by $\mathbf{B}(A)$. If $x \in \mathbf{B}(A)$, then the complement of x is $\neg x$. More precisely, we have:

Theorem 2.2 *The following are equivalent conditions for each element x of an MV-algebra A:*

1. $x \in \mathbf{B}(A)$.
2. $x \vee \neg x = 1$
3. $x \wedge \neg x = 0$
4. *For each $y \in A$, $x \vee y = x \oplus y$*
5. *For each $y \in A$, $x \wedge y = x \odot y$*

The above theorem implies that $\mathbf{B}(A)$ is a subalgebra of the MV-algebra A. In particular, it follows that *Boolean algebras can be characterized as the MV- algebras which satisfy the equation $x \oplus x = x$.*

MV-algebras have been considered by several authors under different presentations. We are going to describe a few of them.

A *Wajsberg algebra* (Rodríguez [37, 20]), or an *NC-algebra* (Komori [23, 25]) is an algebra $(A, \rightarrow \neg, 1)$ with a binary operation $\rightarrow$, a unary operation $\neg$ and a constant 1 fulfilling the following equations:

W1) $x \to (y \to x) = 1$

W2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$

W3) $(x \to y) \to y = (y \to x) \to x$

W4) $(\neg x \to \neg y) \to (y \to x) = 1$

Theorem 2.3 ([20]) *If $(A, \oplus, \neg, 0)$ is an MV-algebra and we define the binary operation $\to$ by $x \to y =_{def} \neg x \oplus y$, then the system $(A, \to, \neg, 1)$ is a Wajsberg algebra, and $\neg x \to y = x \oplus y$. Conversely, if $(A, \to, \neg, 1)$ is a Wajsberg algebra, and if we define $0 =_{def} \neg 1$ and $x \oplus y =_{def} \neg x \to y$, then the system $(A, \oplus, \neg, 0)$ is an MV-algebra, and $x = \neg x \oplus y \to y$.*

A *bounded commutative BCK-algebra* [47, 22] is an algebra $(A, *, 0, 1)$ with a binary operation $*$ and two constants 0 and 1 fulfilling the following equations:

Y1) $(x * y) * z = (x * z) * y$

Y2) $x * (x * y) = y * (y * x)$

Y3) $x * x = 0$

Y4) $x * 0 = x$

Y5) $x * 1 = 0$

Bounded commutative BCK-algebras were considered by several authors, (see [44, 38, 39, 46]. Bosbash [7] considered them under the name of *bricks*. We refer the reader to [4] for an interesting account of the relations between BCK-algebras and partially ordered grupoids (see also [46]).

Theorem 2.4 ([20, 34]) *For each MV-algebra $(A, \oplus, \neg, 0)$, the system $(A, \ominus, 0, 1)$ is a bounded commutative BCK-algebra. Moreover, $\neg x = 1 \ominus x$ and $x \oplus y = 1 \ominus ((1 \ominus x) \ominus y))$. Conversely, if $(A, *, 0, 1)$ is a bounded commutative BCK-algebra, and if we define $\neg x =_{def} 1 * x$ and $x \oplus y =_{def} 1 * ((1 * x) * y)$, then $(A, \oplus, \neg, 0)$ is an MV-algebra, and $x \ominus y = x * y$.*

3 MV-algebras and lattice ordered abelian groups.

An *ideal* of an MV-algebra A is a subset I of A fulfilling the following conditions:

I1) $0 \in I$.

I2) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$.

I3) If x, y are in I, then $x \oplus y \in I$.

By (11) it follows that each ideal of A is an ideal of the lattice $\mathbf{L}(A)$. An ideal I of A is called *prime* provided that it is prime as an ideal of $\mathbf{L}(A)$: $I \neq A$ and $x \wedge y \in I$ implies $x \in I$ or $y \in I$. If C is either a distributive lattice or an MV-algebra, then $Spec(C)$ will denote the set of prime ideals of C.

Let A be an MV-algebra. The set of ideals of A, ordered by inclusion, is an algebraic lattice, which we denote by $\mathcal{I}(A)$. Let $\mathbf{Con}(A)$ be the algebraic lattice of all congruence relations on A. Chang [9] proved that the correspondence $\Theta \mapsto J(\Theta) = 0/\Theta = \{x \in A \mid (x, 0) \in \Theta\}$ establishes an isomorphism J from $\mathbf{Con}(A)$ onto $\mathcal{I}(A)$. The inverse of J is given by: $J^{-1}(I) = \{(x, y) \in A \times A \mid (x \ominus y) \oplus (y \ominus x) \in I\}$, for each ideal I of A. As usuqll, we are going to write A/I in place of $A/J^{-1}(I)$.

Chang [9] proved that an ideal I of an MV-algebra A is prime if and only if the quotient A/I is an MV-chain, and that the intersection of all prime ideals of A is the trivial ideal $\{0\}$. From these results, by standard techniques of universal algebra, he obtained:

Theorem 3.1 **(Chang)** *Each non-trivial MV-algebra A is a subdirect product of MV-chains.*

Let A be a totally ordered MV-algebra. On the set $\mathbf{Z} \times A \setminus \{1\}$ define the binary operation $+$, the unary operation $-$ and the binary relation $\leq$ as follows:

$$(m, x) + (n, y) = \begin{cases} (m + n, x \oplus y) & \text{if } x \oplus y < 1 \\ (m + n + 1, x \odot y) & \text{if } x \oplus y = 1 \end{cases}$$

$$-(m, x) = \begin{cases} (-m, 0) & \text{if } x = 0 \\ (-(m + 1), \neg x) & \text{if } 0 < x < 1 \end{cases}$$

$$(m, x) \leq (n, y) \text{ if and only if } m < n \text{ or } m = n \text{ and } x \leq y$$

The following is a fundamental result of Chang [10]:

Theorem 3.2 **(Chang)** *For each totally ordered MV-algebra A, the system* $G_A = (\mathbf{Z} \times A \setminus \{1\}, +, -, (0,0), \leq)$ *is a totally ordered abelian group, and* $\Gamma(G_A, (1, 0)) \cong A$.

By taking into account that the direct product of a family of totally ordered groups is a lattice ordered group, Lacava [26] observed that from theorems 3.1 and 3.2 it follows that each MV-algebra is of the form $\Gamma(G, u)$ for a suitable lattice ordered abelian group G and a suitable $0 < u \in G$. This result was perfectioned by Mundici [32] as follows.

Recall that an element u in an l-group G is called a *unit* provided that for each x in G there is a natural number n such that $|x| < nu$. Let $\mathcal{G}$ be the category whose objects are the pairs (G,u) such that G is an l-group and u is a unit of G, and whose morphisms are the l-groups homomorphisms which preserve the corresponding units. Let $\mathcal{M}$ be the category of MV-algebras and homomorphisms.

We already noted that for each object (G,u) in $\mathcal{G}$, $\Gamma(G,u)$ is an object in $\mathcal{M}$. Let (G,u) and (H,v) be objects in $\mathcal{G}$ and $h: G \to H$ an l-group homomorphis such that $h(u) = v$. It is easy to check that h maps $[0,u]$ into $[0,v]$. Hence, if we denote by $\Gamma(h)$ the restriction of h to $[0,u]$, we have that $\Gamma(h): \Gamma(G,u) \to \Gamma(H,v)$ is an MV-algebra homomorphism. It is plain that Γ is a functor from $\mathcal{G}$ into $\mathcal{M}$. Mundici [32] proved that Γ is invertible:

Theorem 3.3 (Mundici) *The functor Γ establishes a natural equivalence between the categories $\mathcal{G}$ and $\mathcal{M}$.*

The following theorem is proved in [2, Corollaire A.1.7]:

Theorem 3.4 *Each l-group is a homomorphic image of a sub l-group of a direct product of copies of $\mathbf{Z}$, the additive group of the integers.*

By using this result and Theorem 3.3, it is proved in [14] that each MV-algebra is an homomorphic image of a subalgebra of a product of MV-algebras of the form $\Gamma(\mathbf{Z},n)$, whith $0 < n \in \mathbf{Z}$.

Since for each $n \geq 2$, $\Gamma(\mathbf{Z},n-1) \cong \mathbf{L}_n$, it follows that the equational class of MV-algebras is generated by the algebras $\mathbf{L}_n$, for $n \leq 2$. On the other hand, it is easy to check that an equation holds in the MV-algebras $\mathbf{L}_n$, for all $n \leq 2$ if and only it holds in the algebra $\Gamma(\mathbf{Q},1)$ (cf 1.3). Then we have the following theorem, that was originally proved by Chang [10] by using theorems 3.1 and 3.2 and the following result from model theory: *a universal sentence of the first order language of totally ordered abelian groups holds in a totally ordereded abelian group if and only if it holds in* $\mathbf{Q}$.

Theorem 3.5 (Chang) *The equational class of MV-algebras is generated by the MV-algebra $\Gamma(\mathbf{Q},1)$, i.e. an equation holds in all MV-algebras if and only if it holds in $\Gamma(\mathbf{Q},1)$.*

Chang used Theorem 3.5 to prove the Lukasiewicz conjecture mentioned at the end of §1. We are going to sketch this proof.

On the set **Form** of propositional formulas, define the relation $\equiv$ as follows: $P \equiv Q$ if and only if $(P \to Q)$ and $(Q \to P)$ are both derivable from Lukasiewicz axioms L1)–L4) by the rule of modus ponens. Chang [9] (see also [32, §4]) proved that $\equiv$ is an equivalence relation on **Form**, and that the quotient **Form**$/\equiv$ becomes an MV-algebra with the following definition of the operations:

$$P/\equiv \oplus Q/\equiv =_{def} (\neg P \to Q)/\equiv$$

$$\neg(P/\equiv) =_{def} (\neg P)/\equiv$$

$$0 =_{def} \neg(p_0 \to p_0)/\equiv .$$

Moreover Chang showed that for each $P \in \mathbf{Form}$, $P/\equiv = 1$ if and only if P is derivable from L1)–L4) by modus ponens.

Hence to prove Lukasiewicz conjecture we have to prove that if a propositional formula P is an $L_{\aleph_0}$-tautology, then the equivalence class $P/\equiv = 1$.

Suppose that $p_{i_1}, \ldots p_{i_k}$ are the propositional variables which occur in a formula P. If we replace each occurrence of $p_{i_j}/\equiv$ in $P/\equiv$ by the symbol x_j, $j = 1, \ldots k$, we obtain an expresion $\hat{P}(x_1, \ldots, x_k)$ that can be evaluated in any MV- algebra. Suppose $P/\equiv \neq 1$. Then the k-variable equation $\hat{P} = 1$ does not hold in the MV-algebra $\mathbf{Form}/\equiv$, and by Theorem 3.5, it does not hold in $\Gamma(\mathbf{Q}, 1)$. Therefore there are rational numbers $r_1, \ldots, r_k$ such that $\hat{P}(r_1, \ldots, r_k) \neq 1$. If $f : \mathbf{Var} \longrightarrow [0,1]$ is any function such that $f(p_{i_j}) = r_j$, for $j = 1, \ldots k$, then it is easy to check that $\nu_f(P) = \hat{P}(r_1, \ldots, r_k)$ (see Theorem 1.1). Consequently, the propositional formula P is not an $\mathbf{L}_{\aleph_0}$-tautology. This completes the proof of Lukasiewicz conjecture.

The MV-algebra $\mathbf{Form}/\equiv$ is called *the Lindembaum algebra of the infinite-valued Lukasiewicz propositional calculus*, and will be denoted by $\mathbf{L}$. As a matter of fact, $\mathbf{L}$ is the free MV-algebra on a denumerable set of generators. More precisely, the equivalence classes of the propositional variables form a set of free generators of $\mathbf{L}$.

By Theorem 3.3 there is an l-group $\mathbf{M}$ and a unit $u \in \mathbf{M}$ such that $\mathbf{L} \cong \Gamma(\mathbf{M}, u)$, and the pair $(\mathbf{M}, u)$ is unique up to isomorphisms in the category $\mathcal{G}$.

The following description of the l-group $\mathbf{M}$ is due to Mundici [32].

A function $f : [0,1]^n \longrightarrow \mathbf{R}$ is called a *McNaughton function over* $[0,1]^n$ provided it satisfies the following conditions:

1. f is continuous.

2. There is a finite number of distinct polynomials of degree one and integral coefficients $\lambda_1, \ldots, \lambda_k$ such that for each $(x_1, \ldots, x_n) \in [0,1]^n$, there is $i \in \{1, \ldots, k\}$ such that $f(x_1, \ldots, x_n) = \lambda_i(x_1, \ldots, x_n)$.

A function $g : [0,1]^\omega \longrightarrow \mathbf{R}$ is called a *McNaughton function over* $[0,1]^\omega$ if for some integer $n \geq 1$, there is a McNaughton function f over $[0,1]^n$ such that for each sequence $\mathbf{x} = (x_0, x_1, \ldots) \in [0,1]^\omega$, $g(\mathbf{x}) = f(x_0, \ldots, x_{n-1})$.

Theorem 3.6 (Mundici) *The MV-algebra* $\mathbf{L} \cong \Gamma(\mathbf{M}, u)$, *where* $\mathbf{M}$ *is the l- group formed by the McNaughton functions over* $[0,1]^\omega$ *with pointwise operations, and* u *is the constant function* 1.

The proof of the above theorem given in [32, Theorem 4.5] depends on a classical theorem of McNaughton [30] which relates propositional formulas on n-variables with McNaughton functions over $[0,1]^n$. The original proof of this theorem is not constructive. Recently, Mundici [36] gave a constructive proof of McNaughton theorem.

4 Equational classes of MV-algebras.

An *l-ideal* of an l-group G is a subgroup I such that if $x \in I$, $y \in G$ and $|y| \leq |x|$, then $y \in I$. The set of l-ideals of an l-group G, ordered by inclusion, is an algebraic lattice, that we will denote by $\mathcal{I}(G)$. It is well known that $\mathcal{I}(G)$ is isomorphic to the congruence lattice of the l-group G.

The next theorem is proved in [18](see also [26]):

Theorem 4.1 *Let G be a lattice-ordered abelian group and u a unit of G. The correspondence $J \mapsto \phi(J) = \{x \in G : |x| \wedge u \in J\}$ defines an isomorphism from the poset $\mathcal{I}(A)$ of ideals of the MV-algebra $A = \Gamma(G, u)$ onto the poset $\mathcal{I}(G)$ of l-ideals of G. The inverse isomorphism is given by the correspondence $H \mapsto \psi(H) = H \cap [0, u]$.*

A totally ordered group is called *archimedean* if for each pair of elements x, y in G such taht $0 < x < y$, there is a natural number n such that $y < nx$. A theorem of Hölder (see [3] or [2]) asserts that a toally ordered group is archimedean if and only if it is isomorphic to a subgroup of $\mathbf{R}$.

Recall that an algebra is called *simple* in case it has exactly two congruence relations. In particular, if C denotes either an MV-algebra or an l-group, then we have that C is simple if and only if $\mathcal{I}(C) = \{\{0\}, C\}$.

It is well known (and easy to check) than an l-group is simple if and only if it is totally ordered and archimedean. Then from theorems 3.3 and 4.1 and Hölder's theorem we obtain:

Theorem 4.2 (Chang) *An MV-algebra A is simple if and only if there is a subgroup S of $\mathbf{R}$ such that $A \cong \Gamma(S, 1)$.*

The above theorem asserts that, essentially, the simple MV-algebras are the subalgebras of $\Gamma(\mathbf{R}, 1)$, i.e. the MV-algebras that we considered in §1. In particular, the algebras $\mathbf{L}_n \cong \Gamma((1/n-1)\mathbf{Z}, 1)$ are simple for each $n \geq 2$.

An algebra is called *semisimple* if it is a subdirect product of simple algebras. Since $\Gamma(\mathbf{Q}, 1)$ is a simple MV-algebra, Theorem 3.5 implies that *the equational class of MV-algebras is generated by a simple algebra.* Despite this fact, there are MV-algebras which are not semisimple. Indeed, we are going to produce a family of subdirectly irreducible but not simple MV-algebras.

Given an l-group G, let $\Lambda(G)$ be the lexicographic product $\mathbf{Z} \otimes A$. It is well known that $\Lambda(G)$ is an l-group (see, for instance, [3, Chapter XIII, §2, Lemma 3]). For each l-group homomorphism $h : G \longrightarrow H$, define $\Lambda(h) : \Lambda(G) \longrightarrow \Lambda(H)$ by the prescription $\Lambda(h)((m, a)) = (m, h(a))$ for each $(m, a) \in \mathbf{Z} \otimes G$. It is easy to verify that Λ is a functor from the category l-groups into itself.

For each integer $n \geq 1$ and each x of an l-group G, (n, x) is a unit of $\Lambda(G)$, and hence $\Gamma(\Lambda(G), (n, x))$ is an MV-algebra.

The algebra $\Gamma(\Lambda(G),(n,x))$ can be described as the set

$$\{(0,t) \mid 0 \leq t \in G\} \cup \bigcup_{i=1}^{n-2} \{(i,t) \mid t \in G\} \cup \{(n-1,t) \mid t \in G, t \leq x\}$$

with the operations:

$$(i,s) \oplus (j,t) = \begin{cases} (i+j, s+t) & if \quad i+j < n-1 \\ (n-1, \min\{x, s+t\}) & if \quad i+j = n-1 \\ (n-1, x) & if \quad i+j > n-1 \end{cases}$$

$$\neg(i,t) = (n-1-i, x-t)$$

and

$$0 = (0,0)$$

(see [38, 39, 25]). It is easy to check that the correspondence $J \mapsto \{(0,t) \mid 0 \leq t \in J\}$ establishes an order isomorphism from $\mathcal{I}(G)$ onto the set of *proper* ideals of the MV-algebra $A = \Gamma(\Lambda(G),(n,x))$. In particular, $\{(0,t) \mid t \in G\}$ is the only maximal ideal of A.

For the above remarks we get that when G is a subgroup of $\mathbf{R}$, $\Gamma(\Lambda(G),(n,0))$ is a subdirectly irreducible but non simple MV- algebra.

For each $n \geq 2$, let $\mathbf{K}_n = \Gamma(\Lambda(\mathbf{Z}),(n-1,0))$. The algebra $\mathbf{K}_2$ was introduced by Chang [9] as an example of a non semisimple MV-algebra. These algebras, together with the algebras $\mathbf{L}_n$ play an important role in the characterization of the equational classes of MV- algebras. Indeed, we have the following:

Theorem 4.3 (Komori) *For each proper and not trivial equational subclass C of the equational class of MV-algebras, there are two finite sets I, J of natural numbers such that $I \cup J \neq \emptyset$ and the algebras $\{\mathbf{L}_{n_i}\}_{i \in I}$ and $\{\mathbf{K}_{n_j}\}_{j \in J}$ generate the class C.*

To prove the above theorem, Komori [25] uses the first order theory of a class of totally ordered abelian groups that he had expressly introduced in [24]. In [14], it is shown how Theorem 4.3 can be derived in an algebraic way from theorems 3.3 and 3.4.

For each $n \geq 2$, the equational class of MV-algebras generated by the algebra $\mathbf{L}_n$ is denoted by $\mathcal{V}_n$. The classes $\mathcal{V}_n$ were studied by Grigoglia [21]. In particular, he gave sets of equations to characterize each of them. These sets of equations can be considered as axiomatizations of the corresponding n-valued Lukasiewicz systems of propositional calculi. Observe that $\mathcal{V}_2$, which coincides with the class of Boolean algebras, is characterized by the equation $x \oplus x = x$. The class $\mathcal{V}_3$, the class of *Lukasiewicz three-valued algebras*, is characterized by the equation $x \oplus x \oplus x = x \oplus x$. For $n \geq 4$, more complex systems of equations are requiered to characterize the class $\mathcal{V}_n$ (see also [37]).

The classes $\mathcal{V}_n$ have been considered in [12, 13] as Heyting algebras with some additional unary and binary operations, under the name of *proper n-valued Lukasiewicz algebras*.

Recently Di Nola and Lettieri [19] gave equational characterizations for all equational classes of MV-algebras.

5 Order structure of MV-algebras.

For each integer $n \geq 2$, the algebra $\mathbf{L}_n$ is a totally ordered MV-algebra with exactly n elements. Moreover, it is not hard to see that each totally ordered set C with $n \geq 2$ elements admits a unique structure of an MV-algebra, and that C with this structure is isomorphic to $\mathbf{L}_n$ (see, for instance, [20, Theorem 19]).

Since each finite MV-algebra is a direct product of a finite number of finite MV-chains (see, for instance,[44] or [20]), it follows that a finite partially ordered set admits an MV- algebra structure if and only if it is a direct product of a finite number of finite chains, and in this case the MV-algebra structure is uniquely determined.

On the other hand, there are infinite chains admitting two non isomorphic MV-algebras structures. To give examples, we need the following well known (and easy to prove) result:

Lemma 5.1 *Let S and T be two subgroups of $\mathbf{R}$ such that $1 \in S$ and $1 \in T$. There is an ordered-group isomorphism h from S onto T such that $h(1) = 1$ if and only if $S = T$, and in this case h is the identity.*

Corollary 5.2 *If S and T are as in the Lemma, we have that $\Gamma(S,1) \cong \Gamma(T,1)$ if and only if $\Gamma(S,1) = \Gamma(T,1)$, i.e. two subalgebras of the algebra $\Gamma(\mathbf{R},\mathbf{1})$ are isomorphic if and only if they are equal.*

By a classical result of Cantor, every two countable dense totally ordered sets with no first or last element are isomorphic. Therefore $\Gamma(\mathbf{Q},\mathbf{1})$ and $\Gamma(\mathbf{A},\mathbf{1})$ are order isomorphic, but by the above corollary, they are not isomorphic MV-algebras. Therefore we have an infinite chain admitting two non isomorphic MV-algebra structures.

Note that 1 plays an essential role in Lemma 5.1. Indeed, the correspondence $p \mapsto p/(n-1)$ defines an ordered-group isomorphism from $\mathbf{Z}$ onto $(1/n-1)\mathbf{Z}$ which maps the unit $(n-1)$ to 1. Hence we have $\Gamma((1/n-1)\mathbf{Z},1) \cong \Gamma(\mathbf{Z},\mathbf{n-1}) \cong \mathbf{L}_n$.

Another consequence of Corollary 5.2 is that *the set of (non isomorphic) simple algebras is non denumerable.* Indeed, for each irrational number α such that $0 < \alpha < 1$, $S_\alpha = \{m + n\alpha \mid m,n \in \mathbf{Z} \textit{ and } 0 \leq m + n\alpha \leq 1\}$ is a subalgebra of $\Gamma(\mathbf{R},\mathbf{1})$, and $S_\alpha = S_\beta$ if and only if $\alpha = \beta$ or $\alpha = 1 - \beta$.

Boolean algebras are simple examples of MV-algebras that are uniquely determined by their natural orders. It is proved in [15] that the algebras in the equational classes $\mathcal{V}_3$ and $\mathcal{V}_4$ are also uniquely determined by their natural orders.

In [17] we consider a class of MV-algebras which are uniquely determined by their natural orders, and that contains all the finite MV-algebras and, more generally, all the algebras in the equational classes $\mathcal{V}_n$, for each $n \geq 2$.

An MV-algebra A is called *liminar* provided A/J is finite for each prime ideal J of A.

The main result of [17] is as follows:

Theorem 5.3 *Let A be a liminar MV-algebra. Then $L = \mathbf{L}(A)$ is a bounded distributive lattice which satisfies the following two conditions:*

i) *The prime ideals of L occur in disjoint finite chains.*

ii) *For each $f \in L$ and each $r \in \mathbf{Q}$, there is an element $b \in \mathbf{B}(L)$ such that for each minimal prime ideal J, $b \in J$ if and only if*

$$r = \frac{\mathrm{card}\{K \in \mathrm{Spec}(L) \mid J \subseteq K \text{ and} f \notin J\}}{\mathrm{card}\{K \in \mathrm{Spec}(L) \mid J \subseteq K\}}$$

Conversely, if a bounded distributive lattice L satisfies conditions i) and ii), then there is an MV-algebra A, unique up to isomorphisms, such that $\mathbf{L}(A) \cong L$, and moreover, A is liminar.

The class of liminary MV-algebras is in correspondence with the class of *liminary C^*- algebras with Boolean spectra.* See [17] for details.

It is shown in [18] that an MV-algebra is liminar if and only if it is a Boolean product of a family of finite MV-chains.

6 The prime spectra of MV-algebras.

Recall that if C denotes either an MV-algebra or a bounded distributive lattice *Spec*(C) denotes the set of prime ideals of C. Analogously, if C stands for an l-group, *Spec*(C) denotes the set of prime l-ideals of C. In any case, *Spec*(C) is the set of meet-irreducible elements of the algebraic lattice $\mathcal{I}(C)$. We are going to consider *Spec*(C) ordered by inclusion.

We say that an ordered set X is *MV-representable* provided that there is an MV-algebra A such that X is isomorphic to *Spec*(A).

It is easy to see that the mappings ϕ and ψ of Theorem 4.1 define an order isomorphism between *Spec*$(\Gamma(G,u))$ and *Spec*(G) for each unit u of the l-group G.

Therefore the ordered set X is MV-representable if and only if there is an l-group G with an order unit u such that X is isomorphic to *Spec*(G).

A *root system* is an ordered set X such that for each $z \in X$, $[z) = \{x \in X : x \geq z\}$ is a totally ordered subset of X. A *spectral root system* is a root system X fulfilling the following two conditions:

RS1) Each totally ordered subset of X has supremun and infimun in X.

RS2) If x, y are elements of X such that $x < y$, then there are s, t in X such that $x \leq s < t \leq y$, and there is no element of X between s and t.

The following theorem is proved in [18]:

Theorem 6.1 *A partially ordered set X is MV-representable if and only if it is a spectral root system.*

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R. Cignoli
Departamento de Matemàtica
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria
1428 Buenos Aires (Argentine)

There are 2^{χ_0} existentially closed non elementary equivalent countable groups.

Anatole Khelif

Plan of the proof

I We will prove that there are 2^{χ_0} models of ZFC such that their integers are the true integers and such that their second order arithmetic ((w, p(w)) are not elementary equivalent. We will admit the existence of an inaccessible cardinal and we will obtain 2^{χ_0} countable models without forcing. The hypothesis of the existence of an inaccessible cardinal can be eliminated if we replace ZFC by a large enough finite fragment of ZFC (We can prove in ZFC the existence of models of some finite fragment of ZFC).

II In all these models of ZFC we will find "existentially saturated" groups such that the second order arithmetics of these models are interpretable in these groups.

<u>LEMMA 1</u>

If we admit ZFC + "There exists an inaccessible cardinal", there are 2^{χ_0} models of ZFC such that their integers are the true integers and such that their second order arithmetic are not elementary equivalent.

<u>Proof :</u>

If we admit ZFC + "There is an inaccessible cardinal". There exists an uncountable standard model ($\in$ is well founded) M_o of ZFC + V=L. Thus by Löweneihem - Skölem there are χ_1 countable standard models of ZFC + V=L . Let A be the set of the countable standard models (up to isomorphism of ZFC + V=L. Let < the following relation : if M_1 and $M_2 \in A$, we have $M_1 < M_2$ if and only if every ordinal of M_1 is an ordinal of M_2 and $M_1 \neq M_2$. Then < induce a well order on A and (A,<) is isomorphic to ω_1 , $\in$). Since every element of A is isomorphic to an element of M_o, we can say that in " $M_o \models$ "There are uncountably many standard models of ZFC + V=L". Since Löweneihem Skölem is a theorem of ZFC, $M_o \models$" There are χ_1 standard countable models of ZFC + V=L". By Löweneihem Skölem in the universe (not in M_o). There is a countable standard model of ZFC + V=L which is elementary equivalent to M_o. We can conclude that the set { $x \in A$, $x \models$ "There are χ_1 countable standard models of ZFC + V=L"} is non empity. Let N_o be the smallest element of this set for the relation <.Then $N_o \models$ " There is not an uncountable standard model of ZFC + V=L" and $N_o \models$ " There are χ_1 countable standard models of ZFC + V=L".

But in ZFC + V=L, there exists a definable bijection from ω_1 into the reals. Let R be such a bijection, we can suppose that R is absolute. Then $N_o \models$ "Every standard model x of ZFC + V=L contains (relatively to x) countably many standard models (which are element of x) up to isomorphism of ZFC + V=L.

Since the set of these models, which are elements of x and A, is well ordered by < ; it is isomorphic to an ordinal α_X which is countable relatively to x. In this case, R (α_X) is a real which is an element of x.

If $x \neq y$ (x and y transitive models of ZFC + V=L and elements of N_o) R (α_X) $\neq$ R (α_y).

We know that two different reals have not the same binary development. Thus $N_o \models$" There are ω_1 standard models of ZFC + V=L which are not elementary equivalent even if we consider only the formulas with integers and reals.

But if we identify the set of the formulas to w, the set T of the second order arithmetics which are in models (not necessary standard) of ZFC where the integers are the true integers is an analytic subset of $\{0,1\}^{w}$. Thus the cardinal of T is either $\leq \chi_{o}$ or $2^{\chi_{o}}$.

But in N_{o} this cardinal is $\geq \chi_{1}$, thus it is $2^{\chi_{o}}$. By the absolutness of the cardinal of an analytic set ; we can conclude that, in the universe, there are $2^{\chi_{o}}$ countable models of ZFC + V=L such that their integers are the true integers and such that their second order arithmetics are not elementary equivalent.

LEMMA 2

Let M be a countable model of ZFC + V=L such that the integers in M are the true integers. Then there exists G existentially closed group such that $G \in M$ and such that the second order arithmetic of M is interpretable in G without parameters.

Proposition 1

If $G \in M$, G group and M model of ZFC containing the true arithmetic then $M \models$ "G is existentially closed iff G is existentially closed."

Proof :

G is existentially closed is equivalent to the fact that every existential formula which is consistant with the non quantified formulas with parameters in G and with the group theory holds in G. But consistent means finitely consistent. Since M contains the true arithmetic the consistentness in M and in the universe are equivalent. Thus if $G \in M$, G is existentially closed iff $M \models$" G is existentially closed." (N-B- if M doesn't contains the true arithmetic and $M \models$ "G is existentially closed;" G is never existentially closed because two elements of different non standard orders are not conjugate).

Poposition 2

Let us call " existentially saturated" a group G of cardinal $> \chi_o$ such that :

a) Every group M of cardinal < G has an injective embeddement in G

b) Let H and H' be sub-groups of G, if card H < card G and if ζ is an isomorphisme from H to H' there is $y \in G$ such that for every $x \in H$ $\zeta(x) = y^{-1} xy$.

It is obviously equivalent to the fact that every existential type on a subset of G of cardinal < card G is satisfied in G. If we admit the continuum hypothesis (which holds in every model of V=L) such a group exists and it is existentially closed. We can even say that in this case, up to isomorphism, there exists only one existentially saturated group of cardinal 2^{χ_o}.

If follows from the theorem of Löwenheimen Skölem and the fact that every elementary sub-structure of an existentially closed group is existentially closed that an " existentially saturated" group is the union of an increasing clain of existentially closed sub-groups. Thus it is existentially closed.

Proportion 3

Let M be a model of ZFC and $G \in$ such that $M \models$ "G is existentially saturated", then the second order arithmetic of M is interpretable in G without parameters.

Proof :

In an " existentially saturated" group, which is an existentially closed group, two elements of same order are conjugate. We can write that an element u of G is of infinite order by $u \neq e$

$\exists$ v, v^{-1} u $v=u^2$ (the order of u isn't pair)

$\exists$ w, $w^2 \neq Id \wedge u^{-1} wu = w^{-1}$ (the order of u isn't impair)

An element u' of G is a powerth of u (u^n, $n \in \mathbb{Z}$) if and only if u' belongs to the centralizator of the centralizator of u which we name CC(u). We can identily the conjugary classes of the couple (u, u') to relative integers. We can already say that the first order arithmetic is interpretable in G, the product of two elements of CC(n) can be assimilate to an addition and a morphism from CC(u) to it self inducted by a conjugaison is equivalent to a multiplication.

If u has an infinite order, for every subset P of $\mathbb{Z}$ (in M), there is an element v and an element w such that if u' $\in$ CC(u), w commutes to u'^{-1} vu' if and only if the class of (u,u') belongs to P. So, the second order arithmetic is interpretable in G without parameters.

Proof of lemma 2:

If G belongs to M and if the integers of M are the true integers, according to proportion 1, G is existentially closed. Thus, according to proposition 3, the second order arithmetic is interpretable in G without parameters.

Conclusion:

According to lemma 1, there is $2^{\aleph_0}$ countable models of ZFC + V=L countaing the true arithmetic such that their second order arithmetics are not elementary equivalent.

But in this models and relatively to them, the "existentially saturated" groups of cardinal $2^{\aleph_0}$ are not elementary equivalent.

Since these models are countable, these groups are countable. According to propositions 1 and 2, there are existentially closed. Thus, there exist $2^{\aleph_0}$ countable existentially closed groups which are not elementary equivalent.

Case of the division rings:

Theorem :

Let k be a countable field, there are $2^{\aleph_0}$ non elementary equivalent division rings over k.

Proof :

We have the amalgamation property for the division rings [1]. Then we can conclude that if we assume GCH, we have an "existentially saturated" division ring over k in the same sense that the "existentially saturated" groups. In an "existentially saturated" division ring A over k, x is transcendental over k if and only if there is u belonging to a such that $u^{-1}xu = x^2$ and if there exists y different from 1 such that $x^{-1}yx = y^{-1} \neq y$. This is also a consequence of the amalgamation property. If x is transcendental over k, x' is a power of x (there exists $n \in \mathbb{Z}$ such that $x^n = x'$) if and only if x' belongs to the centralizator of the centralizator of x and if (x^{α}, x'^{α}) is conjugate to (x, x') with $\alpha = 2$ if charasteristic of $k \neq 2$, $\alpha = 3$ if charasteristic of $k = 2$. Then, in the same way that for the "existentially saturated" groups we can interprete te first order arithmetic and the seond order arithmetic. By using lemma 1 and proposition 1 with "division ring" at the place of group we can conclude that there are $2^{\aleph_0}$ non elementary equivalent division rings over k.

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Joining $k-$ and $l-$ recognizable sets of natural numbers

ROGER VILLEMAIRE

Département de mathématiques et d'informatique
U.Q.A.M., C.P. 8888 succ. A, Montréal (Québec), CANADA H3C 3P8

Summary. We show that the first order theory of $< \mathbb{N}, +, V_k, V_l >$, where $V_r : \mathbb{N} \backslash \{0\} \rightarrow \mathbb{N}$ is the function which sends x to $V_r(x)$, the greatest power of r which divides x and k, l are multiplicatively independent (i.e. they have no common power) is undecidable. Actually we prove that multiplication is definable in $< \mathbb{N}, +, V_k, V_l >$. This shows that the theorem of Büchi cannot be generalized to a class containing all k- and all l-recognizable sets.

Introduction. As J.R. Büchi showed (see section 3.), a subset of $\mathbb{N}^n$ represented in base k is recognizable on the alphabet $\{0, 1, \ldots, k-1\}^n$ if and only if it is definable in the first-order theory of $< \mathbb{N}, +, V_k >$, where $V_k(x)$ is the greatest power of k which divides x. This shows that the class of k-recognizable subsets of $\mathbb{N}^n$ ($n \in \mathbb{N}$) is closed under intersection, complementation and projection. Hence a set is in the smallest class containing all k-recognizable sets and closed under intersection, complementation and projection if and only if it is definable in $< \mathbb{N}, +, V_k >$.

A. Joyal asked to which extend it could be possible to generalize the above result joining k- and l-automata. I proved that if one takes the smallest class closed under intersection, complementation and projection which contains all k- and all l-recognizable subsets of $\mathbb{N}^n$ ($n \in \mathbb{N}$) (hence the definable subsets of $< \mathbb{N}, +, V_k, V_l >$), then it contains multiplication. Therefore there is no machine specializing Turing machines by which exactly the sets in this class are recognized. Hence one cannot hope to generalize Büchi's theorem in this way.

In the first three sections we give definitions an results about automata, recognition and logic. In section 4. we reduce the main theorem to some technical result, which we prove in the last section.

1.Automata. Let Σ be an *alphabet*, i.e. a finite set. Σ^* will denote the set of *words* of finite length on Σ containing the *empty* word λ formed of no symbol. Any subset L of Σ will be called a *language* on the alphabet Σ.

DEFINITION.*Let Σ be an alphabet. A* Σ-automata $\mathcal{A}$ *is a quadruplet* (Q, q_0, Γ, T) *where*

Q is a finite set, called the set of states,

q_0 *is an element of* Q*, called the* initial state,
Γ *is a subset of* Q *called the set of* final states
and finally T *is a function of* $Q \times \Sigma$ *to* Q*, called the* transition function.

The transition function T can be extended to a function $T^* : Q \times \Sigma^* \to Q$ in the following way:

$$T^*(q, \sigma) = T(q, \sigma) \text{ for } \sigma \in \Sigma$$
$$T^*(q, \alpha\sigma) = T(T^*(q, \alpha), \sigma) \text{ for } \alpha \in \Sigma^* \text{ and } \sigma \in \Sigma$$

Furthermore we have the following definitions.

DEFINITION.*A word* $\alpha \in \Sigma^*$ *is said to be* accepted *by the* Σ*-automata* (Q, q_0, Γ, T) *if* $T^*(q_0, \alpha) \in \Gamma$.

DEFINITION.*A language* L *on* Σ *is said to be* Σ-recognizable *if there exists a* Σ*-automata such that the set of words accepted by this automata is exactly* L.

2.Recognition over $\mathbb{N}$**.** Let Σ_k be the alphabet $\{0, 1, \ldots, k-1\}$. For $n \in \mathbb{N}$ let $[n]_k$ be the word on Σ_k which is the inverse representation of n in base k, i.e. if $n = \Sigma_{i=0}^{s} \lambda_i k^i$ with $\lambda_i \in \{0, \ldots, k-1\}$, then $[n]_k = \lambda_0 \cdots \lambda_s$.

It is also possible to represent tuples of natural numbers by words on $(\Sigma_k^n)^*$ in the following way. Let $(m_1, \ldots, m_n) \in \mathbb{N}^n$. Add on the right of each $[m_i]_k$ the minimal number of 0 in order to make them all of the same length and call these words ω_i. Let $\omega_i = \lambda_{i1} \cdots \lambda_{is}$ where $\lambda_{ij} \in \Sigma_k$. We represent $(m_1, \ldots, m_n)$ by the word $(\lambda_{11}, \lambda_{21}, \ldots, \lambda_{n1})\ (\lambda_{12}, \lambda_{22}, \ldots, \lambda_{n2}) \cdots (\lambda_{1s}, \lambda_{2s}, \ldots, \lambda_{ns}) \in (\Sigma_k^n)^*$.

DEFINITION.*We say that a set* $X \subseteq \mathbb{N}^n$ *is k-recognizable if it is* Σ_k^n*-recognizable.*

3.Büchi's Theorem. Let $P_k(x)$ be the predicate (i.e. subset) on $\mathbb{N}$ defined by "x is a power of k". Let also as we said before $V_k : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ be the function which sends x to $V_k(x)$, the greatest power of k which divides x.

In [2,Theorem 9] Büchi states that a subset of $\mathbb{N}^n$ is k-recognizable if and only if it is definable in the first-order structure $< \mathbb{N}, +, P_k >$, i.e. defined by formulas built up from $=, +, P_k$ using $\wedge$ ("and"), $\neg$ ("not"), $\exists$ ("there exists a natural number such that ..."). Unfortunately, as remarked by McNaughton in [7], the proof is incorrect. Furthermore the statement has been disproved by Semenov in [11, Corollary 4]. Thanks to the work of Bruyère [1], we know that the ideas of Büchi can be used to show the following theorem. (See [1] for a proof among the lines of Büchi's, [8] for a different proof or also [14]).

THEOREM3.1.**Büchi's Theorem** *A set* $X \subseteq \mathbb{N}^n$ *is k-recognizable if and only if it is definable in the first order structure* $< \mathbb{N}, +, V_k >$.

There is another version of Büchi's Theorem in terms of weak monadic logic. Before we speak of it, let us give a useful definition and lemma.

DEFINITION.*Let $X_{k,j}(x,y)$ denote the relation* "x is a power of k and the corresponding digit in the representation of y in basis k is j", *for $k \in \mathbb{N}$ and $j \in \{0,1,\ldots,k-1\}$. We have the following result.*

LEMMA3.2.*The relation $X_{k,j}(x,y)$ is definable in $< \mathbb{N}, +, V_k >$ for $j = 1,\ldots,k-1$.*

PROOF:$X_{k,j}(x,y)$ is defined by the formula

$$V_k(x) = x \wedge \exists z, t\,[z < x \wedge V_k(t) > x \wedge y = z + jx + t] \vee \exists z[z < x \wedge y = z + jx]$$

Here jy represents $\underbrace{y + \cdots + y}_{j-\text{times}}$ which is a term in the language.

This holds since $V_k(x) = x$ is equivalent to x being a power of k and furthermore $z < x$ and $V_k(t) > x$ means that z has 0 as coefficient for all powers of k greater or equal to x and t has coefficient 0 for all powers of k smaller or equal to x.

Usually Büchi's Theorem is stated using the weak monadic theory of $< \mathbb{N}, S >$, where S is the successor function on $\mathbb{N}$. The weak monadic theory of $< \mathbb{N}, S >$ is the extension of first order logic by allowing also the use of the *weak monadic quantifiers* $\forall X$ and $\exists X$, which are interpreted as "*for all finite subsets of $\mathbb{N}$*" and "*there exists a finite subset of $\mathbb{N}$*" respectively. We will write WM $< \mathbb{N}, S >$ for this structure. Hence usually Büchi's Theorem is stated a follows.

THEOREM3.3.**Büchi's Theorem monadic version** *A set $X \subseteq \mathbb{N}^n$ is 2-recognizable if and only if it is definable in the weak monadic structure WM $< \mathbb{N}, S >$.*

Let us show that this second form of Büchi's Theorem is equivalent to the first one for $k = 2$. We will give an bi-interpretation of WM $< \mathbb{N}, S >$ in $< \mathbb{N}, +, V_2 >$. First of all, any formula $\varphi(X_1,\ldots,X_s,x_1,\ldots,x_t)$ of WM $< \mathbb{N}, S >$, where $X_1,\ldots,X_s$ are monadic variables (i.e. they represent finite sets) and $x_1,\ldots,x_t$ are first-order variables, is equivalent to a formula with no first-order variables, since one can replace an element by a singleton containing it. Let us now show that there is a bijection η between the subsets of $\mathbb{N}$ and the natural numbers such that for any formula $\varphi(X_1,\ldots,X_s)$, there exists a formula φ^* with the property that $\varphi(X_1,\ldots,X_s)$ holds in WM $< \mathbb{N}, S >$ if and only if $\varphi^*(\eta(X_1),\ldots,\eta(X_s))$ holds in $< \mathbb{N}, +, V_2 >$. And furthermore that for any formula $\psi(x_1,\ldots,x_s)$ there exists a formula ψ^* such that $\psi(x_1,\ldots,x_s)$ holds in $< \mathbb{N}, +, V_2 >$ if and only if $\psi^*(\eta^{-1}(x_1),\ldots,\eta^{-1}(x_s))$ holds in WM $< \mathbb{N}, S >$.

Define $\eta(X) = \sum_{i \in X} 2^i$ and let φ be a formula in the language of WM $< \mathbb{N}, S >$. Replace in it $S(n)$ by $2^n + 2^n$, $X(n)$ (i.e. $x \in X$) by $X_{2,1}(2^n, x)$ and $\exists X$, $\forall X$ by $\exists x$, $\forall x$ and call this new formula φ^*. It is easy to show that the above property holds.

Conversely starting from ψ a formula in the language of $< \mathbb{N}, +, V_2 >$, replace in it $x + y = z$ by

$$\exists R[X(0) \wedge Y(0) \leftrightarrow R(S(0))] \wedge$$

$\forall x(R(S(x)) \leftrightarrow$ " at least two of $X(x)$, $Y(x)$, $R(x)$ hold ")$\wedge$
$\forall x(Z(x) \leftrightarrow$ " only one or all three of $X(x)$, $Y(x)$, $R(x)$ hold ")

In this formula R stands for the " carry over " in the addition of $\sum_{i \in X} 2^i$ and $\sum_{i \in Y} 2^i$. This formula can be easily expressed in the language of WM $< \mathbb{N}, S >$.

Finally replace $V_2(x) = y$ by " Y is a singleton contained in X and for all $x \in \mathbb{N}$ smaller than the element of Y, $X(x)$ does not hold ".

Here also this can easily be expressed by a formula of WM $< \mathbb{N}, S >$ as soon as one note that $x < y$ for x, y natural numbers is equivalent of " every finite subset containing y and closed under the inverse of S must contain x ". The formula ψ^* so obtained has now the required property as one can easily check.

Note that in the translation of φ into φ^*, if we replace 2^n by k^n and $X_{2,1}$ by $X_{k,1}$ we get and interpretation of WM $< \mathbb{N}, S >$ in $< \mathbb{N}, +, V_k >$. We will use this fact later.

4. A question of A.Joyal.

DEFINITION. *Two natural numbers k, l are said to be* multiplicatively dependent *if there exists natural numbers n, m such that $k^n = l^m$.*

We have the following facts.

• If k and l are multiplicatively dependent then any set $X \subseteq \mathbb{N}$ which is k-recognizable is also l-recognizable (see [4, Corollary 3.7]).

• A set $X \subseteq \mathbb{N}$ which is a union of a finite set with finitely many arithmetic progressions is k-recognizable for any $k \in \mathbb{N}$ (see [4, Proposition 3.4)

• For k, l multiplicatively independent a set which is k- and l-recognizable is a finite union of a finite set with finitely many arithmetic progressions, hence it is m-recognizable for any m (see [3] or also [6] and [9]).

Therefore for k, l multiplicatively independent the class of k- and the class of l-recognizable sets of natural numbers are as far apart as they can be. This is quite unfortunate from a computational point of view, since recognition depends on the basis. A. Joyal asked if we can find a concept of "machine" and of "recognition" extending k-recognition and l-recognition for k, l multiplicatively independent.

Let $\mathcal{K}$ be the smallest class containing all k-recognizable and all l-recognizable subsets of $\mathbb{N}^n$ (for $n \in \mathbb{N}$) and closed under intersection, complementation and projection. We show that $\mathcal{K}$ contains all the arithmetical hierachy (i.e. the closure of the class of recursive relations under projection and complement), hence that there is no machine model specializing Turing machines by which exactly the sets in $\mathcal{K}$ are recognized. More precisely we show the following.

THEOREM4.1.*The structures* $< I\!N, +, V_k, V_l >$ *for* k, l *multiplicatively independent and* $< I\!N, +, \cdot >$ *are inter-definable (i.e. multiplication can be defined in terms of* V_k, V_l *and* V_k, V_l *can be defined in terms of* $\cdot$, $+$*).*

PROOF:Since any recursive function is definable in $< \mathbb{N}, +, \cdot >$ and V_k,V_l are recursive, it follows that V_k,V_l are definable in $< \mathbb{N}, +, \cdot >$. This settles one direction.

For the other direction let $k^{\mathbb{N}}$ be the set of powers of k. We will first show the following results.

LEMMA4.2.*For any* k, l *multiplicatively independent there exists a strictly increasing function* $h : k^{\mathbb{N}} \to k^{\mathbb{N}}$ *definable in* $< I\!N, +, V_k, V_l >$ *such that the following condition holds.*

$(*)$ $h(k \cdot x) > k \cdot (h(x))$ *for infinitely many* $x \in k^{\mathbb{N}}$ *and furthermore there exists a* $d \in I\!N$ *such that for any consecutive power of* k, k^n,k^m *satisfying the above inequality* $m - n \leq d$.

We will give the proof of Lemma4.2 in the last section.

COROLLARY4.3.*Let* $h : k^{\mathbb{N}} \to k^{\mathbb{N}}$ *be a strictly increasing function satisfying* $(*)$*. The multiplication of powers of* k *is definable in* $< I\!N, +, V_k, h >$.

PROOF:We first need to extend the interpretation of WM $< \mathbb{N}, S >$ in $< \mathbb{N}, +, V_k >$ we gave in section 3. to an interpretation of WM $< \mathbb{N}, S, h^* >$ in $< \mathbb{N}, +, V_k, h >$, where $h^* : k^{\mathbb{N}} \to k^{\mathbb{N}}$ is defined by $h(k^n) = k^{h^*(n)}$. This is easily obtain by replacing h^* by h. Note now that addition is WM $< \mathbb{N}, S, h^* >$ will be interpreted as the multiplication of powers of k in $< \mathbb{N}, +, V_k, h >$.

Using this interpretation it is sufficient to show the following.

LEMMA4.4.*Let* $h^* : I\!N \to I\!N$ *be a strictly increasing function such that* $h^*(S(x)) > S(h^*(x))$ *for infinitely many* $x \in I\!N$*. Suppose furthermore that there exists a* $d \in I\!N$ *such that for any consecutive natural numbers* x, y *satisfying the above inequality* $x - y \leq d$*. Then the addition of natural numbers is definable in WM* $< I\!N, S >$.

PROOF:This lemma is a slight generalization of the result [13, Theorem 2] of W. Thomas. We will follow the proof of Thomas modifing it to prove the above lemma. The technique is due to C.C. Elgot and M.O. Rabin and the interested reader should have a look at their nice paper [5].

The first important fact is to notice that if we can quantify over finite binary relations over $\mathbb{N}$ then we can define addition in the following way.

$$\forall E \subseteq \mathbb{N} \times \mathbb{N} \; [(x, y) \in E \wedge \forall u, v \; (u, v) \in E \to (u + 1, v - 1) \in E] \to (z, 0) \in E$$

The above formula holds if and only if $x+y=z$, since the part in bracket means that E contains $\{(x,y),(x+1,y-1),\ldots,(x+y,0)\}$. Hence if each finite binary relation satisfying this condition contains $(z,0)$, we must have that $z=x+y$.

Therefore we want to show that there is in WM $<\mathbb{N},S,h^*>$ a formula $F(X,x,y)$, such that for any finite binary relation $E\subseteq\mathbb{N}\times\mathbb{N}$ there is $X^E\subseteq\mathbb{N}$ for which $F(X^E,x,y)$ holds in WM $<\mathbb{N},S,h^*>$ if and only if $(x,y)\in E$.

Suppose we can define in WM $<\mathbb{N},S,h^*>$ disjoint sets K_i which union equals $\mathbb{N}$ and one-to-one functions $f_i:\mathbb{N}\rightarrow K_i$, $i=1,\ldots,d$ such that $f_i^{-1}(x)$ is infinite for all $x\in\mathbb{N}$. We will show that this implies the existence of an $F(X,x,y)$ with the above property.

Let us see how we can define quantification over finite binary relations in WM $<\mathbb{N},S,f_i,K_i;i=1,\ldots,d>$. We need the following definitions.

- $Nxt(X,x,y)=X(x)\wedge X(y)\wedge x<y\wedge\forall z[x<z<y\rightarrow\neg X(z)]$.

Hence $Nxt(X,x,y)$ means that x and y are in X and that y is the successor of x in this set, i.e. there is no element of X between x and y.

- $Od(X,x)=X(x)\wedge\exists Y\ Y(x)\wedge$ " Y contains the smallest element of X " $\wedge$ $\forall y,z,t[Nxt(X,y,z)\wedge Nxt(X,z,t)\wedge Y(y)\rightarrow\neg Y(z)\wedge Y(t)]$.

This can be written as a first-order formula and it means that x is in X and that there is a odd number of elements in this set which are smaller or equal to x.

We can now define $F(X,x,y)$ by the following formula.

$\exists u,v[X(u)\wedge X(v)\wedge Od(X,u)\wedge Nxt(X,u,v)\wedge\bigwedge_{i=1}^{d}(K_i(x)\rightarrow f_i(u)=x)\wedge\bigwedge_{i=1}^{d}(K_i(y)\rightarrow f_i(v)=y)]$.

To see that this formula has the required property, let $E=\{(x_1,y_1),\ldots,(x_k,y_k)\}$ be an arbitrary finite relation on $\mathbb{N}$. Let x_i be in $K_{\alpha(i)}$ and y_i be in $K_{\beta(i)}$. Take n_1 to be such that $f_{\alpha(1)}(n_1)=x_1$, choose $n_2>n_1$ such that $f_{\beta(1)}(n_2)=y_1$; this is possible since $f_{\beta_1}^{-1}(y_1)$ is infinite. Choose $n_3>n_2$ such that $f_{\alpha(2)}(n_3)=x_2$ and so on up to n_{2k}. Let $X^E=\{n_1,\ldots,n_{2k}\}$. Then $F(X^E,x,y)$ holds if and only if $(x,y)\in E$.

Therefore the last thing to show is that we can define such f_i and K_i. Let $K_i=\{x\in\mathbb{N};S(h^*(x))\notin\mathrm{Im}h^*\}$, $K_2=\{x\in\mathbb{N};x\notin K_1$ and $S^{(2)}(h^*(x))\notin\mathrm{Im}h^*\}$, $\ldots$, $K_d=\{x\in\mathbb{N};x\notin K_1,\ldots,x\notin K_{d-1}$ and $S^{(d)}(h^*(x))\notin\mathrm{Im}h^*\}$ (here $S(i)$ is the iteration of the function S). Furthermore let $f_i:\mathbb{N}\rightarrow K_i$ be defined in the following way. Let y_i be the first element of K_i.

$$f_i(x)=\begin{cases} y & \text{if } x=h^{*(m)}(S^{(i)}(h^*(y))) \text{ for some } m\in\mathbb{N} \text{ and some } y\in K_i,\\ y_i & \text{otherwise.}\end{cases}$$

It is clear by definition that the sets K_i are disjoint and by $(*)$ that their union is $\mathbb{N}$. Let us show that the functions f_i are well defined. Suppose that $h^{*(m)}(S^{(i)}(h^*(y)))=h^{*(m')}(S^{(i)}(h^*(y')))$ for some m, $m'\in\mathbb{N}$ and y, $y'\in K_i$. Since h^* is one-to-one it follows that $h^{*(m-m')}(S^{(i)}(h^*(y)))=S^{(i)}(h^*(y'))$ (we can suppose without loss of generality that $m>m'$). Since $S^{(i)}(h^*(y'))$ is not in $\mathrm{Im}h^*$

by definition of K_i, we must have that $m = m'$. Hence $S^{(i)}(h^*(y)) = S^{(i)}(h^*(y'))$ and $h^*(y) = h^*(y')$, therefore $y = y'$.

Furthermore the sets $f_i^{-1}(x) = \{y; y = h^{*(m)}(S^{(i)}(h^*(y))), m = 0, 1, \ldots\}$ is infinite for every $x \in \mathbb{N}$ and $i = 1, \ldots, d$. Finally, $f_i(x)$ is definable in WM $< \mathbb{N}, S, h^* >$ by the following formula.

$$\forall X[X(x) \wedge \forall z[X(h(z)) \rightarrow X(z)] \rightarrow X(S^{(i)}(h^*(y)))].$$

This holds since the formula says that any X which contains x and is closed under the inverse of h^* must contain $S^{(i)}(h^*(y))$. This completes the proof.

Let α, β be two words on Σ_k. The concatenation $\alpha^\wedge\beta$ is the word obtained by the letters of α followed by the letters of β. For n, m in $\mathbb{N}$ we will write $n^\wedge m$ for the natural number which corresponds to the concatenation $[n]_k^\wedge[m]_k$. More precisely if $l(n)$ is the length of $[n]_k$ then $n^\wedge m = n + k^{l(n)} \cdot m$.

We now have the following result.

LEMMA4.5.*The concatenation in base k is definable in $< \mathbb{N}, +, V_k, h >$.*

PROOF:$z = x^\wedge y$ holds if and only if

$\exists u(V_k(u) = u \wedge u > x \wedge$"$u$ is the smallest natural number with this property"
$\wedge \forall t[t < u \wedge V_k(t) = t \rightarrow \bigwedge_{j=0}^{k-1}(X_{kj}(t, z) \leftrightarrow X_{kj}(t, x))]$
$\wedge \forall t[V_k(t) = t \rightarrow \bigwedge_{j=0}^{k-1}(X_{kj}(t \cdot u, z) \leftrightarrow X_{kj}(t, y))])$

Since $t \cdot u$ is a product of powers of k this formula define the concatenation in $< \mathbb{N}, +, V_k, h >$ (by Lemma4.3).

By the following result of J.W. Thatcher [12] it follows that any recursive function is definable in $< \mathbb{N}, +, V_k, V_l >$.

LEMMA4.6.*Any recursive function is definable in $< \mathbb{N}, +, ^\wedge >$ where $^\wedge$ is the concatenation in base k.*

PROOF:See [12, Theorem 2] and also the footnote on page 183 in the same paper.

COROLLARY4.7.*The multiplication of natural numbers is definable in $< \mathbb{N}, +, V_k, ^\wedge >$, hence by Lemma4.2 and 4.5 it is definable in $< \mathbb{N}, +, V_k, V_l >$.*

The only thing which remains to be shown is Lemma4.2.

5.Definability in $< \mathbb{N}, +, V_k, V_l >$. We will now prove that for any k, l multiplicatively independent there exists a strictly increasing definable function satisfying $(*)$, this will give a proof to Lemma4.2.

We will often use the following fact: For k, n in $\mathbb{N}$, k and k^n are multiplicatively dependent, hence any set which is k-recognizable is k^n-recognizable, and vice versa (see [4,Corollary 3.7]). This means that V_k is definable in $< \mathbb{N}, +, V_{k^n} >$ and V_{k^n} is definable in $< \mathbb{N}, +, V_k >$, therefore we can consider $< \mathbb{N}, +, V_k >$ and $< \mathbb{N}, +, V_{k^n} >$ to be the same.

For the remainder of this section we will fix the following: k and l are multiplicatively independent natural numbers and Supp(x) will be the set of prime divisors of x. Furthermore $k = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ and $l = p_1^{\beta_1} \cdots p_n^{\beta_n}$ where the p_i are prime numbers.

We will consider three cases.

Case 1) Suppose Supp$(k) \not\subset$ Supp(l) and Supp$(l) \not\subset$ Supp(k). We can suppose without loss of generality that $k > l$ since we can replace k by one of its multiple. In this case we can easily define the multiplication of a power of k with a power of l.

LEMMA5.1.*Let $g : k^{\mathbb{N}} \times l^{\mathbb{N}} \rightarrow \mathbb{N}$ be the multiplication i.e. $g(x, y) = x \cdot y$. The function g is definable in $< \mathbb{N}, +, V_k, V_l >$.*

PROOF:The function $g(x, y) = z$ is defined be the formula saying "z is the smallest natural number such that $V_k(z) = x$ and $V_l(z) = y$".

LEMMA5.2.*Let $f : k^{\mathbb{N}} \rightarrow l^{\mathbb{N}}$ be such that $f(x)$ is the smallest power of l greater than x. The function f is strictly increasing and definable in $< \mathbb{N}, +, V_k, V_l >$.*

PROOF:We will show that for any two powers of k there is a power of l inbetween. Take k^r and let l^s be the greatest power of l smaller than k^r . Then $l^{s+1} > k^r$ and furthermore $l^{s+1} < k^r l < k^{r+1}$ since $l < k$ by hypothesis. Hence $k^r < l^{s+1} < k^{r+1}$. Since f is obviously definable in $< \mathbb{N}, +, V_k, V_l >$, this completes the proof.

LEMMA5.3.*Let $u : \mathbb{N} \rightarrow k^{\mathbb{N}}$ be such that $u(x)$ is the greatest power of k smaller than x. The function u is definable in $< \mathbb{N}, +, V_k >$.*

PROOF:Obvious.

LEMMA5.4.*Let $h : k^{\mathbb{N}} \rightarrow k^{\mathbb{N}}$ be such that $h(x) = u(g(x, f(x)))$. Then $h(x) = x^2$ (i.e. for powers of k only) and the function h is strictly increasing.*

PROOF:By definition $u(g(x, f(x))) = u(x \cdot f(x)) = x \cdot u(f(x))$, for $x \in k^{\mathbb{N}}$. By definition of f, we have $u(f(x)) = x$, hence $u(g(x, g(x))) = x \cdot x$. Therefore $h(x) = x^2$

and h is strictly increasing. Since h is obviously definable in $< \mathbb{N}, +, V_k, V_l >$, this completes the proof.

CONCLUSION.*Here* $h(k \cdot x) = k^2 \cdot x^2 > k \cdot x^2 = k \cdot (h(x))$ *for all* $x \in k^{\mathbb{N}}$, *hence* $(*)$ *holds.*

Case 2) Suppose $\text{Supp}(l) \subset \text{Supp}(k)$ and for any $p_i, p_j \in \text{Supp}(l)$, $\frac{\alpha_i}{\beta_i} = \frac{\alpha_j}{\beta_j} = \frac{\alpha}{\beta}$, where α, $\beta \in \mathbb{N}$. Hence $k^\beta = l^\alpha u$, $u \neq 1$, $\alpha \neq 0$, $(l, u) = 1$. Since k, k^β and l, l^α are multiplicatively dependent we can replace k^β by k and $l^\alpha$by l and assume that $k = lu$.

LEMMA5.5.*Let* $f : k^{\mathbb{N}} \to l^{\mathbb{N}}$ *be as in Case 1. The function* f *is strictly increasing. Furthermore there exists a* $d \in \mathbb{N} \setminus \{0\}$ *such that* $f(xk^d) \geq f(x)l^{(d+1)}$. *As before this function is definable in* $< \mathbb{N}, +, V_k, V_l >$.

PROOF:It follows as in the proof of Lemma 5.2 that f is stricly increasing.

For the second claim take d to be the smallest natural number such that $u^d > l$. Since $f(x)$ is the smallest power of l greater than x, we have that $\frac{f(x)}{l} < x$. Hence $\frac{f(x)k^d}{l} < xk^d$, so $f(x)l^d = \frac{f(x)l^d l}{l} \leq \frac{f(x)l^d u^d}{l} < xk^d$. Therefore $f(xk^d) > f(x)l^d$.

LEMMA5.6.*Let* $g' : l^{\mathbb{N}} \to k^{\mathbb{N}}$ *be the function which sends* l^m *to* k^m. *The function* g' *is multiplicative (i.e.* $g'(x \cdot y) = g'(x) \cdot g'(y)$*) and strictly increasing. Furthermore* g' *is definable in* $< \mathbb{N}, +, V_k, V_l >$.

PROOF:Since $(l, u) = 1$ it follows that $V_l(k^n) = V_l(l^n u^n) = V_l(l^n) = l^n$. Hence we can define $g'(x) = y$ by the formula $V_l(y) = x$. The remaining of the proof is obvious.

LEMMA5.7.*Let* $h = g' \circ f : k^{\mathbb{N}} \to k^{\mathbb{N}}$. *The function* h *is strictly increasing. Furthermore for all* x *in* $k^{\mathbb{N}}$, $h(k^d \cdot x) > k^d \cdot h(x)$. *(The* d *is the one of Lemma 5.5). Finally* h *is definable in* $< \mathbb{N}, +, V_k, V_l >$.

PROOF:Since f and g' are strictly increasing by Lemma 5.5 and 5.6 respectively it follows that h is strictly increasing.

Let us show that for all $x \in k^{\mathbb{N}}$, $h(k^d \cdot x) > k^d \cdot h(x)$. This is the same as showing that for all $x \in k^{\mathbb{N}}$ $g'(f(xk^d)) > g'(f(x))k^d$. Since by Lemma 5.5 we have that for any x in $k^{\mathbb{N}}$ $f(xk^d) \geq f(x)l^{(d+1)}$, it follows (applying Lemma 5.6) that $g'(f(xk^d)) \geq g'(f(x)l^{(d+1)})$. Furthermore by Lemma 5.6 $g'(f(xk^d)) \geq g'(f(x))g'(l^{(d+1)})$ hence $g'(f(xk^d)) > g'(f(x))k^d$ since g' is increasing by Lemma 5.6. Therefore $h(k^d \cdot x) > k^d \cdot h(x)$.

Finally it follows by Lemma 5.5 and 5.6 that h is definable in$< \mathbb{N}, +, V_k, V_l >$.

CONCLUSION.*By Lemma 5.11 we know that $h(k^d \cdot x) > k^d h(x)$ for all $x \in k^{\mathbb{N}}$. Since h is strictly increasing $h(k \cdot x) \geq k \cdot h(x)$. Therefore it follows from $h(k^d \cdot x) > k^d \cdot h(x))$ that for some $y \in \{x, k \cdot x, \ldots, k^d \cdot x\}$, $h(k \cdot y) > k \cdot h(y)$. Hence $(*)$ holds.*

Case 3) Let Supp$(l) \subset$ Supp (k) and for some p_i, p_j, $\frac{\alpha_i}{\beta_i} < \frac{\alpha_j}{\beta_j}$. Hence $m < n$ with as before $k = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ and $l = p_1^{\beta_1} \cdots p_m^{\beta_m}$. We can suppose without loss of generality that $\frac{\alpha_1}{\beta_1} = \min\{\frac{\alpha_i}{\beta_i}; i = 1, \ldots, m\}$ and $\frac{\alpha_m}{\beta_m} = \max\{\frac{\alpha_i}{\beta_i}; i = 1, \ldots m\}$. Furthermore since k^{β_1},k and l^{α_1}, l are multiplicatively dependent we can suppose without loss of generality that $\frac{\alpha_1}{\beta_1} = 1$, hence $\frac{\alpha_m}{\beta_m} > 1$.

LEMMA5.8.*Let $f' : k^{\mathbb{N}} \to l^{\mathbb{N}}$ be the function which sends k^r to l^s, where $s = \lceil \frac{r\alpha_m}{\beta_m} \rceil$ (the smallest natural number greater or equal to $\frac{r\alpha_m}{\beta_m}$). The function f' is definable in $< \mathbb{N}, +, V_k, V_l >$ and strictly increasing.*

PROOF:We will show that $f'(x) = y$ can be defined by the formula " y is the smallest power of l such that $\forall u[V_l(u) \geq y \to V_k(u) \geq x]$".

Let $x = k^r$ and $y = l^s$. Take $u = p_1^{\gamma_1} \cdots p_n^{\gamma_n}$ some natural number, where some γ_i can be zero. There is no loss of generality in assuming that u in the above formula is of this form since any prime factor different of $p_1, \ldots, p_n$ would not change the value of $V_l(u)$ and $V_k(u)$.

Now $V_k(u) = V_k(p_1^{\gamma_1} \cdots p_n^{\gamma_n}) = k^{\min\{\lfloor \frac{\gamma_i}{\alpha_i} \rfloor; i=1,\ldots,n\}}$ and in the same way $V_l(u) = l^{\min\{\lfloor \frac{\gamma_i}{\beta_i} \rfloor; i=1,\ldots,m\}}$. Hence $V_l(u) \geq y \to V_k(u) \geq x$ is equivalent to "$\min\{\lfloor \frac{\gamma_i}{\beta_i} \rfloor; i = 1, \ldots, m\} \geq s$ implies that $\min\{\lfloor \frac{\gamma_i}{\alpha_i} \rfloor; i = 1, \ldots, n\} \geq r$". Furthermore this holds exactly if "for all i, $\gamma_i \geq s\beta_i$" implies "for all i, $\gamma_i \geq r\alpha_i$". Therefore $\forall u[V_l(u) \geq y \to V_k(u) \geq x]$ holds if and only if $r\alpha_i \leq s\beta_i$ for all i. Hence " y is the smallest power of l such that $\forall u[V_l(u) \geq y \to V_k(u) \geq x]$" if and only if $s = \lceil r\frac{\alpha_m}{\beta_m} \rceil$. The function f' is strictly increasing since $\frac{\alpha m}{\beta_m} > 1$. This completes the proof.

LEMMA5.9.*Let $g'' : l^{\mathbb{N}} \to k^{\mathbb{N}}$ be the function which sends l^r to k^r. The function g'' is definable in $< \mathbb{N}, +, V_k, V_l >$ and strictly increasing.*

PROOF:Since $\frac{\beta_1}{\alpha_1} = 1$, we can argue as in the proof of Lemma 5.8 to show that $g''(x) = y$ can be defined by the formula " y is the smallest power of k such that $\forall u V_k(u) \geq y \to V_l(u) \geq x$". The function is strictly increasing by definition.

LEMMA5.10.*Let $h = g'' \circ f'$. The function h is strictly increasing and $h(k^r) = k^s$, with $s = \lceil \frac{r\alpha_m}{\beta_m} \rceil$.*

PROOF:This is obvious from Lemma5.9.

CONCLUSION.*Here for any* $x = k^{u\cdot\beta_m - 1}$, $1 < u \in I\!N$ *we have that* $h(k \cdot x) = h(k^{u\cdot\beta_m}) = k^{u\cdot\alpha_m} = k\cdot k^{u\cdot\alpha_m - 1} = k\cdot k^{\lceil u\cdot\alpha_m - 1\rceil} \geq k\cdot k^{\lceil u\cdot\alpha_m - \frac{\alpha_m}{\beta_m}\rceil} = k\cdot k^{\lceil (u\cdot\beta_m - 1)\cdot\frac{\alpha_m}{\beta_m}\rceil}$ $= k \cdot h(x)$. *Hence* $(*)$ *holds with* $d = \beta_m$.

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PROBLEME DE JAMES AX

Hervé Carrieu[1]

1.– INTRODUCTION,DEFINITION

1.1 Définition, Proposition. *Un corps K est dit pseudo-algébriquement clos (PAC) s'il vérifie l'une des deux propositions équivalentes suivantes :*

(i). *Toute variété V (absolument irréductible) définie sur K admet un point K-rationnel.*

(ii). *Il existe K_0 sous corps de K avec $K_0 \subset K$ extention algébrique, tel que toute courbe plane définie sur K_0 posséde un point K-rationnel.*

1.2 Exemples.

(a). Toute extension algébrique d'un corps PAC est PAC.

(b). Toute extension algébrique infinie d'un corps fini est PAC.

(c). K est PAC s'il satisfait $\theta_i, i = 1, 2, \cdots$ où les θ_i sont des énoncés du langage de la théorie des corps, (voir par exemple [2] p.132);

PROBLEME DE AX : *Est-ce que tout corps pseudo-algébriquement clos et parfait est C_1 ?*

On rappelle qu'un corps K est dit $C_i(d)$ si tout polynôme homogène P de degré d en n variables avec $n > d^i$ admet un zéro non trivial dans K. Un corps K est dit C_i, s'il est $C_i(d)$, pour tout d.

1.3 Remarques :

(a). Tout corps fini est C_1, (Théorème de Chevalley, voir [5])

(b). $C_i(d)$ est une propriété élémentaire.

On introduit alors la notion de corps faiblement C_i.

2.– PROPRIETE C_i-FAIBLE.

2.1 Définition : *Un corps K est dit faiblement C_i si, étant donné P polynome homogène en n variables de degré d, avec $n > d^i$, l'ensemble algébrique $V(P)$ des zéros de P dans $\mathbb{P}^n$ contient une K-variété absolument irréductible.*

2.2 Remarques :

(a). Il est clair que si un corps est C_i, alors il est faiblement C_i, (le zéro non trivial K-rationnel constitue la K-variété absolument irréductible).

(b). Si K est parfait, alors la K-variété absolument irréductible est <u>définie</u> sur K, ([1],p.74).

[1]Laboratoire de Logique Mathématique, couloir 45.55, 5$^{\text{ième}}$ étage. Université PARIS VII JUSSIEU 75005 PARIS.

(c). Pour un corps K parfait et PAC, les conditions "K est C_i" et "K est faiblement C_i" sont équivalentes.

On donne maintenant quelques propriétés importantes relatives aux corps faiblement C_i.

2.3 Lemme : *Un corps K est faiblement C_i si tout polynome P homogène de degré d en n variables admet un zéro non trivial $(\mathbf{x})$ tel que l'extention $K \subset K(\mathbf{x})$ soit primaire, (i.e. $K(\mathbf{x})$ est linéairement disjoint de K_s sur K).*

On déduit alors de ce lemme une propriété de descente

2.4 Propriété-Corollaire : *Soit $K \subset L$ une extention primaire et L un corps faiblement C_i, alors K est faiblement C_i.*

Démonstration : Soit $P \in K[X_0, \cdots, X_n]$ homogène de degré d avec $n \geq d^i$; $P \in K[\mathbf{X}]$ et soit alors $(\mathbf{x})$ un zéro de P avec $L \subset L(\mathbf{x})$ primaire. On a $K \subset L(\mathbf{x})$ extention primaire (par transitivité) et donc $K \subset K(\mathbf{x})$. ∇

Tout comme la propriété C_i, la propriété faiblement C_i se comporte "relativement bien" pour les extentions, plus précisément :

2.5 Proposition : Si K est un corps C_i (resp. faiblement C_i) et L une extention de K de degré de transcendance j, alors L est C_{i+j} (resp. faiblement C_{i+j}).

Démonstration : Voir par exemple [3],p.21 – 22.

3.- PROCEDURE DE DECOMPOSITION.

On considère $V(P)$, zéros d'un polynome $P \in K[\mathbf{X}]$ ou plus généralement un sous ensemble algébrique A de $\mathbb{A}^n$ ou $\mathbb{P}^n$ défini par des polynomes à coefficients dans K. On va donner une procédure qui permet de savoir si cet ensemble A contient une sous-K-variété absolument irréductible.
On considère $A = \cup V_i$ la décomposition sur K de A et pour chaque i, $V_i = \cup_j W_{ij}$ la décomposition sur $\tilde{K}$. Pour tout i, $\{W_{ij}\}$ représente un système complet de variétés conjuguées sur K, et donc $\cap_j W_{ij}$ est invariant sous l'action du groupe de Galois absolu $G(K) = \mathrm{Gal}(K_s/K$. $\cap_j W_{ij}$ est alors un ensemble algébrique défini par des polynomes à coefficients dans K, ([1],p.74) on le notera U_i. Posons $A^{(1)} = \cup U_i$ et réitérons cette décomposition pour $A^{(1)}$. On construit ainsi une suite décroissante $A \supset A^{(1)} \supset \cdots \supset A^{(m)} \supset A^{(m+1)}$ qui est stationnaire ($K[\mathbf{X}]$ est Noethérien). On note $A^\star$ sa limite. On notera que $A^\star$ peut être vide et dire que $A^\star$ est non vide signifie que A contient une K-variété absolument irréductible.
Plus précisément soient $W_1, W_2, \cdots, W_s$ les diverses composantes sur $\tilde{K}$ de $A, A^{(1)}, A^{(2)}, \cdots$ et notons L la plus petite extention Galoisienne de K telle que chaque W_i soit définie sur L (L est appelé *corps de décompostion* de A sur K), alors on a :

3.1 Proposition : *Soit M une extention de K telle que $M \cap L = K$. Alors $A^\star$ est non vide si et seulement si A contient une M-variété absolument irréductible.*
Démonstration : Si $A^\star$ est non vide, il est clair que A contient une K-sous variété absolument irréductible. qui est aussi une M-variété.

Réciproquement supposons que A contienne W, M-variété absolument irréductible. Alors il existe i, j tels que $W \subset W_{ij}$. Les composantes $\{W_{ij}\}$ sont conjuguées sur K et tout $\sigma \in \mathrm{Gal}(L/K)$ se prolonge en un M-automorphisme $\tilde{\sigma}$ de ML, car dire que $M \cap L = K$ avec L extension Galoisienne de K équivaut à dire que M et L sont linéairement disjointes sur K, d'où le prolongement ,d'ailleurs unique, de σ.
$\{W_{ij}\}$ sont alors définies sur ML et conjuguées sur M, or W est invariante sous l'action de $\tilde{\sigma}_i$, avec $\sigma_i \in \mathrm{Gal}(L/K)$, donc $W \subset \cap_j W_{ij}$. Ainsi $W \subset A^{(1)}$ et par induction $W \subset A^\star$. ∇

4.– LES CORPS $\tilde{K}(\sigma)$

Dans [5], J.Ax a prouvé que tout corps PAC et parfait K, est C_1 dés que $G(K)$ est abélien. La proposition suivante nous fournit des exemples corps PAC et parfaits avec $G(K)$ non abéliens. On rappelle qu'un corps K est hilbertien si pour tout m, étant donné m polynômes $f_1, \cdots, f_m \in K[T, X_1, \cdots, X_n]$ on peut trouver une infinité d'éléments a de K tels que $f_1(a, \mathbf{X}), f_2(a, \mathbf{X}), \cdots, f_m(a, \mathbf{X})$ restent irréductibles sur K.

Exemple : Tout corps de nombres est hilbertien, tout corps de fonction algébrique d'une variable est un corps fini et hilbertien, (voir [2]).

Pour tout $(\sigma_1, \cdots, \sigma_l) \in \mathrm{Gal}(K)^l$, on notera

$$\tilde{K}(\sigma_1, \cdots, \sigma_l) = \{x \in \tilde{K} \;/\; \sigma_i(x) = x, \; i = 1, \cdots, l\}$$

On confond ici $\sigma_i \in \mathrm{Gal}(K_s/K)$ avec son unique prolongement à $\tilde{K}$.

4.1 Proposition : *Soit K dénombrable et hilbertien,*
(a). Pour tout l, et pour presque tout $(\sigma) \in G(K)^l$, $\tilde{K}(\sigma)$ est PAC, parfait et de groupe de Galois $G(K) = \hat{F}_l$.
(b). Soit θ un énoncé dans $\mathcal{L}(K)$, et $l \in \mathbb{N}$, alors θ est vraie dans $\tilde{K}(\sigma)$ pour presque tout (σ) de $G(K)^l$ si et seulement si θ est vraie dans tout corps PAC, parfait de groupe de Galois $\hat{F}_l$ et contenant K.
Démonstration : [2],p.257 – 258.

Remarques : "Presque tout" est pris au sens de la mesure de Haar sur le groupe $G(K)$. $\hat{F}_l$ est le groupe profini libre à l générateurs ,(voir par exemple [2],p.183).

La proposition suivante met en relation les propriétés C_i, faiblement C_i avec les corps $\tilde{K}(\sigma)$.

4.2 Proposition : *Soit K hilbertien dénombrable. Sont équivalentes*
(a). K est faiblement C_i.
(b). Pour tout l et presque tout $(\sigma) \in G(K)^l$, $\tilde{K}(\sigma)$ est C_i.

Démonstration : $(a) \Rightarrow (b)$. Pour tout l et presque tout $(\sigma) \in G(K)^l$, $\tilde{K}(\sigma)$ est un corps PAC parfait d'aprés 4.1(a) et est faiblement C_i comme extention algébrique de K, ainsi $\tilde{K}(\sigma)$ est C_i.

$(b) \Rightarrow (a)$. Soit $P \in K[X_0, \cdots, X_n]$. On applique la procédure de décomposition à $A = V(P)$. Soit L le corps de décomposition de $V(P)$ sur K et $\sigma'_1, \sigma'_2, \cdots \sigma'_l$ un système de générateurs de $\mathrm{Gal}(L/K)$. On peut trouver (d'aprés 4.1(a) et l'hypothése (b)) $\sigma_1, \cdots, \sigma_l$ qui prolongent $\sigma'_1, \sigma'_2, \cdots \sigma'_l$ et tels que $\tilde{K}(\sigma_1, \cdots, \sigma_l)$ soit C_1. Ainsi, $A = V(P)$ contient une $\tilde{K}(\sigma_1, \cdots, \sigma_l)$-variété absolument irréductible (i.e. le zero non trivial dans $\tilde{K}(\sigma)$). Or $L \cup \tilde{K}(\sigma) = K$ et donc d'aprés la proposition 3.1, A^* est non vide, A contient alors une K-variété absolument irréductible : K est faiblement C_i. $\bigtriangledown$

4.3 Proposition : *Soit K hilbertien, dénombrable et faiblement C_i. Alors toute extension de K est faiblement C_i et donc, tout corps PAC parfait contenant K est C_i.*

Démonstration : Soit F une extension de K que l'on peut supposer dénombra ble ,(les coefficient d'un $P \in F[\mathbf{X}]$ sont dans une extension finie de K). Soit K' extentsion transcendante pure de K vérifiant $K \subset K' \subset F$ et F algébrique sur K'. K' est dénombrable hilbertien, et d'aprés 4.1(a), $\tilde{K}'(\sigma)$ est PAC et parfait pour tout l et presque tout(σ). Alors d'aprés 4.1(b). $K'(\sigma)$ est C_i, et par 4.2, K' est faiblement C_i, ainsi F, extension algébrique de K', est faiblement C_i. $\bigtriangledown$

4.4 Proposition : (a). *Tout corps PAC parfait L de caractéristique positive p est C_2.*

(b). *Tout corps PAC parfait contenant un corps algébriquement clos est C_1.*

Démonstration : (a). Si L est algébrique sur $\mathbb{F}_p$, L est C_1 car $\mathbb{F}_p$ est C_1. Sinon L contient $\mathbb{F}_p(t)$ avec t transcendant sur $\mathbb{F}_p$ et $\mathbb{F}_p(t)$ hilbertien dénombrable est C_2, donc faiblement C_2. L est alors faiblement C_2 et donc C_2.
(b). Supposons que L contient Ω algébriquement clos. Ω est C_0 et L contient $\Omega(t)$ avec t transcendant sur Ω. $\Omega(t)$ est C_1 et donc faiblement C_1. Ω peut être supposé dénombrable, $\Omega(t)$ est alors hilbertien dénombrable. L est donc faiblement C_1, donc C_1. $\bigtriangledown$

On voit donc que le problème de AX peut se poser de la façon suivante : *Est-ce que tout corps est faiblement C_1 ?*

On déduit alors des résultats précédents

4.4 Proposition : *Soit K un corps et d en entier positif. K admet une extension finie K_d telle que K_d soit faiblement $C_1(d)$. .*

Démonstration : Soit $T = \{$ axiomes des corps PAC, parfait contenant $K \}$, (voir [2]), et soit Δ le diagramme de $\tilde{K}$. Un modèle de $T \cup \Delta$ est alors (4.4(b)) un corps PAC parfait et C_1. La propriété $C_1(d)$ se prouve à l'aide de $T \cup \Delta$ et donc à l'aide de $T \cup \Delta_0$ où Δ_0 est une partie finie de Δ. Notons K_d l'extension finie de K engendrée par tous les éléments de $\tilde{K}$ intervenant dans Δ_0. Donc tout corps parfait PAC contenant K_d est $C_1(d)$.
Si K est hilbertien dénombrable il en est de mme de K_d qui est donc faiblement $C_1(d)$ d'aprés 4.2

Sinon on peut trouver dans K un corps dénombrable hilbertien E qui admet une extension finie E_d faiblement $C_1(d)$ et telle que tout corps PAC parfait contenant E_d est $C_1(d)$. On vérifie alors que $K_d = KE_d$ est faiblement $C_1(d)$. $\bigtriangledown$

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The representation of ℓ-groups over Priestley spaces

D. Gluschankof F. Lucas

1 Introduction

Two natural generalizations of that of direct product are that of Boolean product (see [3]) and that of Hahn product (see [6]). The two type of products have been used for the representation of abelian lattice-ordered groups (ℓ-groups). The first one, for the representation of projectable abelian ℓ-groups (see [13], [23] and [16]). The last one appears through the Conrad-Harvey-Holland Theorem (see [7]). This theorem proves that any abelian ℓ-group can be embedded, as an ℓ-subgroup into a Hahn-product of totally ordered archimedean groups. In spite of its depth, the Conrad-Harvey-Holland Theorem is limited for the study of model-theoretic properties of the represented groups. That is, if the group is isomorphical to a Hahn-product, it is possible to study some of those properties (see, for example [14]). But, if it is a strict subgroup of the Hahn-product, little can be said. On the other side, the Boolean product representation gives a lot of results for the projectable ℓ-groups (see [21], [23], [16] and [11]). But it cannot be used for the non-projectable case.

In this article, we will study a generalization of the notion of Boolean product which will give a representation for a wider class of ℓ-groups, permitting a model-theoretic study of that groups and, maybe, will be useful for other classes of ordered structures.

As a sideresult we obtain a factorization of the representation for MV-algebras as weak Boolean products presented by Cignoli and Torrens in [4].

In all this article "group" and "ℓ-group" will stand for "abelian lattice-ordered group" and "o-group" for "totally ordered abelian group".

2 Priestley spaces, root-systems, Hahn-products and projectability

In this section we recall some definitions and results from [17] and [18] for the Priestley spaces and from [1] and [14] for root-systems, Hahn-products, projectability and r-projectability.

A *Priestley space* is a triple $\langle X, \tau, \leq \rangle$ where $\langle X, \tau \rangle$ is a topological space and $\leq$ is a partial order on X such that, for any $x, y \in X$ such that $x \not\leq y$ there exist a clopen increasing set U (that is, a clopen set which coincides with its final section) and a clopen decreasing set L such that $x \in U$, $y \in L$ and $U \cap L = \emptyset$. An equivalent definition is that of an ordered space having a base of increasing (decreasing) clopen subsets. In the original definition of Priestley's there is the additional condition that $\langle X, \tau \rangle$ must be compact. We prefer to have a little more generality.

It is immediate that a Boolean space (the spectrum of a Boolean algebra) with the trivial order is a Priestley space. The Stone representation theorem (every Boolean algebra is isomorphic to the algebra of continuous functions on its spectrum) is generalized by the following result of Priestley:

Every distributive lattice is isomorphic to the lattice of continuous functions on the Priestley space of its ordered spectrum. In particular, if the lattice has 0 and 1, its Priestley space is compact.

A Priestley space is called an I_d *-space* (I_i*-space*) if the closed initial (final) section of every open set is open. An I*-space* is an I_d -space and I_i-space. The notions of C_d*-space*, C_i*-space* and C*-space* are the analogous ones replacing "open set" by "closed set". A space which has all those properties is called a CI*-space*. Finally, if "open set" ("closed set") in the antecedents of the definitions is replaced by "compact-open set" ("compact set") the analogous properties will be called weak. The initial (final) section of a subset A will be denoted by $A\downarrow$ ($A\uparrow$). A subset of a Priestley space will be called *d-compact* (*i-compact*) if each covering by a family of decreasing (increasing) open sets can be refined to a finite subcovering.

Recall that a structure G on a language $\mathcal{L}$ is called a *Boolean product* of the family of $\mathcal{L}$ structures $(L_x)_{x \in X}$ if

i) G is a subdirect product of the family $(L_x)_{x \in X}$.

ii) X admits a Boolean space topology such that

- ii.a For each atomic formula $\varphi(v_1, \ldots, v_n)$ of $\mathcal{L}$ and elements $g_1, \ldots, g_n \in G$, the set $[\![\varphi(g_1, \ldots, g_n)]\!] = \{x \in X \ / \ L_x \models \varphi(g_1(x), \ldots, g_n(x))\}$ is clopen.
- ii.b For each $g, g' \in G$ and clopen set $Y \subseteq X$ there exists an element of G coinciding with g on Y and with g' on $X \setminus Y$.

A *root-system* is a partially-ordered set such that the final section of each of its elements is totally ordered.

Given a root-system I and a family of o-groups $(L_i)_{i\in I}$, consider the product group $G = \prod_{i\in I} L_i$. For each element g of G, define its *support* $\mathsf{supp}(g) = \{i \in I \ / \ g(i) \neq 0\}$ and its *maximal support* $\mathsf{ms}(g) = \{i \in \mathsf{supp}(g) \ / \ \forall j (j > i \Rightarrow g(j) = 0\}$. If a and b are two subsets of I (for example the maximal supports of two elements), we have a natural order between them given by:

$$a \preccurlyeq b \Longleftrightarrow \forall i \in a \exists j \in b (i \leq j).$$

The *Hahn-product* $\Lambda(I, (L_i)_{i\in I})$ is the subgroup of G given by

$$\{g \in G \ / \ \forall i \in \mathsf{supp}(g) \ \textit{the set} \ [i, \rightarrow) \cap \mathsf{supp}(g) \text{ is anti well-ordered}\}. \qquad (\dagger)$$

Under such a condition, the Hahn-product can be ordered by $g \geq 0$ if and only if, for all i belonging to $\mathrm{ms}(g)$, we have $g(i) \geq 0$. This order gives an ℓ-group structure to $\Lambda(I, (L_i)_{i\in I})$. In fact, for any subgroup G' of G whose elements satisfy the property

$$\forall g (\forall i \in \mathsf{supp}(g) \ \exists j \in \mathsf{ms}(g)(i \leq j)) \qquad (*)$$

a partial-order can be defined as in the case of the Hahn-product. However, this partial order is not necessarily a lattice order.

If there exists a totally ordered group L such that, for all $i \in I$, L_i is isomorphic to L, then $\Lambda(I, (L_i)_{i\in I})$ is called a *Hahn-power* and denoted $\Lambda(I, L)$

For any ℓ-group, the set of its prime ℓ-ideals with the order given by set-theoretic inclusion is a root-system. The quotient of an ℓ-group by a prime ℓ-ideal is a totally ordered group.

Two elements x and y of an ℓ-group G are said to be *orthogonal* and denoted $x \perp y$ if $| x | \wedge | y | = 0$ (where $| x |$ is defined as $(x \vee 0) + (-x \vee 0)$) (we shall also denote with x_+ (x_-) the positive (negative) part of x: $x \vee 0$ $(-x \vee 0)$).

It is easy to verify that, if $f, g \in G$, an ℓ-subgroup of a Hahn-product, $f \perp g$ if and only if for all $x \in \mathsf{supp}(f)$ and $y \in \mathsf{supp}(g)$, x is not comparable with y.

The *polar* of x is the ℓ-ideal of G

$$x^\perp = \{g \in G \ / \ x \perp g\}.$$

The *double polar* of x is the ℓ-ideal

$$x^{\perp\perp} = \{g \in G \ / \ \forall y (y \in x^\perp \Rightarrow y \perp g)\}.$$

An ℓ-group G is called *projectable* if for all $x, y \in G$ there exist $x_0 \in y^{\perp}$ and $x_1 \in y^{\perp\perp}$ such that $x = x_0 + x_1$.

An ℓ-group G is called *r-projectable* if

i) $\forall x, y \exists z, w(z \perp w \;\&\; x = z + w \;\&\; \forall u, v((u \perp v \;\&\; w = u + v) \Rightarrow (u \notin y^{\perp} \;\&\; v \notin y^{\perp})))$.

ii) $\forall x, y \exists z(\mathsf{ms}(z) = \mathsf{ms}(x) \cap \mathsf{ms}(y))$.

A projectable ℓ-group is always r-projectable, a Hahn-product is r-projectable. If the Hahn-product $\Lambda(I, (L_i)_{i \in I})$ is projectable then the root-system I is trivialy ordered. Since for our development we shall be interested only on ℓ-groups satisfying condition (i) of r-projectability, we shall say that those groups are *ri-projectable*

For any ℓ-group G, and non-zero $g \in G$, call g^0 to the set

$$\{x \in g^{\perp\perp} \;/\; \exists y, z \in g^{\perp\perp}(y^{\perp\perp} \neq g^{\perp\perp} \;\&\; z^{\perp\perp} \neq g^{\perp\perp} \;\&\; x = y + z)\}$$

Which is a prime ideal. We have that, if G is a Hahn-product, then g^0 is the lex-kernel of $g^{\perp\perp}$ (see [11]). Denote by g^* the quotient group $g^{\perp\perp}/g^0$. Since each g^0 is a prime ideal the family $\{g^* \;/\; g^{\perp\perp} \neq g^0 \;\&\; g \in G \setminus \{0\}\}$, ordered by the set-theoretical inclusion is a root-system.

If $G = \Lambda(X, (L_x)_{x \in X})$, we have that there exists an embedding of the root-system $\{g^* \;/\; g^{\perp\perp} \neq g^0 \;\&\; g \in G \setminus \{0\}\}$ into the root-system X. If all the points of the root-system X are branching points then the correspondence is bijective.

3 Priestley powers

In [15], Martínez studied the Priestley space of the underlying distributive lattice of a Wajsberg algebra and, by endowing such space with a binary operation, he arrived to give a representation theorem for Wajsberg algebras in terms of what he called a Wajsberg space. Since there is a categorical bijection between the categories of Wajsberg algebras and ℓ-groups with strong unit, we have that his results hold also for this last category. For our representation theorems we shall look to a Priestley space inherent to the ℓ-group structure and we dispense of additional operations.

Consider now a Priestley space $\langle X, \tau, \leq \rangle$ such that the ordered set $\langle X, \leq \rangle$ is a root system (in that case it will be called a *Priestley root-system*) and a totally ordered group L.

Given an element $g \in \mathcal{C}(\langle X, \tau \rangle, L)$ (the ℓ-group of continuous functions on L, where L is endowed with the discrete topology) we have that its support is clopen. If Y is a compact-open subset of X, we shall denote by $g_{|Y}$ the map given by

$$g_{|Y}(x) = \begin{cases} g(x) & \text{if } x \in (Y \uparrow \cap \mathsf{supp}(g)) \downarrow \\ 0 & \text{if } x \notin (Y \uparrow \cap \mathsf{supp}(g)) \downarrow \end{cases}$$

Denote by $Y(g)$ the set $(Y \uparrow \cap \mathsf{supp}(g)) \downarrow \cap \mathsf{supp}(g)$. Observe that the sets $Y(g)$ and $X \setminus (Y(g) \uparrow)$ are orthogonal, in the sense that, given $y \in Y(g)$ and $x \in X \setminus (Y(g) \uparrow)$ we have neither $y \leq x$ nor $x \leq y$. Hence, it is possible to well define $g_{|Y} \cup f_{|X \setminus (Y(g)\uparrow)}$ as the map which coincides with $g_{|Y}$ and $f_{|X \setminus (Y(g)\uparrow)}$ on their respective supports. In particular, it can be proved the equality $g_{|Y} \cup f_{|X \setminus (Y(g)\uparrow)} = g_{|Y} + f_{|X \setminus (Y(g)\uparrow)}$.

Definition 1 *Let $\langle X, \tau, \leq \rangle$ be a Priestley root-system and L a totally ordered group. The* Priestley power $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ *is the group of continuous functions with compact support on $\langle X, \tau \rangle$ with values in L where this group is considered with the discrete topology.*

The following Lemma (and its proof) is totally analogous to the corresponding Lemma for the case of Boolean powers (see [2, Theorem 5.6]).

Lemma 1 *Let $\langle X, \tau, \leq \rangle$ be a Priestley root-system and L a totally ordered group. The group $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ satisfies the following properties:*

i) For all compact-open subset Y of X and $a \in L$ the constant map to a on its support Y belongs to $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$.

ii) For all $f, g \in \mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ and clopen set $Y \subseteq X$ the element which coincides with f on Y and with g on $X \setminus Y$ belongs to $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$. (Patchwork property). In particular we have

ii') For all $f, g \in \mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ and clopen set $Y \subseteq X$ the element $f_{|Y} \cup g_{|X \setminus (Y(f)\uparrow)}$ belongs to $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$. (Orthogonal patchwork property).

As in the case of Boolean powers and Boolean products, for f and g in $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$, if $\Diamond$ is one of the binary relation symbols $=$, $<$, $>$, $\leq$ or $\geq$ we shall denote with $[\![f \Diamond g]\!]$ the subset $\{x \in X \ / \ f(x) \Diamond g(x)\}$ of X.

Remark 1 Observe that, in a Priestley power, for any clopen set Y and element g the set $Y(g) = ((Y \uparrow \cap \mathsf{supp}(g)) \downarrow) \cap \mathsf{supp}(g)$ must be compact-open.

Lemma 2 *Let $\langle X, \tau, \leq \rangle$ be a Priestley space and $a \in X$. The closed initial and final sections of a ($(\leftarrow, a]$ and $[a, \rightarrow)$ respectively) are closed sets.*

Proof: Let $b \notin (\leftarrow, a]$. There exist disjoint clopen sets $U(b)$ and $L(b)$ such that the first one is increasing, the second decreasing, $b \in U(b)$ and $a \in L(b)$. In particular we have that $(\leftarrow, a] \cap U(b) = \emptyset$. So, we have $(\leftarrow, a] = \bigcap_{b \notin (\leftarrow, a]} L(b)$ which is closed. For the final section the proof is analogous. ∎

Lemma 3 *Let $\langle X, \tau, \leq \rangle$ be a Priestley root-system and L an o-group.*

i) $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ satisfies property (*) and hence it has an order compatible with the group structure.

ii) If X has the I_d property then, for all $g \in \mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ we have that $\mathsf{ms}(g)$ is compact.

Proof: i) Let be $g \in \mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ and $a \in \mathsf{supp}(g)$. Consider the subset of $\mathsf{supp}(g)$, $I = \{x \in \mathsf{supp}(g) \ / \ a \leq x\}$. Since X is a root-system, we have that I is totally ordered. Consider now, for each $x \in I$ the closed subsets $A_x = [x, \rightarrow) \cap \mathsf{supp}(g)$. We have that, for each finite subset J of I, the intersection $\bigcap_{x \in J} A_x$ is non-empty. Then, by compacity of $\mathsf{supp}(g)$ we have that $\bigcap_{x \in I} A_i \neq \emptyset$. Hence there exists $y \in \mathsf{supp}(g)$ such that $a \leq y$, $\bigcap_{x \in I} A_i = A_y$ and then $y \in \mathsf{ms}(g)$. Hence we can conclude that property (*) holds.

ii) Let $\mathcal{U}$ be a family of open sets covering $\mathsf{ms}(g)$. Property I_d implies that, for each $U \in \mathcal{U}$ the set $U \downarrow$ is open. By what was proved in (i), we have $\mathsf{supp}(g) = \mathsf{supp}(g) \cap (\mathsf{ms}(g) \downarrow)$. Hence we have that the family of open sets $\{U \downarrow \ / \ U \in \mathcal{U}\}$ cover $\mathsf{supp}(g)$. Since $\mathsf{supp}(g)$ is compact, there exists a finite subfamily $\{U_1 \downarrow, \ldots, U_n \downarrow\}$ still covering $\mathsf{supp}(g)$. We have then that $\{U_1, \ldots, U_n\}$ covers $\mathsf{ms}(g)$ proving its compacity. ■

Proposition 4 *Let $\langle X, \tau, \leq \rangle$ be a I_d and weak C_d Priestley root-system and L an o-group. The Priestley power $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ with its natural order induced by property (*) is an ℓ-group.*

Proof: To prove that an ordered group is a lattice-ordered group it suffices to show that, for each element g there exist $g_+ = g \vee 0$ (see [1, 1.2.9]).

For each element $g \in \mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ consider the following subsets of its support:

$$\mathsf{ms}_+(g) = \{x \in \mathsf{ms}(g) \ / \ g(x) > 0\} \qquad \mathsf{ms}_-(g) = \{x \in \mathsf{ms}(g) \ / \ g(x) < 0\}$$

$$\mathsf{supp}_+(g) = (\mathsf{ms}_+(g) \downarrow) \cap \mathsf{supp}(g) \qquad \mathsf{supp}_-(g) = (\mathsf{ms}_+(g) \downarrow) \cap \mathsf{supp}(g).$$

Consider the (finite) non-zero elements of $\mathrm{Im}(g)$ with their order: $a_0 < \ldots < a_n < 0 < a_{n+1} < \ldots < a_m$. We have $[\![g < 0]\!] = \bigcup_{i=1}^{n} g^{-1}(a_i)$ and $[\![g > 0]\!] = \bigcup_{i=n+1}^{m} g^{-1}(a_i)$ implying that both are compact-open sets. Since $\mathsf{ms}_+(g) = \mathsf{ms}(g) \cap [\![g > 0]\!]$ and $\mathsf{ms}_-(g) = \mathsf{ms}(g) \cap [\![g < 0]\!]$ we have that those sets are compact (and hence closed). By the weak C_d and I_d properties we obtain that $\mathsf{supp}_+(g)$ and $\mathsf{supp}_-(g)$ are clopen. Since they are subsets of the compact set $\mathsf{supp}(g)$ we conclude that both are clopen and compact. Now, by the patchwork property, define g_+ equal to the restriction of g to $\mathsf{supp}_+(g)$ and analogously g_- as its restriction to $\mathsf{supp}_-(g)$. So we can conclude that $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ is an ℓ-group. ■

Proposition 5 *A Priestley-power such that its Priestley root-system satisfies the properties weak C, weak I_i and I_d is ri-projectable.*

Proof: Let $f, g \in \mathcal{PR}(\langle X, \tau, \leq \rangle, L)$. By the weak CI property we have that the set $Y = (((\mathsf{supp}(g) \uparrow) \cap \mathsf{supp}(f)) \downarrow) \cap \mathsf{supp}(f)$ is a compact-open subset of $\mathsf{supp}(f)$. By property (ii) of Lemma 1, call f_1 to the element $f_{|Y}$ and $f_2 = f - f_1$. We shall first prove that f_1 is orthogonal to f_2:

Let be $x \in \mathsf{supp}(f_1)$. By the construction of Y, there exists $x_1 \in \mathsf{supp}(g)$ and $x_2 \in \mathsf{ms}(f)$ such that $x, x_1 \leq x_2$. If $y \in \mathsf{supp}(f)$ and y is comparable with x, then it is less or equal than x_2, implying that it belongs to $\mathsf{supp}(f_1)$. For the converse, if $x \in supp(f_2)$ then it belongs to the complement of Y. Since Y is an increasing-decreasing subset of $\mathsf{supp}(f)$, we have that x cannot be comparable with any element of Y.

Suppose now that there exist f_3, f_4 such that $f_3 \perp f_4$ and $f_2 = f_3 + f_4$. By orthogonality, we have that $\mathsf{supp}(f_3), \mathsf{supp}(f_4) \subseteq \mathsf{supp}(f_2) = \mathsf{supp}(f) \setminus Y$ and then neither f_3, nor f_4 are orthogonal with f_2. So we have property (i) of r-projectability. ■

Remark 2 In Lemma 1 we have separated the property of orthogonal patchwork because it corresponds exactly to the ri-projectability as can be seen in the proof of the above Proposition.

Remark 3 Observe that, if the order on the Priestley space is trivial and the space is compact, then the definition of Priestley power coincides with that of Boolean power. Dropping the condition of compacity, we would have a lattice-power in the sense of Weispfenning's (see [22]).

4 Priestley products

In the same way that the notion of Boolean product generalize that of Boolean power, we shall define the notion of Priestley product as a generalization of that of Priestley power.

Definition 2 *Let $\langle X, \tau, \leq \rangle$ be a Priestley root-system and, for each $x \in X$, let be L_x a totally ordered group. We say that an ℓ-group G is a* Priestley product *of the family $(L_x)_{x \in X}$ if and only if:*

i) G is a subdirect product of the family $(L_x)_{x \in X}$.

ii.a For each atomic formula $\varphi(v_1, \ldots, v_n)$ of $\mathcal{L}$ and elements $g_1, \ldots, g_n \in G$, the set $[\![\varphi(g_1, \ldots, g_n)]\!] = \{x \in X \ / \ L_x \models \varphi(g_1(x), \ldots, g_n(x))\}$ is clopen. In particular, for each element $g \in G$ the set $[\![g \neq 0]\!] = \mathsf{supp}(g)$ is clopen and compact.

ii.b($\perp$) For each $g, g' \in G$ and clopen set $Y \subseteq X$ such that Y and $X \setminus Y$ are orthogonal there exists an element of G coinciding with g on Y and with g' on $X \setminus Y$.

The class of all Priestley products over the space $\langle X, \tau, \leq \rangle$ and the family $(L_x)_{x \in X}$ will be denoted by $\Gamma(\langle X, \tau, \leq \rangle, (L_x)_{x \in X})$.

Remark 4 • We shall introduce later an enriched language for ℓ-groups, hence, there shall appear new atomic formulas and the meaning of the class of Priestley products shall change. So, when it will be important to stress the language, its name will appear as a superscript of the Γ.

- If we replace point ii.b($\perp$) by

  ii.b For each $g, g' \in G$ and clopen set $Y \subseteq X$ there exists an element of G coinciding with g on Y and with g' on $X \setminus Y$.

  we will have the class of *strong Priestley products*, denoted by $\Gamma_s(\langle X, \tau, \leq\rangle, (L_x)_{x\in X})$.

A first –trivial– example of Priestley product is a Priestley power $\mathcal{PR}(\langle X, \tau, \leq\rangle, L)$. By posing $L_x = L$ for all x, it is easy to verify that it belongs to $\Gamma_s(\langle X, \tau, \leq\rangle, (L_x)_{x\in X})$.

A more interesting example is the following: Let $\langle X, \tau, \leq\rangle$ be a compact Priestley root-system. Let $((L_x)_{x\in X}, (\varphi_{xy})_{x<y})$ be an inductive family of totally ordered groups and monomorphisms. Call L to the inductive limit of the family and, for each x, call L'_x to the image of L_x in L. Consider now the subset G of the Priestley power $\mathcal{PR}(\langle X, \tau, \leq\rangle, L)$ given by

$$G = \{g \ / \ \forall x \in X \ (g(x) \in L'_x)\}.$$

First, we show that G is an ℓ-subgroup of $\mathcal{PR}(\langle X, \tau, \leq\rangle, L)$:
It is clear that $0 \in G$. If $g, h \in G$ then, for each $x \in X$, we have $g(x), h(x) \in L'_x$ and hence $(g + h)(x) = g(x) + h(x) \in L'_x$. For the join, we have that, for each $x \in X$, $(g \vee h)(x) = g(x)$ or $(g \vee h)(x) = h(x)$ and hence we can conclude that $(g \vee h)(x) \in L'_x$.
Second, we prove that $G \in \Gamma_s(\langle X, \tau, \leq\rangle, (L'_x)_{x\in X})$:
G is a subdirect product of the family $(L'_x)_{x\in X}$: For each $x \in X$ and $a_x \in L_x$ we have an image $a' \in L'_x \subseteq L$. The set $\{x\}\uparrow$ is clopen in X and, for each $x' \in \{x\}\uparrow$ we have that there exists a unique $a_{x'} \in L_x$ such that its image in L is a'. So, in the Priestley power there exists the map which takes the value a' on $\{x\}\uparrow$ and 0 in its complement. It is clear that this map belongs to G. The property ii.a of the definition of Priestley product is satisfied by G because it is already satisfied by $\mathcal{PR}(\langle X, \tau, \leq\rangle, L)$. For property ii.b the argumentation is analogous, observing that if $Y \subseteq X$ is clopen and $g, h \in G$, if k is the element of the Priestley power which coincides with g over Y and with h over $X \setminus Y$ then $k(x) = g(x)$ or $k(x) = h(x)$ and hence $k(x) \in L'_x$, implying that $k \in G$.

Remark 5 As in the case of Priestley powers, we have that the Priestley products of totally ordered groups satisfy the condition of ri-projectability (equivalent, in those cases to the orthogonal patchwork property).

5 Projective limits of o-groups

Let be $\langle I, \leq\rangle$ an ordered set, $(L_i)_{i\in I}$ a family of (non-ordered) groups and, for each pair $i < j$ an homomorphism $\varphi_{ij} : L_i \longrightarrow L_j$ such that, for all $i < j < k$, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$.

We recall that the *projective limit* of the system is a group $G = \lim_{\leftarrow} L_i$ and a family of homomorphisms $(\pi_i : G \longrightarrow L_i)_{i \in I}$ which commute with the φ_{ij}'s and that, for any other group H and homomorphisms $(\rho_i)_{i \in I}$, there exists an unique homomorphism $h : H \longrightarrow G$ making the diagram commute. Since, if there are $i < j$ such that φ_{ij} is an isomorphism, L_i and L_j can be identified, we can suppose that all the quotients are proper.

In this work we shall be interested in the case where the family $(\varphi_{ij})_{i<j \in I}$ is composed of onto maps.

If now each L_i is an ℓ-group, we have that, if G is the projective limit of this family in the category of ℓ-groups, then, since the kernels of homomorphisms are ℓ-ideals, the ordered set $\langle I, \leq \rangle$ must be isomorphic to a subset of the lattice of ℓ-ideals of G. If each L_i is totally ordered, since the o-groups can be caracterized as the quotients of an ℓ-group by a prime ℓ-ideal, we have that $\langle I, \leq \rangle$ must be isomorphic to a subset of the root-system of prime ℓ-ideals of G and so, a root-system. In fact this characterizes the projective limits (by epimorphisms) of o-groups in the category of ℓ-groups.

Proposition 6 *Let $\langle I, \leq \rangle$ be an ordered set, $(L_i)_{i \in I}$ a family of o-groups and $(\varphi_{ij} : L_i \longrightarrow L_j)_{i<j}$ a family of epimorphisms of ordered groups satisfying the conditions of compatibility. Then $\langle I, \leq \rangle$ is in fact a root-system and the projective limit in the category of (non-ordered) groups admits a natural ℓ-group structure.*

Proof: By the considerations above, we have that, for $i, j, k \in I$, if $i < j, k$, then both L_j and L_k are (ordered) quotients of the o-group L_i and hence they are comparable, implying that $\langle I, \leq \rangle$ is a root-system. Let G be the projective limit and $(\pi_i)_{i \in I}$ the projections. We define a subset K of G by stating $g \in K$ if and only if, for each $i \in I$, $\pi_i(g) \geq 0$. It is immediate that K is closed for the addition and convex, so it is a cone, defining an order $\leq$ on G. Now, call g_+ to the element of $\prod_I L_i$ given by

$$\pi_i(g_+) = \begin{cases} \pi_i(g) & \text{if } \pi_i(g) \geq 0 \\ 0 & \text{if } \pi_i(g) < 0 \end{cases}$$

(g_- is defined analogously). If we consider $H = \langle g_+ \rangle$ as a subgroup of $\prod_I L_i$ and, for each $i \in I$, $\rho_i : H \longrightarrow L_i$ the restriction of π_i, we have that the homomophism $h : H \longrightarrow G$ is injective, implying that g_+ belongs to G. It is easy to verify that g_+ is the least upper bound of g and 0, implying that the order $\leq$ is a lattice order. ■

Lemma 7 *A projective limit in the conditions of the above stated Proposition satisfies the condition of ri-projectability.*

Proof: Let be $x, y \in G = \lim_{\leftarrow} L_i$. Call I^* to the set $(\{\mathsf{supp}(y) \uparrow \cap \mathsf{supp}(x)\} \uparrow) \downarrow$. And define $z = x \restriction_{I \setminus I^*}$ and $w = x - z$. The construction of I^* assures that I^* is orthogonal with its complement, implying $z \perp w$. Now suppose $w = u + v$ for two orthogonal elements u and v, then $\mathsf{supp}(u), \mathsf{supp}(v) \subseteq \mathsf{supp}(y) \subseteq I^*$. In particular, neither u nor v are orthogonal to y, which implies that property (i) of r-projectability is verified. ■

Observe that, in general, such a projective limit does not satisfy property (*). For example, consider a totally ordered abelian group G such that there exists a value V which is neither a succesor nor the last element in the ordered set of convex subgroups. Consider the ordered set of the respective quotients of the group by those convex subgroups but without considering G/V. It is immediate that those groups plus the canonical projections give a projective system whose limit is G and, if $g \in G$ is such that V is the value of g, the support of g, looked at as an element of the projective system, is not upper bounded. However, if a projective limit satisfies property (*) it is possible to show a relationship with Hahn products.

Proposition 8 *Let $\Lambda(I,(G_i)_{i\in I})$ be a Hahn product. There exist a projective family $((L_i)_{i\in I},(\varphi_{ij})_{i<j})$ of totally ordered groups such that $\Lambda(I,(G_i)_{i\in I})$ can be embedded into $\lim_{\leftarrow} L_i$.*

Proof: Call H to $\Lambda(I,(G_i)_{i\in I})$. For each $i \in I$ we consider the sets

$$\langle i\rangle = \{g \in H \ / \ \mathsf{supp}(g) \cap (\{i\}\uparrow) = \emptyset\} \quad \text{and}$$

$$\overline{\langle i\rangle} = \{g \in H \ / \ \mathsf{supp}(g) \cap (\{i\}\uparrow) = \{i\}\}.$$

It is not difficult to verify that both sets are prime ideals of H. Hence the quotients $L_i = H/\langle i\rangle$ are totally ordered. If $i < j$ we have that $\langle i\rangle \subset \langle j\rangle$ implying the existence of a canonical onto morphism $\varphi_{ij} : L_i \to L_j$. Call G to the projective limit of the family $((L_i)_{i\in I},(\varphi_{ij})_{i<j})$ and consider the map $f : H \to \prod_I L_i$ defined by $f(h)(i) = h/\langle i\rangle$. Since, for each $h \neq 0$ there exists $i \in \mathsf{ms}(h)$ we have that f is one-to-one. Observe that the projective limit G can be seen as an ℓ-subgroup of $\prod_I L_i$. Since if $i < j$ we have that $f(g)(j) = \varphi_{ij}(f(g)(i))$ we conclude that $f(H) \subseteq G$. ■

The proof of the above proposition gives us a partial converse:

Proposition 9 *Let $((L_i)_{i\in I},(\varphi_{ij})_{i<j})$ be a projective family of totally ordered groups where $\langle I,\leq\rangle$ is a root-system and each φ_{ij} an onto map. Call G to the projective limit of the family. If G satisfies property (*) then there exists a subset J of I and a family of totally ordered groups $(G_j)_{j\in J}$ such that G can be subdirectly embedded (as a non-ordered group) into $\prod_J G_j$ and the Hahn product $\Lambda(J,(G_j)_{j\in J})$ can be embedded (as an ℓ-group) into G. If, moreover, G satisfies property (†) then it is isomorphic to the Hahn product.*

Proof: Since G has property (*), for each $g \in G$, if $g \neq 0$ the set $\mathsf{ms}(g)$ is not empty. Call J to $\bigcup_{g\in G} \mathsf{ms}(g)$. For each $j \in J$ consider the sets $\langle j\rangle$ and $\overline{\langle j\rangle}$ which, as in the above Proposition, are prime ideals of G. Hence, the ℓ-group $G_j = \overline{\langle j\rangle}/\langle j\rangle$ must be totally ordered. The map $\sigma : G \to \prod_J G_j$ defined by $\sigma(g)(j) = g/\langle j\rangle$ is a group homomorphism. It is one-to-one because, if $g \neq 0$ there exists $j \in \mathsf{ms}(g) \subseteq J$ and hence $g \notin \langle j\rangle$ implying that $g/\langle j\rangle \neq 0$.
The embedding $\Lambda(J,(G_j)_{j\in J}) \hookrightarrow G$ is constructed as in the above proposition. The proof that this embedding is onto in case of G satisfying property (†) is immediate from the definition of Hahn-product. ■

6 Topological projective limits

The notion of projective limit is a generalization of that of product, the last one being when the order on the set I is trivial. The notion of Boolean product is also a generalization of that of product, the later one being when the topology of the space is discret. Considering ordered topological spaces we shall define another construction involving totally ordered groups.

Definition 3 *Let $\langle X, \tau, \leq \rangle$ be a Priestley root-system, $(L_x)_{x\in X}$ a family of o-groups and, for each $x, y \in X$ such that $x < y$, $\varphi_{xy} : L_x \longrightarrow L_y$ an ℓ-epimorphism (the family of epimorphisms satisfying the condition of compatibility). We say that a subgroup G of the projective limit* $\lim_{\leftarrow} L_x$ *is a* topological projective limit *of the family (which will be denoted by $G \in T(\langle X, \tau, \leq \rangle, (L_x)_{x\in X}))$ if and only if:*

i) For each $x, y \in X$ such that $x \leq y$, $a \in L_y$ and $b \in \varphi_{xy}^{-1}(a)$ there exists $g \in G$ such that $\pi_x(g) = b$ (and hence $\pi_y(g) = a$).

ii) For all $g \in G$, $\mathsf{supp}(g)$ is a decreasing d-compact clopen set.

iii) For all $g \in G$ the subsets $[\![g > 0]\!]$ and $[\![g < 0]\!]$ are clopen.

iv) For all $f, g \in G$ and clopen set $Y \subseteq X$ the subset $Y(f)$ is d-compact and clopen and the element $f_{|Y} \cup g_{|X\setminus(Y(f)\uparrow)}$ belongs to G. (Orthogonal patchwork property).

A subgroup of the projective limit satisfying only conditions (i), (ii) and (iii) will be called a weak topological projective limit *and the class of such groups will be denoted by $T_w(\langle X, \tau, \leq \rangle, (L_x)_{x\in X})$. If it satisfies the general patchwork property it will be called a* strong topological projective limit *and its class denoted by $T_s(\langle X, \tau, \leq \rangle, (L_x)_{x\in X})$. However we will not deal with this last class.*

Remark 6 Condition (i) of the above definition is analogous to that of subdirect product for the case when the order on X is trivial. It would be possible to call it a subprojective limit but we think that it would only lead to complicate our language.

Remark 7 Observe that, for a more general case, it will be necessary –as in the case of Boolean products– to replace condition (iii) by

iii') For all atomic formula $\varphi(v_1, \ldots, v_n)$ and elements $g_1, \ldots, g_n \in G$, the set $[\![\varphi(g_1, \ldots, g_n)]\!]$ is clopen.

Lemma 10 *Let G be a topological projective limit of a family $(L_x)_{x\in X}$ of totally ordered abelian groups.*

i) G satisfies property (*).

ii) For any $g \in G$, we have $\mathsf{supp}_+(g) = [\![g > 0]\!]$ and $\mathsf{supp}_-(g) = [\![g < 0]\!]$

iii) G is an ℓ-subgroup of the projective limit.

iv) For any d-compact clopen decreasing set $Y \subseteq X$ there exists $g \in G$ such that $Y = \mathsf{supp}(g)$.

Proof: i) The proof is analogous to that of Lemma 3 i), observing that in that proof we use the compacity with respect to increasing closed sets, which is exactly the d-compacity.
ii) Since G satisfies condition (*) we have that $\mathsf{supp}_+(g)$ is well defined and equals $(\llbracket g > 0 \rrbracket \cap \mathsf{ms}(g)) \downarrow$. Let $x \in \mathsf{supp}_+(g)$, there exists then $y \in \mathsf{ms}(g)$ such that $g(y) = \pi_y(g) > 0$. Since $\pi_y(g) = \varphi_{xy}(\pi_x(g))$ and the mappings are order morphisms, we conclude that $g(x) = \pi_x(g) > 0$, implying that $y \in \llbracket g > 0 \rrbracket$. The converse is immediate and for $\mathsf{supp}_-(g)$ the proof is analogous.
iii) Since G is a subgroup of the projective limit, it is ordered. To prove that it is an ℓ-subgroup it suffices to show that, for each $g \in G$, the element g_+ belongs to G. By using the orthogonal patchwork property and what was proved in (i), we have that $g_+ = g_{|\mathsf{supp}_+(g)}$ is an element of the group.
iv) Observe that property (*) implies that $Y = m(Y) \downarrow$, where $m(Y)$ is the set of maximal elements of Y. Using property (i) of Definition of topological projective limits, for each $y \in m(Y)$ there exists $g_y \in G$ such that $g_y(y) > 0$ and $g_y(x) = 0$ for all $x > y$. By property (iii) of the Definition and part (i) of this Lemma, we have that, for each $y \in m(Y)$, the set $\llbracket g_y > 0 \rrbracket$ is clopen and decreasing and the family $(\llbracket g_y > 0 \rrbracket)_{y \in m(Y)}$ covers Y. By d-compacity of Y there is a finite family of disjoint sets $Y_1, \ldots, Y_n$, where $Y_i = \llbracket g_{y_i} > 0 \rrbracket$ for some $y_i \in m(Y)$ covering Y. Now, using $n-1$ times the orthogonal patchwork property, call g to the element of G which coincides with g_{y_i} on Y_i for each $i = 1, \ldots, n$. We have that $\mathsf{supp}_+(g) = \mathsf{supp}(g) = Y$. ■

For a Hahn-product $G = \Lambda(X, (L_x)_{x \in X})$ we have that, for each $x \in X$, the subset $< x >= \{h \in G \ / \ \mathsf{supp}(h) \cap (\{x\} \uparrow) = \emptyset\}$ is a prime ℓ-ideal of G , implying that the quotient $G/ < x >$ is totally ordered. In particular $< x >$ is a value for all $g \in G$ such that $x \in \mathsf{ms}(g)$. Observe that, if G is only a subgroup of $\prod_X L_x$ satisfying the property (*), we have that G is a partially ordered group and also each $< x >$ is a prime convex subgroup and $G/ < x >$ is totally ordered.

Proposition 11 *Let $\langle X, \tau, \leq \rangle$ be a I_d and weak C_d Priestley root-system and L an o-group. Call G to the Priestley power $\mathcal{PR}(\langle X, \tau, \leq \rangle, L)$. There exists a family $(L_x)_{x \in X}$ of o-groups and a family of compatible epimorphisms $(\varphi_{xy} : L_x \longrightarrow L_y)_{x<y}$ such that G is isomorphic to a member of $T(\langle X, \tau, \leq \rangle, (L'_x)_{x \in X})$.*

Proof: For each $x \in X$, call L_x to the quotient $G/ < x >$ and π_x to the canonical epimorphism. Suppose $x < y$, since X is a root-system, we have that $\{y\} \uparrow$ is a proper final section of $\{x\} \uparrow$ and then $\langle x \rangle$ is a proper (prime) ℓ-ideal of $\langle y \rangle$, defining a natural projection φ_{xy} which commutes with π_x and π_y. By construction and the property of universality of the projective limit, we have that G is isomorphic to a subgroup of it. We know that G verifies (as a Priestley power) property (*). Since the supports of its elements are not the same for our new construction, we have to check that it still satisfies that property. Let be $g \in G$ and $x \in X$ such that $\pi_x(g) \neq 0$. This implies that $g \notin \langle x \rangle$ and then there exists $y \geq x$ such that $y \in \mathsf{supp}(g)$ (in the Priestley power). By Lemma 3 i), there exists $z \in \mathsf{ms}(g)$ such that $y \leq z$. Now, for any $w > z$ we have that $g \in \langle w \rangle$ but $g \notin \langle z \rangle$, implying that z belongs to the maximal support of g construed as an element of the projective limit. Hence we can conclude that G satisfies property (*). The above proof shows that the support of g as an element of the projective limit is the initial section of its support as an element of the Priestley power, implying that the maximal supports are the same.

Property (i) of the definition of topological projective limits holds true by the construction of the projective system. Properties (ii) and (iii) hold because, for any $g \in G$ $\mathsf{supp}(g)$, supp_+ and $\mathsf{supp}_-(g)$ (construed in the Priestley power) are compact-open. Since X satisfies the properties I_d and weak C_d, we have that the respective supports in the projective limit (the respective initial sections) are also clopen and, being closed initial sections of compact sets, d-compact. The patchwork property, being applied to orthogonal sets, is transferred directly from the Priestley power to the projective limit. ■

Proposition 12 *Let $G = \Lambda(I, (L_i)_{i \in I})$. There exists a Priestley root-system $\langle X, \tau, \leq \rangle$ which –as a root-system– extends I, a family of o-groups $(L'_x)_{x \in X}$ (such that, for any $x \in I$, we have that L_x is isomorphic to a subgroup of L'_x) and such that G is isomorphic to an element of $T(\langle X, \tau, \leq \rangle, (L'_x)_{x \in X})$.*

Proof: Let $\mathcal{V}$ be the family of all values of elements of G (that is, the ℓ-ideals maximal for the condition of not containing an element of the group) and its successors (that is, the ℓ-ideals minimal for the condition of containing an element of the group). Since any value is a prime ℓ-ideal and any ℓ-ideal greater than a prime one is also prime, we have that the set $\mathcal{V}$, with the order of the inclusion forms a root-system. Define, then, for each $V \in \mathcal{V}$, the totally ordered group $L'_V = G/V$. We shall define on $\mathcal{V}$ the topology τ whose two semibases of clopen sets are:

$$(\hat{g} = \{V \in \mathcal{V} \ / \ g \notin V\},\ \check{g} = \{V \in \mathcal{V} \ / \ g \in V\})_{g \in G}.$$

That is, any open set can be writen as $(\bigcup_{j \in J_0} \hat{g}_j \cap \check{h}_j) \cup (\bigcup_{j \in J_1} \hat{g}_j) \cup (\bigcup_{j \in J_2} \check{h}_j)$ for families $(g_j)_{j \in J_0 \cup J_1}$ and $(h_j)_{j \in J_0 \cup J_2}$ of elements of G. The proof of the d-compacity of the sets $\hat{g}$ is analogous to that of the compacity in the classical case (non-ordered spectrum) (see, for example [13, §1]).

We shall first show that $\langle \mathcal{V}, \tau, \subseteq \rangle$ is a Priestley space:
Let $V, W \in \mathcal{V}$ such that $V \not\subseteq W$. Hence, there exists $g \in V \setminus W$. Then, V belongs to the increasing clopen set $\check{g}$ and W to the decreasing clopen set $\hat{g}$. The definitions of those sets, imply that the intersection is empty, proving that $\mathcal{V}$ is a Priestley space.

By the construction of the quotients, observe that, if $V, W \in \mathcal{V}$ such that $V \subset W$ there exists a natural epimorphism $\varphi_{VW} : L'_V \longrightarrow L'_W$ and the family of all such epimorphisms is compatible. Let then call H to the projective limit of the system. Since the natural maps $\rho_V : G \longrightarrow L'_V$ commute avec the φ_{VW}, we have that there exists an ℓ-group homomorphism $\sigma : G \longrightarrow H$ defined by $\sigma(g)(V) = g/V$. Since, for each non-zero

$g \in G$ there exists a value $V \in \mathcal{V}$, we have that $\sigma(g)(V) \neq 0$, implying that σ is an embedding.Since, for each $i \in I$, the set $\langle i \rangle \subseteq G$ is a value, we have that $\{< i > \ / \ i \in I\}$ is a sub root-system of $\mathcal{V}$. By the standard construction of a Hahn product we have that L_i is the quotient of the ℓ-ideal $B(i) = \{g \in G \ / \ i \in \mathsf{ms}(g)\}$ by $< i >$. Hence, we obtain a natural embedding of L_i into L'_i.

The property (*) is consequence of the fact that, for any lattice-ordered group G, element $g \in G$ and ℓ-ideal $A \subset G$ such that $g \notin A$, there exists an ℓ-ideal V extending A and maximal for the condition of not containing g. In fact V is a value of g and, in our case, it belongs to $\mathsf{ms}(g)$.

For property (i), consider two values $V \subset W$, $a \in L_W$ and $b \in \varphi_{VW}^{-1}(a)$. Since L_V is a quotient of G, there exists $g \in G$ such that $\rho_V(g) = b$ and hence $\rho_W(g) = a$. The construction of the embedding σ gives the surjectivity of the quotients.

Property (ii) is consequence of the fact that $\mathsf{supp}(g) = \{V \in \mathcal{V} \ / \ g \notin V\} = \hat{g}$ which is d-compact clopen. This set is, obviously, decreasing.

For property (iii), since G is an ℓ-group, we have that, for all $g \in G$, g_+ and g_- belong to G. So we have $\mathsf{supp}_+(g) = \hat{g_+}$ and $\mathsf{supp}_-(g) = \hat{g_-}$ which are clopen sets. Since by Lemma 10 (ii), we have the identities $[\![g > 0]\!] = \mathsf{supp}_+(g)$ and $[\![g < 0]\!] = \mathsf{supp}_-(g)$ we can conclude that (iii) holds for G.

For the orthogonal patchwork property, it suffices to prove that, for any $g \in G$ and clopen set $Y \subseteq X$ the element $g_{|Y}$ exists. The support of this element must be $Y(g) = (Y \uparrow \cap \mathsf{supp}(g)) \downarrow \cap \mathsf{supp}(g)$ which is d-compact clopen and decreasing. By Lemma 10 (iv), there exists an element $f \in G$ such that $\mathsf{supp}(f) = Y(g)$. So, using condition (i) of r-projectability, we have that there exist unique g_0 and g_1 such that $g = g_0 + g_1$, $g_0 \perp g_1$ and g_1 is maximal for the condition of not being possible to decompose it in two elements, one of them orthogonal to f. The orthogonality of g_0 and g_1 implies that the final sections of $\mathsf{supp}(g_0)$ and $\mathsf{supp}(g_1)$ in $\mathsf{supp}(g)$ are orthogonal. The maximality of g_1 implies that $\mathsf{supp}(g_1)$ is the greatest part of $\mathsf{supp}(g)$ with no subpart orthogonal to $\mathsf{supp}(f)$. Hence, we can conclude that g_1 equals $g_{|Y}$, proving the orthogonal patchwork property. ■

Remark 8 Observe that in the above Proposition, for $g \in \Lambda(I, (L_i)_{i \in I})$, if we call $\mathsf{sh}(g)$ and $\mathsf{msh}(g)$ to its support and maximal support in the Hahn-product, we have that $\mathsf{supp}(g) \cap I \neq \mathsf{sh}(g)$ if I is infinite and not anti well-ordered, because $\mathsf{supp}(g) \cap I$ is an initial section of I. However, we have that $\mathsf{msh}(g) = \mathsf{ms}(g) \cap I$.

Observe that in the above Proposition, we have only used that G is a Hahn-product for proving the embedding of the root-system. The rest the proof is absolutely general for ri-projectable ℓ-groups. However, if we consider any ℓ-group, the proof still holds true for weak topological projective limits. So we can state the

Theorem 13 (Topological Ordered Representation Theorem for ℓ-groups) *any ℓ-group there exists a Priestley root-system $\langle X, \tau, \leq \rangle$, a family $(L_x)_{x \in X}$ of o-groups* *and an ℓ-group $G' \in T_w(\langle X, \tau, \leq \rangle, (L_x)_{x\in X})$ such that the isomorphism $G \cong G'$ holds. Moreover, if G is ri-projectable, then $G' \in T(\langle X, \tau, \leq \rangle, (L_x)_{x\in X})$*

In fact, we have also the converse for the ri-projectable case:

Theorem 14 *Given a Priestley root-system $\langle X, \tau, \leq \rangle$, a family $(L_x)_{x\in X}$ of o-groups, all the members of $T(\langle X, \tau, \leq \rangle, (L_x)_{x\in X})$ are ri-projectable.*

Proof: Let $G \in T(\langle X, \tau, \leq \rangle, (L_x)_{x\in X})$ and $f, g \in G$. Call Y to $\mathsf{supp}(g)$. By the ortohogonal patchwork property, the element $f_0 = f_{|Y}$ belongs to G. Call f_1 to $f - f_0$. It is easy to verify that $f_0 \perp f_1$ and that the decomposition is "optimal" in the sense of the definition of ri-projectability. ■

Remark 9 In [4] Cignoli and Torrens study for MV-algebras (another formulation for Wajsberg algebras) the space of *all* prime ideals endowed with its Stone or/and Priestley topology. They state the problem of characterizing the topological root-systems which are the spectra of MV-algebras. Theorem 13 and the proof of Proposition 12 gives an answer to the analogous question:

Which are the Priestley root-systems which are homeomorphic to the space of values and their successors of an ℓ-group?

Answer: *all.* Consider, for a Priestley root-system $\langle X, \tau, \leq \rangle$, the Priestley power $\mathcal{PR}(\langle X, \tau, \leq \rangle, \mathbb{Q})$. The fact of taking an archimedean o-group assures us that the topological root-system of values and their successors of the Priestley power will be homeomorphic to the original Priestley root-system. This example permits to sharpen the result as

Corollary 15 *For any Priestley root-system there exists a divisible, ri-projectable ℓ-group whose space of values and their successors is homeomorphic to the given Priestley root-system.*

Remark 10 In [13] and [1, Ch. 10; §10.6], Keimel developes a sheaf representation for f-rings (which can be utilised for abelian ℓ-groups) over the space of all prime ℓ-ideals. A main difference with our treatment is that in Keimel's the f-rings (ℓ-groups) associated to each point of the space are not necessarily totally ordered. In Keimel's representation the order of the space plays no particular role but in ours it is essential for defining the order of the ℓ-group.

7 Factorizing the weak Boolean representation

Recall (see [11]) that given an ℓ-group G with a weak unit $u \in G$, we can define in a natural way a Boolean algebra

$$B(G; u) = \{g \in G \;/\; 0 \leq g \leq u \;\&\; g \perp (u - g)\}.$$

In the projectable case, it is proved that for any weak units $u, v \in G$, the isomorphism $B(G;u) \cong B(G;v)$ holds. This Boolean algebra determines the natural Boolean product representation of such groups. If G has a strong unit u (that is, for all $g \in G$ there exists $n \in \mathbb{N}$ such that $g \leq nu$), we call $B(G)$ to $B(G;u)$ since the Boolean algebra does not depend (modulo isomorphism) from the particular strong unit chosen. This $B(G)$ is exactly the same $B(A)$, the Boolean algebra of complemented elements of the MV-algebra $A = G[0,u] = \{g \in G \ / \ 0 \leq g \leq u\}$ (see [4]).

Definition 4 *We say that an ℓ-group G is* uniformly Boolean *if there exists a weak unit u such that, for all weak unit $v \geq u$ the isomorphism $B(G;v) \cong B(G;u)$ holds.*

As in the projectable case, the isomorphism is given by the application $g \longmapsto g \wedge u$ (in the general case this application is only injective).

As it is the case for MV-algebras, we have that a non-trivial uniformly Boolean ℓ-group G is directly indecomposable (does not admit a non-trivial representation as the product of two ℓ-groups) if and only if $B(G) \cong 2$. In their article, Cignoli and Torrens prove that any MV-algebra A is representable as a weak Boolean product (the supports are not necessarily clopen but only open) of directly indecomposable MV-algebras. Moreover, they prove that each Boolean subalgebra of $B(A)$ is the Boolean algebra of such a representation. The essentials of Cignoli and Torrens proof, translated to our context, are the following:

Let G be a uniformly Boolean ℓ-group, $u \in G$ a weak unit such that $B(G;u) \cong B(G)$ and C a Boolean subalgebra of $B(G;u)$. For each prime ideal P of C, the set

$$I(P) = \{g \in G \ / \ \exists n \in \mathbb{N}, a \in P(| g | \leq na)\}$$

is an ℓ-ideal of G. By calling X_C to the set of prime ideals of C with its Boolean topology, the map

$$\alpha : G \longrightarrow \prod_{P \in X_C} G/I(P) \qquad (\alpha(g)(P) = g/I(P))$$

gives a weak Boolean product representation. Such that all the ℓ-groups $(G/I(P))_{P \in X_C}$ are indecomposable if and only if $C = B(G;u)$.

Now, considering the weak topological projective limit representation of such a group, if we call $\mathcal{V}(u)$ to $\hat{u}$, we have that $\langle \mathcal{V}(u), \tau, \subseteq \rangle$ is also a Priestley space, but in this case compact, giving a new topological projective limit representation of G by totally ordered groups constructed in an analogous way to that of Proposition 12. If we call $\mathcal{D}_{01}(G;u) = \mathcal{D}_{01}(\langle \mathcal{V}(u), \tau, \subseteq \rangle)$ to the bounded distributive lattice whose Priestley space is $\langle \mathcal{V}(u), \tau, \subseteq \rangle$, we have that it is isomorphic to the quotient of the interval $[0,u]$ of G by the relation given by: $g \sim g'$ if and only if $\hat{g} = \hat{g}'$. Since for $g, g' \in B(G;u)$ we have $\hat{g} = \hat{g}'$ if and only if $g = g'$, we can think $B(G;u)$ as a 01-sublattice of $\mathcal{D}_{01}(G;u)$. Obviously the same holds for any Boolean subalgebra C of $B(G;u)$.

Since the functor Pr (which gives the Priestley space of a bounded lattice) is contravariant, we have the following topological surjections:

$$\langle \mathcal{V}(u), \tau, \subseteq \rangle \xrightarrow{\mu} \mathsf{Pr}(B(G;u)) \xrightarrow{\nu} \mathsf{Pr}(C).$$

Now, for each $P \in Y = \mathsf{Pr}(C)$, call $v(P)$ to $(\nu \circ \mu)^{-1}(P)$ which is a compact subset of X and then a compact Priestley space. Call G_P to the restriction of G to $v(P)$. We have a natural isomorphism $G_P \cong G/I(P)$. If $h : G \longrightarrow \prod_{x\in X} G_x$ is the topological projective limit representation of G, we can thought of G_P as the topological projective limit representation of $G/I(P)$ in the form $h_P : G/I(P) \longrightarrow \prod_{x\in v(P)} G_x$.
This permits us to state the following:

Proposition 16 *Let G be a uniformly Boolean ℓ-group, $u \in G$ a weak unit satisfying the corresponding property and C a Boolean subalgebra of $B(G;u)$. The Cignoli and Torrens weak Boolean product representation of G over $\mathsf{Pr}(C)$ can be factorized through the weak topological projective limit representation of G over $X = \langle \mathcal{V}(u), \tau, \subseteq \rangle$.*

8 Transfer of elementary equivalence

The Feferman-Vaught theorem (see [9]) says, essentially, that the theory of the product of models is given by that of its factors. In [3], [21], [23] and [11] it is extended for certain particular cases of Boolean products. So it is a natural question to look at its possible extension for Priestley powers and products. We shall restrict ourselves to the study of elementary equivalence and this will be done by the method of back-and-forth or Ehrenfeucht-Fraïssé games.

Recall (see [8]) that given two models A and B for a finite language $\mathcal{L}$ which has no function symbols, we have that $A \equiv B$ (A is elementary equivalent to B) if and only if there exists a family of relations $(\equiv_{n,m})_{n,m\in\omega} \subseteq A^n \times B^n$ with the following properties:

i) If $(a_1, \ldots, a_n) \equiv_{n,0} (b_1, \ldots, b_n)$ then there is a partial isomorphism $f : (a_1, \ldots, a_n) \longrightarrow (b_1, \ldots, b_n)$ for the relation and constant symbols of $\mathcal{L}$.

ii) If $(a_1, \ldots, a_n) \equiv_{n,m} (b_1, \ldots, b_n)$ and $a_{n+1} \in A$ ($b_{n+1} \in B$), there exists $b_{n+1} \in B$ ($a_{n+1} \in A$) such that $(a_1, \ldots, a_n, a_{n+1}) \equiv_{n+1,m-1} (b_1, \ldots, b_n, b_{n+1})$. (Back-and-forth property).

Two such models are said to be ω-elementary equivalent (denoted $A \equiv_\omega B$) if the above family of relations does not depend on the index m. That is, the back-and-forth is uniform.

Taking the language $\mathcal{L} = \langle +, -, 0, \leq \rangle$ of ordered groups, we can transform it in a new language without function symbols $\mathcal{L}' = \langle \mathsf{sum}(\ ,\ ,\ ), 0, \leq \rangle$ where $\mathsf{sum}(f, g, h)$ if and only if

$f + g = h$. We do not need a relation for the minus function because it can be expressed with a formula in $\mathcal{L}'$. So we can use the back-and-forth characterization of elementary equivalence.

The following result is general and almost does not depend on the order structure of the Priestley space.

Proposition 17 *Let $\langle X, \tau, \leq \rangle$ be a Priestley root-system and L and L' two totally ordered groups. If $L \equiv L'$ then their respective Priestley powers $G = \mathcal{PR}(\langle X, \tau, \leq \rangle, L)$ and $G' = \mathcal{PR}(\langle X, \tau, \leq \rangle, L')$ are elementary equivalent.*

Proof: By the Ehrenfeucht-Fraïssé theorem, there exist a family of relations $(E_{n,m})_{n,m \in \omega} \subseteq L^n \times L'^n$ giving the elementary equivalence.
Define a new family of relations $(E'_{n,m})_{n,m \in \omega}$ such that $E'_{n,m} \subseteq G^n \times G'^n$ by:
$(g_1, \ldots, g_n) E'_{n,m} (g'_1, \ldots, g'_n)$ if and only if, for each $x \in X$,
$(g_1(x), \ldots, g_n(x)) E_{n,m} (g'_1(x), \ldots, g'_n(x))$ holds.
We shall prove that this family has the properties (i) and (ii) above.

i) $(g_1, \ldots, g_n) E'_{n,0} (g'_1, \ldots, g'_n)$ implies that, for each $x \in X$, $(g_1(x), \ldots, g_n(x))$ and $(g'_1(x), \ldots, g'_n(x))$ satisfy the same atomic formulas of the language $\mathcal{L}'$. It is immediate, by the definition of the product operations that this holds for the sum on the n-tuples of G and G'. For the order, since $g_i \leq g_j$ if and only if $\mathsf{ms}(g_{i+}) \preccurlyeq \mathsf{ms}(g_{j+})$ and for each $x \in \mathsf{ms}(g_{i+}) \cap \mathsf{ms}(g_{j+})$, $g_i(x) \leq g_j(x)$ we have that $E'_{n,0}$ preserves the order. So we have that $E'_{n,0}$ implies an $\mathcal{L}'$-isomorphism.

ii) Suppose $(g_1, \ldots, g_n) E'_{n,m} (g'_1, \ldots, g'_n)$ and consider $g_{n+1} \in G$. There is a partition of X in a finite set $\mathcal{O}$ of clopen compact sets on which the maps $g_1, \ldots, g_{n+1}, g'_1, \ldots, g'_n$ are constant (this hold because each one of the maps takes only a finite number of values). For each clopen compact $U \in \mathcal{O}$ chose a point $x_U \in U$ and $g'_U \in L'$ such that $(g_1(x_U), \ldots, g_{n+1}(x_U)) E_{n+1,m-1} (g'_1(x_U), \ldots, g'_n(x_U), g'_U)$. Since all the given maps are constant on U, we have that the equivalence holds true for each $x \in U$. Now, by the patchwork property, define the element $g_{n+1} \in G'$ given by $g_{n+1\,|U} = g'_U$ for each $U \in \mathcal{O}$. The construction implies $(g_1, \ldots, g_{n+1}) E'_{n+1,m-1} (g'_1, \ldots, g'_{n+1})$.

So we have the elementary equivalence of those Priestley powers. ■

Remark 11 Observe that for the Proposition above we do not need any additional assumption about the properties of the Priestley root-system. So this result holds true even in the case that the respective Priestley powers are not ℓ-groups (see Proposition 4).

For a Priestley root-system $\langle X, \tau, \leq \rangle$, let $\mathcal{D}(X)$ be the distributive lattice associated with the space given by the Priestley representation theorem of distributive lattices. Observe that $\mathcal{D}(X)$ can be thought of as the set of clopen decreasing or clopen increasing subsets of X. Define now $\mathcal{TR}(X)$ the set of clopen compacts which are the intersection of a decreasing clopen d-compact and an increasing clopen i-compact. Since any decreasing (increasing) clopen is d-compact (i-compact), we have that any nonempty element of $\mathcal{TR}(X)$ is associated to a unique ordered pair of elements of $\mathcal{D}(X)$ and conversely. On $\mathcal{TR}(X)$ we can consider two partial order relations: the set theoretical inclusion $\subseteq$ and the order $\preccurlyeq$ defined on §2.

Proposition 18 *Let $\langle X, \tau, \leq\rangle$ and $\langle X', \tau', \leq\rangle$ be two Priestley root-systems and L and L' two totally ordered groups. If $L \equiv L'$ and $\mathcal{D}(X) \equiv_\omega \mathcal{D}(X')$, then their respective Priestley powers $G = \mathcal{PR}(\langle X, \tau, \leq\rangle, L)$ and $G' = \mathcal{PR}(\langle X', \tau, \leq\rangle, L')$ are elementary equivalent.*

Proof: Let $\mathcal{C}(X)$ ($\mathcal{C}(X')$) be the set of clopen compact sets of X (of X'). Each $Y \in \mathcal{C}(X)$ is the union of a finite family of elements of $\mathcal{TR}(X)$. Let $\mathcal{TR}(X)^*$ be the set of finite sequences of elements of $\mathcal{TR}(X)$ (we proceed analogously with $\mathcal{TR}(X')$). By the consideration above, we have that $\mathcal{D}(X) \equiv_\omega \mathcal{D}(X')$ if and only if $\mathcal{TR}(X) \equiv_\omega \mathcal{TR}(X')$. Hence, it is immediate that the uniform family $(F_n)_{n\in\omega}$ of back-and-forth relations between $\mathcal{D}(X)$ and $\mathcal{D}(X')$ implies the existence of a family $(F^*_{n,m})_{n,m\in\omega}$ of back-and-forth relations between $\mathcal{TR}(X)^*$ and $\mathcal{TR}(X')^*$. By hypothesis, there exists another family $(E_{n,m})_{n,m\in\omega}$ giving the elementary equivalence of L and L'.
As in the proof of the previous Proposition, for each $g_1, \ldots, g_n \in G$ there is a finite partition $\mathcal{O}^*(g_1, \ldots, g_n)$ of X in clopen compact sets such that, g_i ($i = 1, \ldots, n$) is constant for each $U \in \mathcal{O}^*(g_1, \ldots, g_n)$. Observe that, if $k \leq n$ then the partition $\mathcal{O}^*(g_1, \ldots, g_n)$ is finer that the partition $\mathcal{O}^*(g_1, \ldots, g_k)$.
Now, define a family $(E'_{n,m})_{n,m\in\omega}$ of back-and-forth relations between G and G' by:
$(g_1, \ldots, g_n)E'_{n,m}(g'_1, \ldots, g'_n)$ if and only if, there exists an order isomorphism f between $\mathcal{O}^*(g_1, \ldots, g_n)$ and $\mathcal{O}^*(g'_1, \ldots, g'_n)$;
$(\mathcal{O}^*(g_1), \ldots, \mathcal{O}^*(g_1, \ldots, g_n))F^*_{n,m}(\mathcal{O}^*(g'_1), \ldots, \mathcal{O}^*(g'_1, \ldots, g'_n))$, where the implied isomorphism coincides with f and, for each $U \in \mathcal{O}^*(g_1, \ldots, g_n)$, $x \in U$ and $y \in f(U)$ we have that $(g_1(x), \ldots, g_n(x))E_{n,m}(g'_1(y), \ldots, g'_n(y))$ holds.
Let us prove that the family $(E'_{n,m})_{n,m\in\omega}$ is a back-and-forth family for the models G and G':

i) Let $(g_1, \ldots, g_n)E'_{n,0}(g'_1, \ldots, g'_n)$ then f is an isomorphism between $\mathcal{O}^*(g_1, \ldots, g_n)$ and $\mathcal{O}^*(g'_1, \ldots, g'_n)$. For each $U \in \mathcal{O}^*(g_1, \ldots, g_n)$, $x \in U$ and $y \in f(U)$, we have $(g_1(x), \ldots, g_n(x))E_{n,0}(g'_1(y), \ldots, g'_n(y))$ and hence the two n-tuples are immediately isomorphic for the group relation. We shall verify that the isomorphism also holds for the order relation: if $g_i \leq g_j$, for each maximal $U \in \mathcal{O}^*(g_1, \ldots, g_n)$ (that is, an element of the partition maximal for the relation $\preccurlyeq$) and $x \in U$, we have $g_i(x) \leq g_j(x)$, $f(U)$ is maximal in $\mathcal{O}^*(g'_1, \ldots, g'_n)$ and for each $y \in f(U)$, $g'_i(y) \leq g'_j(y)$ holds. So we have the isomorphism also for the order relation.

ii) Suppose $(g_1, \ldots, g_n)E'_{n,m}(g'_1, \ldots, g'_n)$ and consider an element $g_{n+1} \in G$. By the back-and-forth between $\mathcal{TR}(X)$ and $\mathcal{TR}(X')$ there exists a finite sequence $\mathcal{O}^*$ and a bijection $f : \mathcal{O}^*(g_1, \ldots, g_{n+1}) \longrightarrow \mathcal{O}^*$ such that

$(\mathcal{O}^*(g_1),\ldots,\mathcal{O}^*(g_1,\ldots,g_{n+1}))F^*_{n,m}(\mathcal{O}^*(g'_1),\ldots,\mathcal{O}^*(g'_1,\ldots,g'_n),\mathcal{O}^*)$. For each $U \in \mathcal{O}^*(g_1,\ldots,g_{n+1})$ we can chose an $x \in U$ and a $g_U \in L'$ such that for all $y \in f(U)$, $(g_1(x),\ldots,g_n(x),g_{n+1})E_{n+1,m-1}(g'_1(y),\ldots,g'_n(y),g_U)$ holds. Remark that the elements $g_1,\ldots,g_{n+1}$ and $g'_1,\ldots,g'_n$ are constant, respectively, on U and $f(U)$. Now, by the patchwork property, define $g'_{n+1} \in G'$ as coinciding with g'_U on each $U \in \mathcal{O}^*$. So we can conclude $(g_1,\ldots,g_{n+1})E'_{n+1,m-1}(g'_1,\ldots,g'_{n+1})$, implying the elementary equivalence. ■

For Priestley products we have a weaker result. Consider $\mathcal{L}'_d$ the language $\mathcal{L}'$ enriched with the countable set of unary relation symbols $(d_p)_{p\in \text{ Primes}}$ where $d_p(g)$ is to be interpreted as "g is divisible by p". As it was stated in Remark 4, for any ℓ-group $G \in \Gamma^{\mathcal{L}'_d}$ and any $g \in G$, the set $[\![d_p(g)]\!]$ must be clopen.

We recall (see [19]) that the elementary class generated by the totally ordered archimedean groups is that of *regular* totally ordered groups. A group G is regular if and only if either it has a first positive element (in that case $G \equiv \mathbb{Z}$) or, for each prime p and each $g, h \in G$ such that $g < h$, each class of $G/\langle p\rangle$ has a representative in the interval (g,h). Hence, in particular any divisible totally ordered abelian group is regular and elementary equivalent to $\mathbb{Q}$.

Proposition 19 *Let $\langle X,\tau,\leq\rangle$ be a Priestley root-system and, for each $x \in X$, G_x and H_x two elementary equivalent totally ordered regular groups. In that conditions, if $G \in \Gamma_s^{\mathcal{L}'_d}(\langle X,\tau,\leq\rangle,(G_x)_{x\in X})$ and $H \in \Gamma_s^{\mathcal{L}'_d}(\langle X,\tau,\leq\rangle,(H_x)_{x\in X})$ then $G \equiv H$.*

Proof: Since for each $x \in X$, we have the elementary equivalence of the two regular groups G_x and H_x we can use a result inspired by [12, Lemma 2.6] and developped in [11, Lemma 3.1] giving, for the regular totally ordered groups an uniform family of back-and-forth relations $(E_{n,m})_{n,m\in\omega}$. Define a new family $(E'_{n,m})_{n,m\in\omega}$ such that, for each $n \in \mathbb{N}$ we have $E'_{n,m} \subseteq G^n \times H^n$ by:
$(g_1,\ldots,g_n)E'_{n,m}(h_1,\ldots,h_n)$ if and only if for all $x \in X$ we have
$(g_1(x),\ldots,g_n(x))E'_{n,m}(h_1(x),\ldots,h_n(x))$.
We shall prove that this family satisfies properties (i) and (ii) of the back-and-forth:

i) $(g_1,\ldots,g_n)E'_{n,0}(h_1,\ldots,h_n)$ implies that $(g_1(x),\ldots,g_n(x))$ and $(h_1(x),\ldots,h_n(x))$ satisfy the same atomic formulas in $\mathcal{L}'_d$ for each $x \in X$. Since $\mathcal{L}'_d$ is relational this implies that $(g_1,\ldots,g_n)$ and $(h_1,\ldots,h_n)$ also satisfy the same atomic formulas and are then $\mathcal{L}'_d$-isomorphic.

ii) Suppose $(g_1,\ldots,g_n)E'_{n,m}(h_1,\ldots,h_n)$ and consider an element $g_{n+1} \in G$. For each $x \in X$ it is possible to find an element $h^x \in H_x$ such that $(g_1(x),\ldots,g_n(x),g_{n+1}(x))E'_{n+1,m-1}(h_1(x),\ldots,h_n(x),h^x)$. Since H is a subdirect product of the family $(H_x)_{x\in X}$, we have that, for each $x \in X$ there exists an element $\hat{h^x} \in H$ such that $\hat{h^x}(x) = h^x$. For each $x \in X$ call Y_x to the subset of X given by $\{y \in X \ / \ (g_1(y),\ldots,g_n(y),g_{n+1}(y))E_{n+1,m-1}(h_1(y),\ldots,h_n(y),\hat{h^x}(y))\}$. Since the relation $E_{n+1,m-1}$ can be expressed by a quantifier free $\mathcal{L}'_d$-formula, we can conclude that Y_x is clopen. Since $\mathsf{supp}(g_{n+1})$ is compact, there exists a finite subset $\{x_1, \ldots, x_s\} \subseteq X$ such that $\bigcup_{i=1}^{s} Y_{x_i} = \mathsf{supp}(g_{n+1})$. Hence, using $s-1$ times the patchwork property, we can find h_{n+1} such that $(g_1, \ldots, g_{n+1})E'_{n+1,m-1}(h_1, \ldots, h_{n+1})$. ■

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U.R.A. 753
Équipe de Logique-Mathématique
U.F.R. de Mathématiques
Université Paris 7
2, pl. Jussieu
75251 Paris Cedex 05 - France
e-mail: grpord@logique.jussieu.fr

Département de Mathématiques
U.F.R. Structures et Matériaux
Université d'Angers
2, Bd. Lavoisier
49045 Angers Cedex - France
e-mail: gluscha@univ-angers.fr
e-mail: lucas@univ-angers.fr

Corps presque réels clos ou "ARC"

Françoise Delon

Nous appelons ARC (pour "almost real closed") la classe des corps admettant une valuation henselienne telle que le corps de restes soit réel clos. La valuation triviale est autorisée, les corps réels clos sont donc ARC. Plus généralement, d'après le théorème d'Ax-Kochen-Ershov, un corps est ARC ssi il est élémentairement équivalent à un corps de séries formelles généralisées $\mathbb{R}((G))$.

Cette classe est élémentaire car close par ultraproduit et équivalence élémentaire. On en obtiendra une axiomatisation effective si, parmi les valuations avec les propriétés ci-dessus, on peut en trouver une qui soit définissable, uniformément pour tous les corps de la classe. Il suffira alors de transcrire la définition de la classe ARC. La question de la définissabilité d'une valuation, c'est-à-dire de son anneau de valuation, dans un corps a souvent été considérée. Ax est sans doute le premier qui remarque que $k[[X]]$ est définissable dans le corps $k((X))$ (cela lui permet de prouver que l'indécidabilité du corps k se transmet au corps $k((X))$). Il note que, si v est la valuation en X dans $k((X))$,

$$v(x) \geq 0 \text{ ssi } k((X)) \models \exists y\ y^2 = 1 + Xx^2,$$

puis donne un argument pour éliminer le paramètre X (au prix de l'introduction de quelques quantificateurs). Son argument se généralise immédiatement aux corps $R((G))$ où R est réel clos (ou plus généralement vérifie $R = R^2 \cup (-R^2)$) et G ne contient pas de sous-groupe convexe propre 2-divisible: la valuation canonique à valeurs dans G y est definissable dans la seule structure de corps. Une condition restrictive sur G est certainement nécessaire. Supposons en effet que G a un sous-groupe convexe divisible propre, par exemple $G = \mathbb{Z} \times D$ ordonné lexicographiquement, où D est divisible non trivial. Alors, par un isomorphisme canonique,

$$\begin{aligned} \mathbb{R}((G)) &\cong (\mathbb{R}((D)))((X)) \\ &\equiv \mathbb{R}((X)), \text{ par Ax-Kochen-Ershov.} \end{aligned}$$

(isomorphisme et équivalence élémentaire font référence aux structures de corps). Or il n'y a bien sûr dans $\mathbb{R}((X))$ qu'une valuation définissable non triviale. Cette propriété se transfère par équivalence élémentaire à $\mathbb{R}((G))$ où elle signifie que la valuation quotient de la canonique, à valeurs dans $\mathbb{Z}$, est définissable, et aucune autre, en particulier pas la canonique.

Nous avions fait un raisonnement et donné une formule fondés sur les carrés et la 2-divisibilité. On peut faire de même avec un nombre premier quelconque à la place de 2. On obtient ainsi le résultat suivant: **dans le corps $R((G))$, R réel clos, la valuation canonique est définissable ssi il existe un nombre premier p tel que G ne contienne pas de sous-groupe convexe p-divisible propre.**

Que se passe-t-il lorsque G ne remplit pas cette condition? Introduisons le sous-groupe convexe p-divisible maximal D_p de G, ce pour chaque p premier. Soit v la valuation canonique sur $\mathbb{R}((G))$, et $v_p = v/D_p$ la valuation quotient à valeurs dans G/D_p. Chaque v_p est définissable, il y a de plus un bon choix ϕ_p de la formule définissante - qu'on peut bien sûr considérer au-dessus d'un corps quelconque - pour lequel on a: **un corps K est ARC ssi, pour tout premier p, ϕ_p définit sur K une valuation henselienne avec un corps de restes réel et sans extension réelle propre de degré $\leq p$.** Ce critère fournit une axiomatisation de la classe ARC.

Si K est ARC, ϕ_p définit la plus grossière valuation henselienne avec corps de restes clos pour les extensions réelles de degré $\leq p$. Soit v_p cette valuation et v_K=sup v_p. Alors v_K est la plus grossière des valuations henseliennes à corps de restes réel clos, elle est définissable ssi elle coïncide avec une des v_p, et aucune autre valuation à corps de restes réel clos n'est définissable. Qu'elle soit limite de valuations définissables lui confère un comportement intéressant: ainsi, si L est un autre corps, alors

$$K \equiv L \quad \Rightarrow \text{ (L est ARC également et) } \quad v_K K \equiv v_L L.$$

L'implication réciproque est une conséquence d'Ax-Kochen-Ershov:

$$v_K K \equiv v_L L \qquad \Rightarrow \qquad (K, v_K) \equiv (L, v_L)$$

(puisque les deux valuations sont henseliennes et les deux corps de restes réels clos)

$$\Rightarrow K \equiv L \text{ en tant que corps.}$$

On a de la même façon,

$$K \text{ décidable} \quad \text{ssi} \quad v_K \text{ l'est.}$$

Les choses se passent beaucoup moins bien pour les théories incomplètes. Ainsi, pour une classe $\mathcal{K}$ de corps ARC, la décidabilité de Th($\mathcal{K}$) n'implique pas celle de Th($\{v_K K, K \in \mathcal{K}\}$) (intuitivement et très approximativement, à cause de l'existence d'ensembles récursivement énumérables non récursifs). L'implication inverse est toujours exacte, d'après Ax-Kochen-Ershov, qui permet de transférer la décidabilité même dans le cadre des théories incomplètes.

Pour avoir un exemple de corps ARC dans lequel v n'est pas définissable, considérer $\mathbb{R}((G))$ avec

$$G = \prod_{p \text{ premier}} \mathbb{Z}_{(p)}$$

où $\mathbb{Z}_{(p)} = \{ab^{-1}; a \in \mathbb{Z}, b \in \mathbb{N}^*, b \text{ premier à } p\}$ et G est ordonné lexicographiquement, les premiers ayant leur ordre naturel. Alors $D_p \cong \prod_{q<p} \mathbb{Z}_{(q)}$, v_p est à valeurs dans $\prod_{q \geq p} \mathbb{Z}_{(q)}$, et v_K coïncide avec la valuation canonique.

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L'idée d'Ax est exploitée dans (2) et (3). Les résultats présentés ici viennent de (4).

SOME MODEL THEORY
FOR ALMOST REAL CLOSED FIELDS

Françoise Delon
U.F.R. de Mathématiques
Université Paris VII
2 Place Jussieu
75251 Paris cedex 05
France
Humboldt, Freiburg, FRG

Rafel Farré
Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya
Pau Gargallo, 5
08028 Barcelona
Spain

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Some model theory for almost real closed fields

Françoise Delon
U.F.R. de Mathématiques
Université Paris VII
2 Place Jussieu
75251 Paris cedex 05
France
Humboldt, Freiburg, FRG

*Rafel Farré**
Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya
Pau Gargallo, 5
08028 Barcelona
Spain

ABSTRACT. We study the model theory of fields k carrying a henselian valuation with real closed residue field. We give a criteria for elementary equivalence and elementary inclusion of such fields involving the value group of a not necessarily definable valuation. We also characterize the first-order definable convex subgroups of a given ordered abelian group and prove that the definable real valuation rings of k are in correspondence with the definable convex subgroups of the value group of a certain real valuation of k.

§ 1. Introduction

In [J1] B. Jacob introduced the notion of Hereditarily S-Pythagorean fields and studied their model theory. The almost real closed fields include these fields and the results of §3 may be seen as a generalization of some results of [J1] and [J2].

By an almost real closed field we understand a field carrying a henselian valuation with real closed residue field. The simplest example of an almost real closed field is a real closed field. The almost real closed fields appear in a natural way when studying algebraic properties of Hereditarily-Pythagorean field and dealing with closures in certain sense: the generalized real closed fields of Becker (see [B1] and [B2]), chain-closed fields in the sense of Harman (see [H]) and in the sense of Schwartz (see [Sch1]). Another example of almost real closed fields are the Rolle fields of Brown, Craven and Pelling (see [B-C-P]), whose model-theory was studied by F. Delon in [D2]. Other characterisations of almost real closed fields are obtained in [Sch2] and [B-B-G], where the algebraic properties of such fields are studied.

In section 2 we study the almost real closed fields (ARC) and the chain $W(k)$ of real valuations of such a field k. We make special attention to v_0, the first element of the chain with real closed residue field. Also the valuations v_S (the first element with S-Euclidian residue field), for S a set of primes, play an important role. Their value rings are the Jacob rings defined in [J1] and turn out to be first-order definable in the case where S is

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finite. We prove that the theory of ARC fields is elementary and provide an explicit axiom system using the first-order definability of v_S for S finite.

In section 3 we prove the transfer Theorem 3.4. In fact, one of the implications of 3.4 is a consequence of the Ax-Kochen-Ershov theorem and the completeness and model-completeness of the theory of real closed fields. The new fact is that the theory of the field k (without valuation) determines the theory of $v_0(k)$. As a consequence we prove that there is a bijection between theories of ARC fields and certain theories of ordered abelian groups. The theories of ordered abelian groups involved in this bijection are characterized in Proposition 3.9. Moreover, this bijection preserves the completeness and, in certain cases, the decidability. If the valuation ring of v_0 were definable, the results of Theorem 3.4 would be a consequence of this fact. This raises the question whether v_0 has a definable valuation ring.

In section 4 we study the definable real valuation rings. For this purpose, we start giving a characterization in 4.1 and 4.2 of the definable convex subgroups of an ordered abelian group. In Theorem 4.4 we prove that there is a bijection between the definable real valuation rings of k and the definable convex subgroups of the value group of the maximum element of $W(k)$. This result allows us to conclude that v_0 is definable only when $v_0 = v_S$ for some S finite. This criteria does not apply to the valuations v_S for S infinite. v_S may be definable even in the case it is not equal to any $v_{S'}$ for S' finite. In the case where $\dim_p k < \infty$ (definition in §2) the definable elements of $W(k)$ are completely characterized.

§ 2. Almost real closed fields

If (k, v) is a valued field, we are going to denote by A_v, M_v, U_v, $v(k)$, k/v and π respectively the valuation ring, the maximal ideal of A_v, the group of units of A_v, the value group, the residue field of (k, v) and the canonical projection from A_v to k/v. If v, v' are two valuations of k and $A_v \subseteq A_{v'}$ we shall denote it by $v \geq v'$. Then v induces in a natural way a valuation on k/v' which we will denote v/v'. If k carries an order $\leq$ with P its positive cone, v is called convex or compatible with $\leq$ if A_v is convex in k, which is equivalent to $1 + M_v \subseteq P$. Then $\leq$ induces an order in k/v by putting $\pi(a) > 0$ iff $a > 0$ for every $a \in U_v$. If F is a subfield of k, $A_F = \{x \mid |x| \leq y \text{ for a certain } y \in F\}$ is a valuation ring of k with $F \subseteq A_F$. We are going to use the notation v_F for the valuation associated to A_F. Hence π_F is injective on F. If F' is another subfield of k with $A_F = A_{F'}$ then $\pi_F = \pi_{F'}$ is injective on F' and $\pi_F(F')$ is not bounded in $(k/v_F, \leq)$.

We denote

$$V(k) = \{\, v \mid v \text{ a is henselian valuation over } k \text{ with real closed residue field} \,\}$$
$$W(k) = \{\, v \mid v \text{ a is henselian real valuation over } k \,\}.$$

PROPOSITION 2.1. *Let k be any field. Then:*

i) If k is real, then $W(k)$ is linearly ordered.

ii) If v' is a real valuation of k and $v \in V(k)$, $v' \geq v$ implies $v' \in V(k)$.

iii) If v' is any valuation of k, $v \in V(k)$ and $v' \leq v$ then $v' \in V(k)$ iff $v/v'(k/v')$ is divisible.

iv) If $V(k)$ is not empty, then it is a final segment of $W(k)$ with minima and maxima. The minimal element v_0 of $V(k)$ is the valuation of $V(k)$ with no non-trivial convex divisible subgroup in $v_0(k)$ and the maximum element v_1 is the valuation with archimedean real closed residue field.

Proof: i) If k is real then $W(k)$ is not empty, and every henselian valuation over k is convex for every order of k [K-W]. Fix any order of k. The convex subrings of k are totally ordered by inclusion.

ii) and iii) are proved using systematically the following facts:

1. If v is a real valuation of k, then k is real closed iff v is henselian, k/v real closed and $v(k)$ divisible [K-W].

2. For two valuations $v' \geq v$ over k, one has

v' is henselian iff v and v'/v are

$$k/v' \simeq (k/v)/(v'/v)$$

$$v(k) \simeq v'(k)/(v'/v)(k/v) \quad \text{[Ri]}.$$

iv) is a consequence of i), ii) and the fact that given an order $\leq$ of k there exists a convex valuation v with archimedean ordered residue field. □

The value group of any valuation of $V(k)$ is the quotient of $v_1(k)$ by a convex divisible subgroup, and $v_0(k)$ is the quotient of $v_1(k)$ by its biggest convex divisible subgroup. Thus, given two valuations $v, v' \in V(k)$ and any prime number p, $\dim_p v(k) = \dim_p v'(k)$, where $\dim_p v(k)$ is the dimension of $v(k)/pv(k)$ as $\mathbb{F}_p$-vector space. When $V(k) \neq \emptyset$, it allows us to speak of $\dim_p k$ for every prime number p. As we will see, these dimensions by themselves contain much information about the field k.

Definition. We are going to call a field k *almost real closed* (ARC) if $V(k) \neq \emptyset$.

PROPOSITION 2.2. *Let k be a field:*

i) If k is ARC then so is any real algebraic extension of k.

ii) If v is a real valuation of k then k is ARC iff k/v is ARC and v is henselian.

iii) If k is ARC, $W(k) = \{v \mid v$ is a real valuation of $k\}$.

Proof: i) is routine.

ii) If $v' \in V(k)$, it is convex for every order of k, hence comparable with every real valuation of k. We now distinguish the cases $v \geq v'$ and $v' \geq v$, and finish the proof with the same tools as in 2.1 i) and ii).

iii) follows from ii). □

It is easily seen that k is real closed iff it is an ARC field and $\dim_p k = 0$ for every prime number p. k is a Rolle field iff it is an ARC field with $\dim_p k = 0$ for every odd prime number p ([B-C-P]). In a similar way we can characterize chain-closed field and generalized real closed fields. Also it can be shown that there are exactly $2^{\dim_2 k}$ orders in any ARC field k.

We are interested in first-order definable valuations of k, and Jacob valuations ([J1]) will provide us with important examples of those. In order to understand the meaning of these valuations let us introduce some definitions. Let $\mathbb{P}$ be the set of all prime numbers and $S \subseteq \mathbb{P}$. Following [J1] we call a valuation v over a field k *S-Kummer henselian* (S-Kh) if Hensel lemma holds for polynomials of type $x^p - a$ when $p \in S$ and $a \in A_v$. p-Kh will mean $\{p\}$-Kh. When v is real, then v is S-Kh iff for any $a \in U_v$ and $p \in S$, $a \in k^p$ iff $\pi(a) \in (k/v)^p$.

LEMMA 2.3. *Let k be any field and v, v' two valuations of k such that $v' \geq v$ and k/v' has characteristic zero. Then v' is S-Kummer henselian iff v and v'/v are.*

proof: The characteristic zero of the residue fields of v and v' makes the polynomials of type $x^p - a$ having only simple roots when $a \neq 0$. The only non-trivial step is to prove that v is S-Kh when v' is. If $x^p - \pi_v(a)$ has a root $\pi_v(b)$, then $\pi_v(a/b^p) = 1$ and $\pi_{v'}(a/b^p) = 1$. Applying the S-Kh of v' to the polynomial $x^p - a/b^p$ we find $c \in k$ with $c^p = a/b^p$ and $\pi_{v'}(c) = 1$. We will finish if we prove $\pi_v(c) = 1$ because then cb is the desired root of $x^p - a$. It is clear from $(\pi_v(c))^p = 1$, $\pi_{v'}(c) = 1$ and the zero characteristic of k/v'. □

Note that we only need that the characteristic of k/v' does not belong to S.

We call a field k *S-Euclidian* if it is real and $k = \pm k^p$ for every $p \in S$. If $p \neq 2$ this condition is equivalent to $k = k^p$. For $S = \{p\}$ we are going to say p-Euclidian. The usually Euclidian fields are exactly the 2-Euclidian fields in our terminology. If k is a field we define

$$V_S(k) = \{\, v \mid v \text{ is a } S\text{-Kh valuation of } k \text{ with } S\text{-Euclidian residue field} \,\}$$

V_p will denote $V_{\{p\}}$.

LEMMA 2.4. *Let k be a field and v a real valuation of k. Then:*

i) $v \in V_S(k)$ iff for every $x \in k$ and $p \in S$ the following equivalence holds:

$$x \in \pm k^p \text{ iff } v(x) \in pv(k)$$

ii) If $v \in V_S(k)$, $m \in \mathbb{N}^*$ such that all its prime divisors are in S and $x \in k$, the following equivalence holds:

$$x \in \pm k^m \text{ iff } v(x) \in mv(k)$$

iii) The following are equivalent:

$$k \text{ is } S\text{-Euclidian iff } v \in V_S(k) \text{ and } v(k) \text{ is } S\text{-divisible}$$

Proof: Suppose $v \in V_S(k)$ and let $a \in k$ such that $v(a) = pv(b)$ for a certain $b \in k$. Then $v(x/b^p) = 0$ and thus $\pi(a/b^p) \in \pm(k/v)^p$. Applying the S-Kh to the polynomial $x^p \mp a/b^p$ we deduce $a/b^p \in \pm k^p$ whence $a \in \pm k^p$. This proves the only if part of i). For the if part let $a \in K$ with $v(a) = 0$. This implies $a \in \pm k^p$ for $p \in S$ and hence $\pi(a) \in \pm(k/v)^p$ and Hensel lemma holds for the polynomial $x^p - a$ (k/v has not primitive p-roots of unit). This proves $v \in V_S(k)$. ii) follows iterating i). The left to right implication of iii) follows easily keeping in mind that k/v has no primitive p-roots of unit. The other side follows from i). □

Point iii) of the preceding proposition suggests us to call a field *S-almost Euclidian* when $V_S(k) \neq \emptyset$. Clearly an S-Euclidian field is S-almost Euclidian and every S-almost Euclidian field is real.

If k is a real field and v is a 2-Kh valuation then $1 + M_v \subseteq k^2$ and thus v is compatible with any order of k. By the same arguments as for $V(k)$, the set $W_2(k)$ of 2-Kh valuations is totally ordered. If $2 \in S$ then $V_S(k) \subseteq W_2(k)$, whence $V_S(k)$ is also totally ordered.

PROPOSITION 2.5. *Let k be a field and S a set of primes. Then:*

i) If v' is a real valuation of k and $v \in V_S(k)$, $v' \geq v$ implies $v' \in V_S(k)$ and $v'/v \in V_S(k/v)$.

ii) If v' is a valuation of k, $v \in V_S(k)$ and $v' \leq v$ then $v' \in V_S(k)$ iff $v/v'(k/v')$ is S-divisible.

iii) The infimum of two valuations of $V_S(k)$ belongs to $V_S(k)$.

iv) If $V_S(k) \neq \emptyset$, $V_S(k)$ has a minimum element. The minimum element is the unique valuation of $V_S(k)$ without non-trivial S-divisible convex subgroups.

v) If $2 \in S$ and $V_S(k) \neq \emptyset$, $V_S(k)$ is a final segment of $W_2(k)$ with maximum element, i.e. the valuation v_1 of $V_S(k)$ with archimedean residue field.

Proof: The proofs of i) and ii) use the same arguments as the proof of Proposition 2.1 replacing everywhere henselian by S-Kh and real closed by S-Euclidian and using Lemmas 2.3 and 2.4.

Claim. If k carries two independent p-Kh valuations of residual characteristic zero, then $k = k^p$.

Proof of the claim: Let v and v' be two such valuations and $a, b \in k$. From the aproximation theorem for independent valuations, there exists $c \in k$ such that $v(c-a) > v(a)$ and $v'(c-b^p) > v'(b^p)$. From the p-Kh of v' we deduce $c \in k^p$ and from the p-Kh of v we conclude $a \in k^p$.

iii). If the two valuations v and v' are comparable the conclusion is obvious. If v and v' are not comparable, applying the claim to $v/v \wedge v'$ and $v'/v \wedge v'$ ($v \wedge v'$ stands for the infimum of v and v') we deduce that $2 \notin S$ and $k/v \wedge v'$ is S-Euclidian, whence $v \wedge v' \in V_S(k)$. This shows again that when $2 \in S$, $V_S(k)$ is totally ordered.

iv) follows from iii).

v). Take any $v \in V_S(k)$ and any order of the residue field k/v. Take v' the valuation of k/v compatible with this order and with archimedean residue field. Taking v_1 the valuation of k composition of v and v' we have $v_1 \in V_S(k)$ and k/v_1 is archimedean. From the linear order on $W_2(k)$ we deduce that it is the only one. □

Example. Let $n > 1$ be an integer, k a field with n orders existentially closed [vdD], and $S = \mathbb{P} \setminus \{2\}$. Then $k = k^p$ for all $p \in S$, hence $V_S(k) = \{ v \mid v \text{ a real valuation on } k\}$. If k is a bit saturated, its orders are not archimedean and $V_S(k)$ is not linearly ordered. Indeed two distinct orders of k are independent, which means that they define distinct topologies. Now any non-trivial valuation which is convex for some order defines the same topology as this order. Hence the archimedean valuations associated to two different orders on k are not comparable.

We shall denote the first element of $V_S(k)$ and $V_p(k)$ by v_S and v_p respectively. The valuation rings of these valuations are the Jacob rings (see [J1]) defined as follows:

$$\mathcal{O}_1(k,S) = \{\, x \in k \mid x \notin \pm k^p \text{ and } 1 + x \in k^p \text{ for some } p \in S \,\}$$

$$\mathcal{O}_2(k,S) = \{\, x \in k \mid x \in \pm k^p \text{ for all } p \in S \text{ and } x\mathcal{O}_1(k,S) \subseteq \mathcal{O}_1(k,S) \,\}$$

$$\mathcal{O}(k,S) = \mathcal{O}_1(k,S) \cup \mathcal{O}_2(k,S)$$

Remark. When S is finite $\mathcal{O}(k,S)$ is definable by a first-order formula of the language of rings LR= $\{+,-,\cdot,0,1\}$.

PROPOSITION 2.6. *Let $S \subseteq \mathbb{P}$ and k an S-almost Euclidian field, then $\mathcal{O}(k,S)$ is the valuation ring of v_S.*

Proof: If $v \in V_S(k)$ we remark the following facts:
1).- $x \in \mathcal{O}_1(k,S)$ iff $v(x) > 0$ and $v(x) \notin pv(k)$ for some $p \in S$
2).- $v(x) \geq 0$ and $v(x) \in pv(k)$ for all $p \in S$ implies $x \in \mathcal{O}_2(k,S)$
3).- If $v(k)$ has no non-trivial convex S-divisible subgroup then $A_v = \mathcal{O}(k,S)$
Proof of 3): From 1) and 2) we have that $\mathcal{O}_1(k,S) \subseteq A_v \subseteq \mathcal{O}(k,S)$. It remains to prove that if $x \in \mathcal{O}_2(k,S)$ then $v(x) \geq 0$. Suppose $v(x) < 0$; from the fact that $v(k)$ has no S-divisible convex subgroups we deduce the existence of $y \in k$ such that $0 < v(y) \leq -v(x)$ and $v(y)$ is no S-divisible. Thus $y \in \mathcal{O}_1(k,S)$ and $v(xy) \leq 0$, whence $x \notin \mathcal{O}_2(k,S)$. □

As a consequence of Proposition 2.6 we deduce that if S is finite the class of S-AE fields is elementary: take sentences expressing that $\mathcal{O}(k,S)$ is an S-Kh valuation ring with S-Euclidian residue field.

LEMMA 2.7. *Let k be any field and $S \subseteq \mathbb{P}$. Then k is S-AE iff k is S'-AE for every $S' \in P_f(S)$, the set of finite parts of S.*

Proof: The left to right direction is obvious. Conversely, suppose $V_{S'}(k) \neq 0$ for every $S' \in P_f(S)$, and let $A_{S'}$ be the valuation ring of $v_{S'}$. Then $A = \bigcap\{A_{S'} \mid S' \in P_f(S)\}$ is a real valuation ring because it is the intersection of an inverse filtrant family of real valuation rings ($A_{S' \cup S''} \subseteq A_{S'} \cap A_{S''}$). If v is the valuation corresponding to A obviously $v \geq v_{S'}$ and from proposition 2.5 we have that $v \in V_{S'}(k)$ for every $S' \in P_f(S)$ whence $v \in V_S(k)$. □

As a consequence of Lemma 2.7 we deduce that for any S the class of S-AE fields is an elementary class.

Remark. If k is S-AE and $S' \subseteq S \subseteq \mathbb{P}$ then $V_S(k) \subseteq V_{S'}(k)$ and $v_{S'} \leq v_S$. From the fact that the maximum convex S-divisible subgroup of an o.a.g. (ordered abelian group) is the intersection of all the maximum convex S'-divisible subgroups for $S' \in P_f(S)$ it follows that $v_S = \sup_{S' \in P_f(S)} v_{S'}$ and thus $\mathcal{O}(k,S) = \bigcap\{\mathcal{O}(k,S') \mid S' \in P_f(S)\}$ (this is shown in [J1] using the definition of $\mathcal{O}(k,S)$).

Our interest in the sets $V_S(k)$ is when k is ARC. In this case, even when $2 \notin S$, V_S is totally ordered because every real valuation is henselian. Then obviously $V(k) = V_{\mathbb{P}}(k) \subseteq V_S(k) \subseteq W(k) = W_2(k)$ whence $v_S \leq v_{\mathbb{P}} = v_0 \leq v_1$ for every set of primes S.

PROPOSITION 2.8. *The class ARC is elementary and its theory is decidable.*

Proof. This class is closed by ultraproducts and elementary equivalence: if k is ARC and $L \equiv k$, from the Keisler-Shelah theorem k and L have isomorphic ultrapowers $k^U \simeq L^U$; thus L^U carries a henselian valuation v with real closed residue field. Because L is relatively algebraically closed into L^U, $v \restriction L \in V(L)$. Applying the Ax-Kochen-Ershov theorem [We], the theory T of henselian valued fields with real closed residue field is decidable in the language of valued fields. We are using here the decidability of the theory of ordered abelian groups [G] in addition to the decidability of the theory of real closed fields [Ch-K]. The decidability of ARC follows from the decidability of T. □

An Hereditarily-Pythagorean field (do not mistake with the notion of Hereditarily S-Pythagorean in [J1]) is a real field k such that every real algebraic extension is Pythagorean (a sum of squares is a square). E. Becker (see [B1] for example) has characterized Hereditarily-Pythagorean fields as those fields k for which every finite extension is of the form $k(\sqrt[t_1]{a_1}, \sqrt[t_2]{a_2}, \cdots, \sqrt[t_n]{a_n})$ for some $a_1, \cdots a_n \in k$ and $t_1, \cdots t_n \in \mathbb{N}$.

Property iii) in the following proposition provides an explicit axiomatization of ARC.

PROPOSITION 2.9. *For any field k the following are equivalent:*

i) k is an ARC field.

ii) a) every real valuation of k is henselian.

b) k carries an order $\leq$ such that for every archimedean subfield $(F, \leq)$ of $(k, \leq)$, the real closure of $(F, \leq)$ is contained in $(k, \leq)$.

iii) k is $\mathbb{P}$-AE and Hereditarily-Pythagorean.

Proof: i→ii. In order to prove b), let $\leq$ any order of k and F a subfield of k archimedean for this order. Let v_F the valuation associated to F; then v_F has archimedean residue field and thus $v_F \geq v_0$, whence k/v_F is real closed. From the henselian property of v_F there exists $i : k/v_F \to k$ a ring homomorphism such that $F \subseteq \mathrm{Im} i$ and $\pi_F \circ i = Id$. Moreover, taking in k/v_F the order induced by the order of k, i is an homomorphism of ordered rings. Identifying k/v_F with its image we have $(F, \leq) \subseteq (k/v_F, \leq) \subseteq (k. \leq)$ whence the real closure of $(F, \leq)$ is contained in $(k, \leq)$.

ii→i. Let $\leq$ the order satisfiying ii) b) and let v the archimedean valuation associated to this order. From ii) a) v is henselian and hence we can consider (as above) k/v with the induced order as an ordered subfield of $(k, \leq)$. From ii) b) $(k/v, \leq)$ has its real closure into $(k, \leq)$ which in turn is also archimedean. Passing through π to the residue field we conclude that the real closure of $(k/v, \leq)$ is contained into $(k/v, \leq)$ whence k/v is real closed.

i→iii follows from proposition 2.2 i) and the fact that any ARC field is Euclidian and hence Pythagorean.

Claim. k is real closed iff k is $\mathbb{P}$-Euclidian and Hereditarily-Pythagorean.

Proof of the claim: One direction is trivial. For the other we need to prove that k has no proper real algebraic extensions, that is, no proper real extensions of type $k(\sqrt[t]{a})$. If t is odd, from the $\mathbb{P}$-Euclidianity of k we have $a \in k^t$. If t is even it cannot be $-a \in k^t$ because $k(\sqrt[t]{a})$ is real; thus $a \in k^t$.

iii→i. Let $v \in V_{\mathbb{P}}(k)$. k/v is Hereditarily-Pythagorean (because k is) and $\mathbb{P}$-Euclidian, thus real closed. We are going to prove that (k, v) has no proper algebraic immediate extensions and hence v will be henselian. Let (L, v) with $L = k(\sqrt[t]{a})$ be an immediate (and thus real) extension of k. Then $v(a) \in tv(L) = tv(k)$ and hence $a \in \pm k^t$. If t is odd $a \in k^t$ and $L = k$. If t is even, L being real it cannot be $a \in -k^t$, whence $a \in k^t$ and $L = k$. □

Remarks:

1.- It follows from the proof that if ii) b) is true for an order of k it is true for every order of k.

2.- For every $S \subset \mathbb{P}$, the class of S-AE fields has an undecidable theory, because it is weaker than one of the theories T_2^R or T_q^R of [Zi], which are hereditarily undecidable. The same holds for Pythagorean fields. As far as we know, the question of the decidability of Hereditarily-Pythagorean fields and the decidability of $\mathbb{P}$-AE fields are open.

§ 3. A transfer theorem

We are going to use P. Schmitt analysis of ordered abelian groups (o.a.g.). For this purpose we include an appendix with the notation and statements to be used in this and the following sections without proofs (see [S1] for proofs). The proofs of Lemma 3.1 and Theorem 3.2 are inspired in [D-L]. If G is an o.a.g. and n a natural number $n \geq 2$, G_n will denote the maximum convex n-divisible subgroup of G. When working with different o.a.g. we are going to use the notation $A_n^G(g)$ for $A_n(g)$ to indicate we are refering to G. The same convention for $F_n^G(g)$ and $\Gamma_n^G(g)$. If H is a convex subgroup of G, $\overline{g}$ denotes the class of g modulo H, if the congruence we are speaking about is clear from the context.

LEMMA 3.1. *Let G be an ordered abelian group, $n \geq 2$. If $G_n \neq \{0\}$ then $Sp_n(G) \simeq Sp_n(G/G_n) \cup \{a\}$, where a minores $Sp_n(G/G_n) \setminus \{\emptyset\}$, $a > \emptyset$, a satisfying the relations $A(a)$, $\neg F(a)$, $\neg D(a)$ and $\neg\alpha(p, k, m)(a)$ for every relevant p, k, m.*

Proof: If $g \in G_n \setminus \{0\}$ then $A_n^G(g) = \{0\}$. If $g \notin G_n$ then $G_n \subseteq A_n^G(g)$ and bearing in mind that $C \to C/G_n$ provides a bijection between the convex subgroups of G containing G_n and the convex subgroups of G/G_n, $B^{G/G_n}(\overline{g}) = B^G(g)/G_n$ and that $B^G(g)/C \simeq B^{G/G_n}(\overline{g})/C/G_n$ we get $A_n^{G/G_n}(\overline{g}) = A_n(g)/G_n$. $g \notin nG$ iff $\overline{g} \notin n(G/G_n)$ and in this case $G_n \subseteq F_n^G(g)$, since $g \notin G_n + nG$. Then, bearing in mind that for a convex subgroup C of G containing G_n, $g \in C + nG$ iff $\overline{g} \in C/G_n + nG/G_n$ we get $F_n^{G/G_n}(\overline{g}) = F_n^G(g)/G_n$. Thus $h \in \Gamma_{i,n}^G(g)$ iff $\overline{h} \in \Gamma_{i,n}^{G/G_n}(\overline{g})$ for $i = 1, 2$, whence $\Gamma_n^G(g) \simeq \Gamma_n^{G/G_n}(\overline{g})$ follows. Now the application defined by

$$\begin{aligned} Sp_n(G) &\longrightarrow Sp_n(G/G_n)+\{a\} \\ A_n^G(g) &\longmapsto A_n^{G/G_n}(\overline{g}) \ \text{ if } g\notin G_n \\ F_n^G(g) &\longmapsto F_n^{G/G_n}(\overline{g}) \ \text{ if } g\notin nG \\ \{0\} &\longmapsto a \\ \emptyset &\longmapsto \emptyset \end{aligned}$$

is well-defined, bijective and preserves the relations $<$, A, F, D and $\alpha(p,k,m)$. □

THEOREM 3.2. *Let G and H be two ordered abelian groups. Then*

i) $H\equiv G$ *iff* $\begin{cases} H_n=\{0\} \text{ iff } G_n=\{0\} \text{ for all } n\geq 2 \\ H/H_n\equiv G/G_n \text{ for all } n\geq 2\end{cases}$

ii) If $H\subseteq G$*, then* $H\preceq G$ *iff* $\begin{cases} H_n=\{0\} \text{ iff } G_n=\{0\} \text{ for all } n\geq 2 \\ H_n=G_n\cap H \text{ and } H/H_n\preceq G/G_n \text{ for all } n\geq 2\end{cases}$

Proof: The implications from left to right follow from the fact that the maximum convex n-divisible subgroup is definable by a formula without parameters, namely $x\in G_n$ iff $G\models(\forall y)((x\geq 0\wedge 0\leq y\leq x)\vee(x\leq 0\wedge 0\leq y\leq -x)\rightarrow(\exists z)(y=nz))$. For the converse, if $G_n=\{0\}$ for some $n\geq 2$ the other implications are trivial. If $G_n\neq\{0\}$ for all $n\geq 2$ the same happens for H_n, and using Lemma 3.1 and Theorem C of the appendix, i) follows from $Sp_n(H)\simeq Sp_n(H/H_n)+\{a\}\equiv Sp_n(G/G_n)+\{a\}\simeq Sp_n(G)$, where a is defined as in Lemma 3.1. We use here the following claim, a consequence of [F-V].

Claim. Let $\mathfrak{A}$ be a coloured chain with minimum, which we are going to denote by 0. Let $\mathfrak{A}^*$ be the coloured chain consisting in adding to $\mathfrak{A}$ an extra element a ($a\notin\mathfrak{A}$), placed in the order following the minimum ($a>0$, $a<b$ for all $b\in\mathfrak{A}$, $b>0$) and satisfying some fixed set of monadic predicats (the same for all such $\mathfrak{A}$). Then, for every formula $\varphi(x_1,\ldots,x_m,x_{m+1},\ldots,x_n)$ in the language of coloured chains there is a formula $\varphi'(x_1,\ldots,x_m)$ in the language of coloured chains such that for any coloured chain $\mathfrak{A}$ with minimum, and for any $a_1,\ldots,a_m\in\mathfrak{A}$ we have

$$\mathfrak{A}^*\models\varphi(a_1,\ldots,a_m,a,\ldots,a) \quad\text{iff}\quad \mathfrak{A}\models\varphi'(a_1,\ldots,a_m).$$

In particular, if φ is a sentence, φ' is a sentence and $\mathfrak{A}^*\models\varphi$ iff $\mathfrak{A}\models\varphi'$.

For the proof of ii) we use Theorem D of the appendix. From Lemma 3.1 and the claim above, $Sp_n(H)\simeq Sp_n(H/H_n)+\{a\}\preceq Sp_n(G/G_n)+\{a\}\simeq Sp_n(G)$. From $H_n=G_n\cap H$ and $H/H_n\preceq G/G_n$ we have that $nH=nG\cap H$ and thus the elementary inclusion $Sp_n(H)\preceq Sp_n(G)$ is defined by

$$\begin{aligned} Sp_n(H) &\longrightarrow Sp_n(G) \\ A_n^H(h) &\longmapsto A_n^G(\overline{h}) \ \text{ if } h\notin H_n \\ A_n^H(h)=\{0\} &\longmapsto A_n^G=\{0\} \ \text{ if } h\in H_n\setminus\{0\} \\ F_n^H(h) &\longmapsto F_n^G(\overline{h}) \ \text{ if } h\notin nH \\ \emptyset &\longmapsto \emptyset \end{aligned}$$

It remains to prove that the extension $H \subseteq G$ preserves the predicates $M(k)$, $E(n,k)$ and $D(p,r,i)$. If $H \models M(k)(h)$ then $h \notin H_n$ and $C^{H/H_n}(\overline{h}) \simeq C^H(h)$ and thus $H \models M(k)(h)$ iff $H/H_n \models M(k)(\overline{h})$. Doing the same for G, from $H/H_n \preceq G/G_n$ we conclude $H \models M(k)(h)$ iff $G \models M(k)(h)$. Also if $H \models E(n,k)(h)$ then $h \notin nH$ and a similar argument shows that $H \models E(n,k)(h)$ iff $H/H_n \models E(n,k)(\overline{h})$. For $D(p,r,i)$ it suffices to remark that $h \in p^r H$ iff $\overline{h} \in p^r(H/H_p)$ and if $h \notin p^r H$ then $\Gamma_{p^r}^{H/H_p}(\overline{h}) \simeq \Gamma_{p^r}^{H}(h)$. □

Remarks:

1.-Theorem 3.2 in not true if we restrict the conditions for n prime, as shows the following example. Let $H = \bigoplus_{n\in\omega^*} F_n$ and $G = \bigoplus_{n\in\omega^*} E_n$, where ω^* denotes the inverse order of ω and

$$F_n = \begin{cases} \mathbb{Q} & \text{if } n = 0 \\ \mathbb{Z}_2 & \text{if } n \text{ is odd} \\ \mathbb{Z}_3 & \text{if } n > 0 \ n \text{ is even} \end{cases} \qquad E_n = \begin{cases} \mathbb{Q} & \text{if } n = 0 \\ \mathbb{Z}_3 & \text{if } n \text{ is odd} \\ \mathbb{Z}_2 & \text{if } n > 0 \ n \text{ is even.} \end{cases}$$

Here $\mathbb{Z}_2$ and $\mathbb{Z}_3$ denote respectively the rings 2−adic and 3−adic integers. It is easily seen that $H_p \neq \{0\}$ and $G_p \neq \{0\}$ for all prime p, $H/H_p \simeq G/G_p \simeq \{0\}$ for all prime $p \neq 2,3$, and

$$H/H_2 \simeq G/G_2 \simeq \cdots \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2,$$

$$H/H_3 \simeq G/G_3 \simeq \cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3.$$

However $H \not\equiv G$ since $H_2 \subset H_3$ and $G_3 \subset G_2$.

2.-In fact, as suggestet in the example above, the relations $H_p \subseteq H_q$ for $p,q \in \mathbb{P}$ is the extra information we need in order to code Th(G). Theorem 3.2, for elementary equivalence, may be stated as

$$H \equiv G \ \text{ iff } \begin{cases} H_p = \{0\} \text{ iff } G_p = \{0\} \text{ for all } p \in \mathbb{P} \\ H_p \subseteq H_q \text{ iff } G_p \subseteq G_q \text{ for all } p,q \in \mathbb{P} \\ H/H_p \equiv G/G_p \text{ for all } p \in \mathbb{P}. \end{cases}$$

This follows from Theorem 3.2 since for $n \geq 2$ $H_n = H_p$ for some prime $p \mid n$ and this happens at the same time in H and G since $H_n = H_p$ iff $H_p \subseteq H_q$ for all prime q, $q \mid n$. It also holds a similar result for elementary embeddings.

Notation. For any ordered abelian group G, $G_0 = \bigcap_{p\in\mathbb{P}} G_p$, the maximum convex divisible subgroup of G.

COROLLARY 3.3. *Let G and H be two ordered abelian groups. Then*

i) If $H_0 = \{0\}$ and $G_0 = \{0\}$, then

$$H \equiv G \ \textit{ iff } \ H/H_n \equiv G/G_n \textit{ for all } n \geq 2$$

ii) If $H_0 = \{0\}$, $G_0 = \{0\}$ and $H \subseteq G$, then

$$H \preceq G \ \textit{ iff } \ H_n = G_n \cap H \textit{ and } H/H_n \preceq G/G_n \textit{ for all } n \geq 2$$

iii) $H \equiv G$ *implies* $H/H_0 \equiv G/G_0$.
iv) $H \preceq G$ *implies* $H \cap G_0 = H_0$ *and* $H/H_0 \preceq G/G_0$.
v) $G \equiv G/G_0$ *iff either* $G_0 = \{0\}$ *or for all* $p \in \mathbb{P}$ *there is some* $q \in \mathbb{P}$ *such that* $G_q \subset G_p$.
vi) $H/H_0 \equiv G/G_0$ *and* $G \not\equiv H$ *iff for some* $p \in \mathbb{P}$ $H_0 = H_p$, $G_0 = G_p$, $H/H_p \equiv G/G_p$ *and exactly one of* H_p, G_p *is equal to* $\{0\}$.
vii) If $H \subseteq G$, *then* $H \cap G_0 = H_0$, $H/H_0 \preceq G/G_0$ *and* $H \not\preceq G$ *iff for some* $p \in \mathbb{P}$ $H_0 = H_p$, $G_0 = G_p$, $H \cap G_p = H_p$, $H/H_p \preceq G/G_p$, $H_p = \{0\}$ *and* $G_p \neq \{0\}$.

Proof: i) and ii) follow from Theorem 3.2 and the following remark. If $G_0 = \{0\}$, then $G_n = \{0\}$ iff $G_n \subseteq G_q$ for all $p \in \mathbb{P}$ iff $(G/G_{np})_n = \{0\}$ for all $p \in \mathbb{P}$.

iii). It follows from remark 2 to Theorem 3.2 and 3.3 i) bearing in mind that $(G/G_0)_n = G_n/G_0$, whence $G/G_0/(G/G_0)_n \simeq G/G_n$.

iv). $H \preceq G$ implies $H \cap G_p = H_p$ for all $p \in \mathbb{P}$ whence $H \cap G_0 = H_0$. The rest of the proof is as in iii).

v)$\Leftarrow$. If $G_0 = \{0\}$ is clear and the other case follows from remark 2 to Theorem 3.2 as in iii), since $G_p \subseteq G_q$ iff $(G/G_0)_p \subseteq (G/G_0)_q$.

v)$\Rightarrow$. If $G_0 \neq \{0\}$ and for some $p \in \mathbb{P}$ $G_p \subseteq G_q$ for all $q \in \mathbb{P}$ the $G_p \neq \{0\}$ and $(G/G_0)_p = \{0\}$ whence $G \not\equiv G/G_0$.

vi)$\Rightarrow$. If $H/H_0 \equiv G/G_0$ and $H \not\equiv G$ then either $H \not\equiv H/H_0$ or $G \not\equiv G/G_0$, hence by 3.3 v) there is some prime p such that $\{0\} \neq H_0 = H_p$ or $\{0\} \neq G_0 = G_p$. In any case, from $H/H_0 \equiv G/G_0$ we get $H_0 = H_p$ and $G_0 = G_p$, whence $H/H_p \equiv G/G_p$. H_p, G_p may not be equal to $\{0\}$ nor different from $\{0\}$ at the same time since by remark 2 to Theorem 3.2 this would imply $G \equiv H$. The converse is clear since $H_0 = H_p$ and $G_0 = G_p$.

vii) It follows from vi) bearing in mind that in this case $H_0 = H_p$ and $G_0 = G_p$. □

Remark. If we want to deal in the conditions of Corollary 3.3 with prime n only, we can set

i) If $H_0 = \{0\}$ and $G_0 = \{0\}$, then

$$H \equiv G \ \text{ iff } \ \begin{cases} H_p \subseteq H_q \text{ iff } G_p \subseteq G_q \text{ for all } p, q \in \mathbb{P} \\ H/H_p \equiv G/G_p \text{ for all } p \in \mathbb{P} \end{cases}$$

ii) If If $H_0 = \{0\}$, $G_0 = \{0\}$ and $H \subseteq G$, then

$$H \preceq G \ \text{ iff } \ \begin{cases} H_p \subseteq H_q \text{ iff } G_p \subseteq G_q \text{ for all } p, q \in \mathbb{P} \\ H_p = G_p \cap H \text{ and } H/H_p \preceq G/G_p \text{ for all } p \in \mathbb{P} \end{cases}$$

We are going to denote by LOG=$\{0, +, -, \leq\}$, the usual o.a.g.-language, and by TOG the theory of all ordered abelian groups in this language. The following corollary will be useful later.

Corollary 3.4. *For every LOG-sentence* φ *we can find by a recursive procedure* $n, m \in w$, $r_{i,j}, s_{i,j}, t_{i,j} \geq 2$ *and LOG-sentences* $\psi_{i,j}$ *for* $i = 1, \dots, n$ $j = 1, \dots, m$ *such that for any ordered abelian group* G

$$G \models \varphi \ \textit{ iff } \ \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} \left(G/G_{r_{i,j}} \models \psi_{i,j} \vee G_{s_{i,j}} = \{0\} \vee G_{t_{i,j}} \neq \{0\} \right).$$

Proof: From the definability of G_n, it follows that one can translate every LOG-sentence ψ of G_n to a LOG-sentence ψ^n of G in a recursive way. Now we consider Σ the set of sentences of type

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} (\psi_{i,j}^{r_{i,j}} \vee G_{s_{i,j}} = \{0\} \vee G_{t_{i,j}} \neq \{0\}),$$

for some $n, m \in w$, and $r_{i,j}, s_{i,j}, t_{i,j} \geq 2$, LOG-sentences $\psi_{i,j}$ for $i = 1, \dots, n$ $j = 1, \dots, m$. Σ is clearly closed under conjunction and negation. Now let φ be a LOG-sentence and let Γ be

$$\Gamma = \{\psi \in \Sigma \mid TOG \models \varphi \rightarrow \psi\}$$

We are going to prove that $TOG \cup \Gamma \models \varphi$. Indeed, if $G \models TOG \cup \Gamma$ consider $\Gamma_G = \{\psi \in \Sigma \mid G \models \psi\}$, then $\Gamma \subseteq \Gamma_G$. $TOG \cup \Gamma_G \cup \{\varphi\}$ must to be consistent because otherwise $TOG \cup \{\varphi\} \models \neg\psi$ for a certain $\psi \in \Gamma_G$ (Σ is closed under conjunction and negation), a contradiction to $\Gamma \subseteq \Gamma_G$. Let $H \models TOG \cup \Gamma_G \cup \{\varphi\}$, by Theorem 3.2 i), $H \equiv G$ whence $G \models \varphi$. By a compactness argument there exists $\psi \in \Gamma$ such that $TOG \models \varphi \leftrightarrow \psi$ (Γ is closed under conjunction). Since TOG is recursively axiomatizable, by a dove-tailing procedure we can find $n, m \in w$, $r_{i,j}, s_{i,j}, t_{i,j}$ and $\psi_{i,j}$ for $i = 1, \dots, n$ $j = 1, \dots, m$. □

For $n \geq 2$ let us denote $v_n = \max\{v_p \mid p \in \mathbb{P},\ p \mid n\}$, whose valuation ring A_n is definable since $A_n = \bigcap_{p|n} A_p$. Hence, for an ARC field k, $v_n(k) = v_0(k)/v_0(k)_n$. If k and L are ARC fields we denote by v_0^k and v_0^L respectively the first element of $V(k)$ and $V(L)$. Also v_n^k and v_n^L denote the valuation v_n corresponding to k and L.

THEOREM 3.5. *Let k and L be two ARC fields. Then*
i) $k \equiv L$ iff $v_0^k(k) \equiv v_0^L(L)$.
ii) If $k \subseteq L$, then $k \preceq L$ iff v_0^L extends v_0^k and $v_0^k(k) \preceq v_0^L(L)$.

Proof: The right to left implications follow from the Ax-Kochen-Ershov theorem.

i). If $k \equiv L$ then $v_n^k(k) \equiv v_n^L(L)$ for every $n \geq 2$ by the definability of the valuation ring A_n. Now Corollary 3.3 i) implies $v_0^k(k) \equiv v_0^L(L)$.

ii). From $k \preceq L$ it follows $A_n(k) = A_n(L) \cap k$ and $v_n^k(k) \preceq v_n^L(L)$ whence $A_{v_0}(k) = \bigcap_{n\geq 2} A_p(k) = \bigcap_{n\geq 2} A_p(L) \cap k = \left(\bigcap_{n\geq 2} A_p(L)\right) \cap k = A_{v_0}(L) \cap k$. Finally, using Corollary 3.3 ii) we conclude $v_0^k(k) \preceq v_0^L(L)$. □

Remarks:

1.- The same proof (using Theorem 3.2 instead of Corollary 3.3) shows that if in i) we change v_0^k and v_0^L by $v \in V(k)$ and $w \in V(L)$, $v \neq v_0^k$ and $w \neq v_0^L$ the statement remains true. The conditions $v \neq v_0^k$ and $w \neq v_0^L$ are necessary as shows the following example: $k = L = \mathbb{R}((\mathbb{Z} \oplus \mathbb{Q}))$, where $\mathbb{Z} \oplus \mathbb{Q}$ is lexicographically ordered. $V(k) = V(L)$ has two elements v_0 and v_1, with $v_0(k) = \mathbb{Z}$, $v_1(L) = \mathbb{Z} \oplus \mathbb{Q}$ and $v_0(k) \not\equiv v_1(L)$. In the same way if $v \in V(k)$ and $w \in V(L)$, $v \neq v_0^k$, $w \neq v_0^L$, w an extension of v then $k \preceq L$ implies $v(k) \preceq w(L)$. Corollary 3.3 shows that this distinction is essential only in the case when $v_p = v_0$ for some $p \in \mathbb{P}$. As we will see in § 4, this is exactly the case when A_{v_0} is definable.

2.- Part i) of the theorem may be stated as: $k \equiv L$ iff $v_n(k) \equiv v_n(L)$ for all $n \geq 2$ iff $v_0(k) \equiv v_0(L)$. Part ii) may be stated as: If $k \subseteq L$, $k \preceq L$ iff v_n^L extends v_n^k and $v_n(k) \preceq v_n(L)$ for all $n \geq 2$ iff v_0^L extends v_0^k and $v_0(k) \preceq v_0(L)$.

COROLLARY 3.6. *Let $(k, \leq)$ and $(L, \leq)$ be two ordered ARC fields. Then*
i) $(k, \leq) \equiv (L, \leq)$ iff $k \equiv L$.
ii) If $(k, \leq) \subseteq (L, \leq)$, then $(k, \leq) \preceq (L, \leq)$ iff $k \preceq L$.

Proof: If $k \equiv L$ then, by Theorem 3.5, $v_0(k) \equiv v_0(L)$. The residually ordered fields are elementarily equivalent since they are real closed, thus applying the extended version of the Ax-Kochen-Ershov theorem of [Fa2] (Corollary 4.2 iii)) we get $(k, \leq) \equiv (L, \leq)$. The proof of ii) is similar using in this case Corollary 4.2 ii) of [Fa2]. □

Remak. After 3.5 and 3.6 the result may be described as follows. Given a pair of ARC fields, 'They are elementarily equivalent as rings iff they are elementarily equivalent as ordered rings (no matter the order we choose) iff they are elementarily equivalent as valued and ordered fields (provided we are careful distinguishing between v_0 and the other valuations of $V(k)$ and no matter the order we choose). For the valuations we must be careful distinguishing between v_0 or the rest of valuations of $V(k)$ only in the case when $v_0 = v_p$ for some prime p, and in this case the distinction is essential. For elementary embeddings the situation is analogous.

We are going to use the notation $\mathfrak{A} \preceq_1 \mathfrak{B}$ to indicate that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ which is existentially closed in $\mathfrak{B}$.

COROLLARY 3.7.
i) Let k be an ARC field and L a field extension of k. Then $k \preceq_1 L$ iff L carries a real valuation w extension of v_0^k with $v_0(k) \preceq_1 w(L)$.
ii) Let $(k, \leq)$ be an ordered ARC field and $(L, \leq)$ an ordered field extension. Then $(k, \leq) \preceq_1 (L, \leq)$ iff $(L, \leq)$ carries a compatible valuation w extension of v_0^k with $v_0(k) \preceq_1 w(L)$.

Proof:
i). If $k \preceq_1 L$ then there exists a field extension K of L such that $k \preceq K$. Hence K is ARC and $v_0^k(k) \preceq v_0^K(K)$. Take w to be the restriction of v_0^K to L. The converse is an application of the existential version of Ax-Kochen-Ershov (see [K-P]).
ii). The arguments are the same as in the proof of i), using Corollary 4.2 i) of [Fa2]. □

Let us now obtain some consequences of Theorem 3.5. It might be pointed out that the following results may be obtained as well for the language of ordered rings or even the language of valued and ordered fields taking the valuation to be v_0.

THEOREM 3.8. *For every LR-sentence φ there exists a LOG-sentence $\varphi^\diamond$ such that for every ARC field k,*

$$k \models \varphi \text{ iff } v_0(k) \models \varphi^\diamond.$$

Moreover, $\varphi^\diamond$ may be built recursively from φ.

Proof: i) First of all, using the definability of A_n we can translate every LOG-sentence ψ of $v_n(k)$ to an LR-sentence $\overline{\psi}^n$ of k by a recursive procedure. Consider now Σ the set of sentences of type:

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} \left(\overline{\psi}_{i,j}^{r_{i,j}}\right)$$

for some $n, m \in w$, $r_{i,j} \geq 2$ and $\psi_{i,j}$ LOG-sentences for $i = 1, \dots, n$ $j = 1, \dots, m$. Σ is clearly closed under conjunction and negation. Now let φ be a LOG-sentence and let Γ be

$$\Gamma = \{\psi \in \Sigma \mid TARC \models \varphi \rightarrow \psi\},$$

where $TARC$ denotes the theory of ARC fields. We are going to prove that $TARC \cup \Gamma \models \varphi$. For, if $k \models TARC \cup \Gamma$ consider $\Gamma_k = \{\psi \in \Sigma \mid k \models \psi\}$, $\Gamma \subseteq \Gamma_k$. $TARC \cup \Gamma_k \cup \{\varphi\}$ is consistent because otherwise by compacity $TARC \cup \{\varphi\} \models \neg\psi$ for certain $\psi \in \Gamma_k$ and thus $\neg\psi \in \Gamma$, a contradiction. Let $L \models TARC \cup \Gamma_k \cup \{\varphi\}$, by remark 2 following Theorem 3.5, $k \equiv L$ whence $k \models \varphi$. By a compactness argument there exists $\psi \in \Gamma$ such that $TARC \models \psi \leftrightarrow \varphi$. Since $TARC$ is recursively axiomatizable, we can find this ψ through a dove-tailing recursive procedure, i.e., we can find recursively $n, m \in w$, and $r_{i,j} \geq 2$, $\psi_{i,j}$ LOG-sentences for $i = 1, \dots, n$ $j = 1, \dots, m$. Finally, translating $\psi_{i,j}$ to $v_0(k)$, we can find recursively the desired sentence $\varphi^\diamond$. □

Remark. Since G/G_m is interpretable in G/G_n for $m \mid n$ (because of $G/G_m \simeq G/G_n/(G/G_n)_m$), in the set Σ of Theorem 3.8 it suffices to take sentences of type $\overline{\psi}^n$. Thus Theorem 3.8 may be equivalently stated as follows: for every LR-sentence φ we may find, by a recursive procedure, a naturals number $n \geq 2$ and a LOG-sentence $\varphi^{\diamond_n}$ in such a way that for every ARC field k, $k \models \varphi$ iff $v_n(k) \models \varphi^{\diamond_n}$.

It is not possible, however, to translate LOG-sentences of $v_0(k)$ to LR- sentences of k. For example, consider the LOG-sentence expressing $G_n = \{0\}$ for some $n \geq 2$. For any ARC field k, $v_0(k) \models G_n = \{0\}$ iff $k \models \bigwedge_{p \in \mathbb{P}} A_n \subseteq A_p$. If this were firs-order expressible, by a compacity argument, it would be equivalent to a finite conjunction of sentences of type $A_n \subseteq A_p$, which is no possible, since by remark 5 after Theorem 4.4 we can find an ARC field k satisfying this finite set of setences and $k \not\models \bigwedge_{p \in \mathbb{P}} A_n \subseteq A_p$. If we try to apply the same kind of arguments of 3.8 to translate LOG-sentences to LR-sentences, the problem we find is that $\{v_0(k) \mid k \models TARC\} = \{G \mid G \models TOG \text{ and } G_0 = \{0\}\}$ is not an elementary class, as Proposition 3.13 shows.

Now we are going to see that we can translate also theories of ARC fields to theories of ordered abelian groups preserving the completeness and in some cases the decidability.

Definition. Given a theory T of ARC fields, we define $T^\diamond$ as the following theory of ordered abelian groups

$$T^\diamond = \mathrm{Th}\{v_0(k) \mid k \models T\}.$$

Remarks.

1.- For every LR-sentence φ, $\varphi \in T$ iff $\varphi^\diamond \in T^\diamond$.

2.- For every ARC field k, $k \models T$ iff $v_0(k) \models T^\diamond$ iff $v_0(k) \models \varphi^\diamond$ for all $\varphi \in T$.

3.- For every ordered abelian group G with $G_0 = \{0\}$, $\mathbb{R}((G)) \models T$ iff $G \models T^\diamond$ iff $G \models \varphi^\diamond$ for all $\varphi \in T$.

4.- $\diamond$ is a one to one application between theories of ARC fields and theories of ordered abelian groups. In fact $\diamond$ is monotone: $T \subseteq T'$ iff $T^\diamond \subseteq T'^\diamond$.

Al these remarks are immediate consequences of the definition of $T^\diamond$ and Theorems 3.5 and 3.8. If T is a consistent theory of ARC fields, obviously $T^\diamond$ has models G with $G_0 = \{0\}$. Hence, $\diamond$ is not onto, there are consistent theories Σ of ordered abelian groups with $\Sigma \models G_0 \neq \{0\}$: take G with $\{0\} \neq G_p \subseteq G_q$ for some $p \in \mathbb{P}$ and all $q \in \mathbb{P}$ and Σ its theory. The followig proposition characterizes those theories of ordered abelian groups which are under the image of $\diamond$.

PROPOSITION 3.9. *For every theory Σ of ordered abelian groups the following are equivalent:*

i) There is a theory T of ARC fields such that $T^\diamond = \Sigma$,

ii) Σ satisfies the following properties:

(a) $G \models \Sigma$ implies $G/G_0 \models \Sigma$,

(b) $\Sigma = \mathrm{Th}\{G \mid G \models \Sigma$ and $G_0 = \{0\}\}$.

Moreover, if i) is satisfied $\mathrm{Mod}(T) = \{k \mid k$ *carries a henselian valuation v with real closed residue field and* $v(k) \models \Sigma\}$

Proof:

i)⇒ii). (b) is an immediate consequence of the definition of $T^\diamond$. For (a), let $G \models \Sigma$. Then $G \equiv \prod_{i\in I} v_0(k_i)/U$, a certain ultraproduct of $v_0(k_i)$ with $k_i \models T$. Thus, by Corollary 3.3 iii), $G/G_0 \equiv \left(\prod_{i\in I} v_0(k_i)\,/U\right)/\left(\prod_{i\in I} v_0(k_i)\,/U\right)_0$. Now $\prod_{i\in I}(k_i, v_0^{k_i})/U$ is a henselian valued field with real closed residue field and value group isomorphic to $\prod_{i\in I} v_0(k_i)/U$. Thus $v_0(\prod_{i\in I}(k_i, v_0^{k_i})/U) \simeq \left(\prod_{i\in I} v_0(k_i)\,/U\right)/\left(\prod_{i\in I} v_0(k_i)\,/U\right)_0$ and $\prod_{i\in I} k_i/U \models T$, whence $G/G_0 \models \Sigma$

ii)⇒i). We show first that by (a), the class $\mathcal{K} = \{k \mid k$ carries a henselian valuation v with real closed residue field and $v(k) \models \Sigma\}$ is elementary. $\mathcal{K}$ is clearly closed under ultraproducts. If $k \equiv L \in \mathcal{K}$ then k is ARC and $v_0(k) \equiv v_0(L)$. By (a) $v_0(L) \models \Sigma$ whence $v_0(k) \models \Sigma$ and $k \in \mathcal{K}$. Let now $T = \mathrm{Th}(\mathcal{K})$. If we show $T^\diamond = \Sigma$ the proof of the proposition is complete.

$T^\diamond \subseteq \Sigma$. By (b) we must show $T^\diamond \subseteq \mathrm{Th}\{G \mid G \models \Sigma$ and $G_0 = \{0\}\}$. We show $\{G \mid G \models \Sigma$ and $G_0 = \{0\}\} \subseteq \{v_0(k) \mid k \models T\}$. If $G \models \Sigma$ and $G_0 = \{0\}$ then, by definition of T, $\mathbb{R}((G)) \models T$ whence $G = v_0(\mathbb{R}((G))) \in \{v_0(k) \mid k \models T\}$.

$\Sigma \subseteq T^\diamond$. We must show $k \models T$ implies $v_0(k) \models \Sigma$. Since $\mathcal{K}$ is elemetary, $k \models T$ implies that k carries a henselian valuation v with real closed residue field and and $v(k) \models \Sigma$ thus, by (a), $v_0(k) = v(k)/v(k)_0 \models \Sigma$. □

The following propositions show that $\diamond$ preserves some logical properties.

PROPOSITION 3.10. *T is complete iff $T^\diamond$ is complete.*

Proof: If $T^\diamond$ is complete, given $k, L \models T$ then $v_0(k), v_0(L) \models T^\diamond$ whence $v_0(k) \equiv v_0(L)$ and $k \equiv L$, hence T is also complete. For the converse suppose T is complete and $T^\diamond$ is not. Then $H \not\equiv G$ for some $H, G \models T^\diamond$. Now, by Proposition 3.9, $\mathbb{R}((H)), \mathbb{R}((G)) \models T$

whence $\mathbb{R}((H)) \equiv \mathbb{R}((G))$ and $H/H_0 = v_0(\mathbb{R}((H))) \equiv v_0(\mathbb{R}((G))) = G/G_0$. Hence, by corollary 3.3 vi) $\{0\} \neq H_p \subseteq H_q$ or $\{0\} \neq G_p \subseteq G_q$ for some $p \in \mathbb{P}$ and all $q \in \mathbb{P}$. By the completeness of T, $T \models A_p \subseteq A_q$ for all $q \in \mathbb{P}$ whence $T^\diamond \models G_p = \{0\}$, a contradiction to $H, G \models T^\diamond$. $\square$

The same is true for model completeness, but in fact we have:

PROPOSITION 3.11.

i) The only model complete theory of non-trivial ordered abelian groups (in LOG) is the theory of non-trivial divisible ordered abelian groups.

i) The only model complete theory of ARC fields (in LR) is the theory of real closed fields.

Proof:

i). Suppose Σ is a model complete theory of ordered abelian groups and $G \models \Sigma$. Let $n \geq 2$, from $nG \simeq G$ we get $nG \models \Sigma$ whence $nG \preceq G$. This implies $nG = G$, otherwise taking $g \in G \setminus nG$ we have $G \models (\exists y)(ng = ny)$ and $nG \not\models (\exists y)(ng = ny)$.

ii). Suppose T is a model complete theory of ARC fields and $k \models T$. We are going to show that $v_0(k)$ is divisible. $nv_0(k) \simeq v_0(k)$ implies, by Theorem 3.5, $\mathbb{R}((nv_0(k))) \simeq \mathbb{R}((v_0(k))) \equiv k \models T$ whence $\mathbb{R}((nv_0(k))) \preceq \mathbb{R}((v_0(k)))$. By Theorem 3.2 $nv_0(k) \preceq v_0(k)$ and, as before, $nv_0(k) = v_0(k)$. $\square$

Remark. In fact, if TRCF denotes the theory of real-closed fields, $\text{TRCF}^\diamond$ is the theory of the trivial ordered abelian group $\{0\}$, which is also model complete. The theory of non-trivial divisible ordered abelian groups is not under the image of $\diamond$, by Proposition 3.9.

For the relation between the decidability ot T and $T^\diamond$ we have one trivial implication: $T^\diamond$ decidable implies T decidable. It follows from $\varphi \in T$ iff $\varphi^\diamond \in T^\diamond$ for every LR sentence and the recursivenes of $\diamond$. Since not all LOG-sentences are equivalent to one of type $\varphi^\diamond$, the decidability of T does not implie the decidability of $T^\diamond$ in general, as the following example shows.

Example. Let $B \subseteq \mathbb{N}^2$ recursive such that $\pi(B)$ is not recursive, where π denotes the first coordinate projection $\pi : \mathbb{N}^2 \to \mathbb{N}$. Since there is a recursive bijection between $\mathbb{N}$ and $\mathbb{P}$, there is some $A \subseteq \mathbb{P}^2$ such that $\pi(A)$ is not recursive. Let $D = \{(x, y) \in \mathbb{N}^2 \mid x < y\}$. $A \cap D$ is recursive and $\pi(A \cap D)$ is not recursive: $\pi(A \setminus D)$ is recursive since $x \in \pi(A \setminus D)$ iff $(\exists y \leq x)((x, y) \in A)$, thus the recursiveness of $\pi(A \cap D)$ would imply, by $\pi(A) = \pi(A \cap D) \cup \pi(A \setminus D)$, the recursiveness of $\pi(A)$. Let now

$$T = TARC \cup \{A_q \subset A_p \mid (p, q) \in A \cap D\}.$$

1.- By the remark 5 after Theorem 4.4, there is an ARC field k such that $A_q(k) \subset A_p(k)$ for all $p, q \in \mathbb{P}$ with $p < q$. So T is consistent.

2.- Let us now show that T is decidable. By the remark after Theorem 3.8, there is a recursive procedure which, to every LR-sentence φ associates a natural number $n \geq 2$ and a LOG-sentence $\varphi^{\diamond_n}$ in such a way that

$$T \models \varphi \text{ iff } T(n) = \text{Th}\{v_n(k) \mid k \models T\} \models \varphi^{\diamond_n}.$$

Bearing in mind that G_r is interpretable in G_s for $r \mid s$, we can suppose without loss of generality (by replacing n by $n\prod\{p \mid p \in \mathbb{P} \text{ and } p < n\}$) that the set of prime divisors of n is an initial segment of $\mathbb{P}$, say $p_1, \ldots, p_r$. Now we want to prove that $T(n)$ is axiomatized by

$$TGAO \cup \{G_n = \{0\}\} \cup \{G_q \subset G_p \mid (p,q) \in A \cap D \text{ and } p, q \mid n\},$$

where $TGAO$ denotes the theory of ordered abelian groups. It is clear that $T(n)$ implies this set of sentences. Conversely, let $G \models TGAO \cup \{G_n = \{0\}\} \cup \{G_q \subset G_p \mid (p,q) \in A \cap D \text{ and } p, q \mid n\}$ and take

$$F = G \oplus (p_1, \ldots, p_r)^{-\infty}\mathbb{Z} \oplus (p_1, \ldots, p_{r+1})^{-\infty}\mathbb{Z} \oplus \cdots \oplus (p_1, \ldots, p_m)^{-\infty}\mathbb{Z} \oplus \cdots$$

lexicographically ordered, where p_n denotes the n-th. prime number and $s^{-\infty}\mathbb{Z}$ denotes $\bigcup_{k\in\mathbb{N}} s^{-k}\mathbb{Z}$. It is not difficult to verify that $\mathbb{R}((F)) \models T$ and that $v_n(\mathbb{R}((F))) = G$, whence $G \models T(n)$ follows. This proves that $T(n)$ is axiomatized by $TGAO \cup \{\psi_n\}$, where ψ_n is obtained from n by a recursive procedure. Thus

$$T \models \varphi \text{ iff } TGAO \models \psi_n \to \varphi^{\diamond_n},$$

and the decidability of $TGAO$ implies the decidability of T.

3.- However $T^\diamond$ is not decidable, as it will be clear if we prove

$$T^\diamond \models G_p \neq \{0\} \text{ iff } p \in \pi(A \cap D).$$

If $p \in \pi(A \cap D)$ then $T \models A_q \subset A_p$ for some q such that $(p,q) \in A \cap D$, thus $T^\diamond \models G_q \subset G_p$ whence $T^\diamond \models G_p \neq \{0\}$. Conversely, for p_0 prime, $p_0 \notin \pi(A \cap D)$, let, by remark 5 after Theorem 4.4, k be an ARC field such that $A_{p_0}(k) \subset A_q(k)$ for all $q \in \mathbb{P}$ and $A_q(k) \subset A_p(k)$ for all $p, q \in \mathbb{P}$, $p < q$, $p \neq p_0$. From $k \models T$ and $v_0(k) \models G_p = \{0\}$ it follows $T^\diamond \not\models G_p \neq \{0\}$.

However in some cases, for example when T is complete, the decidability of T implies the decidability of $T^\diamond$.

PROPOSITION 3.12. *Let T a theory of ARC fields such that for all $n \geq 2$ $T \models v_0 = v_n$ or $T \models v_0 \neq v_n$, i.e. $T \models \bigwedge_{p\in\mathbb{P}} A_n \subsetneq A_q$ or $T \models \bigvee_{p\in\mathbb{P}} A_p \subset A_n$, in particular if T is complete. Then*

T is decidable iff $T^\diamond$ is decidable.

Proof: Suppose T is decidable. Then $\{n \mid T \models v_0 = v_n\}$ is recursive, because if this set is non-empty, taking m is this set, we get $T \models v_0 = v_n$ iff $T \models A_n \subsetneq A_m$ for all $n \geq 2$. By Corollary 3.4, for every LOG-sentence φ we can find by a recursive procedure $n, m \in w$, $r_{i,j}, s_{i,j}, t_{i,j} \geq 2$ and LOG-sentences $\psi_{i,j}$ for $i = 1, \ldots, n$ $j = 1, \ldots, m$ such that for any ARC field k

$$v_0(k) \models \varphi \text{ iff } \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} \left(v_{r_{i,j}}(k) \models \psi_{i,j} \vee v_0^k = v_{s_{i,j}}^k \vee v_0^k \neq v_{t_{i,j}}^k \right),$$

We can replace, uniformly for $k \models T$, the statements $v_0^k = v_{s_{i,j}}^k$ and $v_0^k \neq v_{t_{i,j}}^k$ by true or false LR-sentences, accordingly to $\{n \mid T \models v_0 = v_n\}$. Moreover, since A_n is definable, we can replace the statements $v_{r_{i,j}}(k) \models \psi_{i,j}$ by $k \models \varphi_{i,j}$ for some LR-sentences $\varphi_{i,j}$, also obtained by a recursive procedure from $r_{i,j}$ and $\psi_{i,j}$. Putting all this facts together we can find, by a recursive procedure, an LR-sentence φ^∇ such that for all $k \models T$

$$v_0(k) \models \varphi \ \text{ iff } \ k \models \varphi^\nabla.$$

Now the decidability of T implies the decidability of $T^\diamond$. $\square$

In particular, for an ARC field k, k is decidable iff $v_0(k)$ is decidable.
We end this section with a characterization of when $\{v_0(k) \mid k \models T\}$ is elementary.

PROPOSITION 3.13. *Given a theory Σ of ordered abelian groups, then*

$$\Sigma \models G_0 = \{0\} \ \textit{ iff } \ \textit{there is some } n \geq 2 \textit{ such that } \Sigma \models G_n = \{0\}$$

Proof: $\Sigma \models G_0 = \{0\}$ iff $\Sigma \models (\forall x)(x = 0 \vee \bigvee_{p \in \mathbb{P}} x \notin G_p)$. By compacity $\Sigma \models (\forall x)(x = 0 \vee \bigvee_{p \in S} x \notin G_p)$ for some finite $S \subseteq \mathbb{P}$. Hence $\Sigma \models G_n = \{0\}$ taking $n = \prod_{p \in S} p$. $\square$

Thus $\{v_0(k) \mid k \models T\}$ is elementary iff $T^\diamond \models G_n = \{0\}$ for some $n \geq 2$ iff $T \models v_0 = v_n$ for some $n \geq 2$.

§ 4. Definable real valuation rings

In section 2 we have studied the valuations v_S and have shown that when S is finite its valuation ring A_S is first-order definable. In this section k will be an ARC field and we will study the first-order definable real valuation rings of k.

In order to give some results on definable real valuation rings of k we first study the definable convex subgroups of an o.a.g.

Notation. If G is an o.a.g., $G^+ = \{g \in G \mid g \geq 0\}$ and if $g \in G$, $|g| = \max\{g, -g\}$.

THEOREM 4.1. *Let G be an o.a.g. and H a convex subgroup of G. If H is definable by a LOG-formula with parameters in G, then there exists $n \in \mathbb{N}$ such that $H = \bigcap_{g \notin H} A_n(g)$.*

Proof: Let $\varphi(\overline{a}, x)$ be a formula with parameters $\overline{a}$ in G defining H (here $\overline{a}$ denotes a finite sequence of elements of G). From Theorem B of the appendix there exist $n \geq 2$, a quantifier-free LOG*-formula $\psi_1(\overline{a}, x)$, an LSP-formula $\psi_0(y_1, \ldots, y_m, z_1, \ldots, z_r)$, LOG-terms $s_i(\overline{a}, x)$ for $i = 1, \ldots, m$ and $t_i(\overline{a}, x)$ for $i = 1, \ldots, r$ such that

$$G \models \varphi(\overline{a}, x) \quad \text{iff} \quad \begin{cases} G \models \psi_1(\overline{a}, x) \\ Sp_n(G) \models \psi_0(C_1(x), \ldots, C_m(x), D_1(x), \ldots, D_r(x)) \end{cases}$$

where $C_i(x) = A_n(s_i(\overline{a}, x))$ and $D_i(x) = F_n(t_i(\overline{a}, x))$.

Since H is a convex subgroup, for every $N \in \mathbb{N} \setminus \{0\}$, $x \in H$ iff $Nx \in H$, thus

$$(1) \quad x \in H \quad \text{iff} \quad \begin{cases} G \models \psi_1(\overline{a}, Nx) \\ Sp_n(G) \models \psi_0(C_1(Nx), \dots, C_m(Nx), D_1(Nx), \dots, D_r(Nx)). \end{cases}$$

Let N be a common multiple of n, p^r and m for every p, r such that $D(p,r,i)$ occurs in ψ_1, and for every m such that for a certain k $E(m,k)$ occurs into ψ_1. For such N, by Proposition A iii) we have

$$D_i(Nx) = F_n(t_i(\overline{a}, 0)) \qquad \text{for } i = 1, \dots, r$$

and

$$D(p,r,i)(sNx + b) \leftrightarrow D(p,r,i)(b)$$
$$E(m,k)(tNx + c) \leftrightarrow E(m,k)(c)$$

for all formulas of this type occuring as subformulas of ψ_1. Hence, replacing such subformulas by logical constants (always true or always false sentences) we can consider $\psi_1(\overline{a}, Nx)$ a quantifier-free formula in the language $\{0, +, -, \leq, M(k) \text{ for } k > 0\}$.

Claim. If $H \neq G$ and G/H is not discrete, we can choose $M \in \mathbb{N} \setminus \{0\}$, $\alpha, \beta \in G^+$, $\alpha \in H$, $\beta \notin H$ such that every atomic subformula of $\psi_1(\overline{a}, Mx)$ is constant in the interval (α, β).

Proof of the claim. Let M be a multiple of N, $M > k$ for any atomic predicate $M(k)$ occuring in the formula ψ_1. If for every atomic subformula θ of $\psi_1(\overline{a}, Mx)$ there exists such an interval in which θ is constant, we only need to take the intersection of all (a finite number) such intervals. The claim is clear for all subformulas of type $sMx \geq b$ or $sMx \leq b$, where $s \in \mathbb{N}$ and $b \in G$. For each subformula of type $M(k)(sMx + b)$ we distinguish three cases.

Case 1. $H \subset A(b)$. Taking $\alpha \in H^+$ and $\beta \in A(b)^+ \setminus H$, for $x \in (\alpha, \beta)$ $A(sMx + b) = A(b)$ and $M(k)(sMx + b) \leftrightarrow M(k)(b)$.

Case 2. $A(b) \subset H$. Then $b \in H$, and taking $\alpha \geq |b|$, $\beta \in G^+ \setminus H$, for $x \in (\alpha, \beta)$ $A(sMx + b) = A(sMx)$ and $M(k)(sMx + b)$ is false because $M > k$.

Case 3. $A(b) = H$. Then, by hypothesis, $C(b)$ is not discrete and there exists $\beta \in B(b)^+ \setminus A(b)$ such that $sM\beta < |b| + A(b)$. Thus taking any $\alpha \in H^+$, $x \in (\alpha, \beta)$ implies $|sMx| < |b| + A(b)$ whence $A(sMx + b) = A(b)$ and $M(k)(sMx + b)$ is false. This ends the proof of the claim.

Let us now go back to the proof of the theorem. In the case $H = G$, $\bigcap\emptyset$ is understood to be equal to G and the case $H = \{0\}$ causes no problem since $\{0\} = \bigcap_{g \in G \setminus \{0\}} A_n(g)$ for any $n \in \mathbb{N}$, thus we may suppose $\{0\} \subset H \subset G$. If G/H is discrete, $H = A_n(g)$ for every integer n and some $g \in G \setminus H$. Otherwise, let us define, for the integer n of (1)

$$H_0 = \bigcap \{A_n(g) \mid g \in G, \text{ and } H \subseteq A_n(g)\}.$$

If we prove $H_0 = H$ we are done, since this implies $H = \bigcap\{A_n(g) \mid g \notin H\}$: $\bigcap\{A_n(g) \mid g \notin H\} \subseteq H$ because $g \notin A_n(g)$ and $H \subseteq \bigcap\{A_n(g) \mid g \notin H\}$ because $g \notin H = H_0$

implies $g \notin A_n(h)$ for some $h \in G$ with $H \subseteq A_n(h)$ and, by Proposition A iv) in the appendix, $H \subseteq A_n(h) \subseteq A_n(g)$. Suppose that $H \subset H_0$, then for some $x, y \in H_0 \setminus H$, $A_n(x) = A_n(y) \subset H$, since by Proposition A v), $A_n(x) \subset A_n(y)$ implies $x \in A_n(y)$ and $H \subseteq A_n(y)$ whence $y \in A_n(y)$, a contradiction. Let us denote $A_n = A_n(x)$ for $x \in H_0 \setminus H$. Also $A_n(x) = A_n$ for $x \in H \setminus A_n$. We apply the claim, and we are now going to restrict the interval (α, β) in such a way that we get the extra condition that $C_i(Mx)$ is constant for $i = 1, \ldots, m$ ($D_i(Mx) = D_i(0)$ is constant for $i = 1, \ldots, r$). This would lead to a contradiction to (1). If $C_i(Mx) = A_n(rMx+b)$ for some $r \in \mathbb{N}$, $r \neq 0$, $b \in G$, we distinguish two cases. If $A_n \subset A_n(b)$, then, by restricting the interval if necessary, we can suppose $\beta \in H_0^+ \setminus H$, thus $x \in (\alpha, \beta)$ implies $A_n(x) \subseteq A_n \subset A_n(b)$ and $A_n(rMx + b) = A_n(b)$. If $A_n(b) \subseteq A_n$ then, by restricting the interval if necessary (if $b \in H$ we take $\alpha \in H^+ \setminus A_n$ such that $rM\alpha > 2|b|$ and $\beta \in H_0^+ \setminus H$; if $b \notin H$ take $\alpha \in H^+ \setminus A_n$ and $\beta \in H_0^+ \setminus H$ such that $2rM\beta < |b|$), we can get that $x \in (\alpha, \beta)$ implies $rMx + b \in H_0 \setminus A_n$, whence $A_n(rMx + b) = A_n$. □

Remarks:

1.- If $\dim_p(G/pG) < +\infty$ for every $p \in \mathbb{P}$ then the only definable convex subgroups of G are the $A_n(g)$ for some $g \in G$ and $n \geq 2$. It follows from the fact that in this case $Sp_m(G)$ is finite for every $m \geq 2$ (see [S1] for example).

2.- It is also true that $H = \bigcup\{B_n(g) \mid g \in H\}$ for the same n as in the thesis of the preceding theorem. For, if $g \in H$ then for every $h \in G \setminus H$, $g \in A_n(h)$ and thus $B_n(g) \subseteq A_n(h)$ by prop. A v). This implies $B_n(g) \subseteq H$ for every $g \in H$.

3.- It is not possible, in general, to write H as a union of A_n's or intersection of B_n's as show the following examples:

Example 1. Let $\alpha = \omega + 1$ and let $G = \bigoplus_{\beta \in \alpha} \mathbb{Z}_\beta$ lexicographically ordered, where $\mathbb{Z}_\beta$ is an isomorphic copy of $\mathbb{Z}$. If $H = \mathbb{Z}_\omega$, then, for every $n \geq 2$, $H = B_n(1_\omega)$ (where 1_β stands for the unit of $\mathbb{Z}_\beta$) and $\bigcup\{A_n(g) \mid g \in G \text{ and } A_n(g) \subseteq H\} = \{0\}$.

Example 2. Let the order type $\alpha = 1 + \omega^*$, where ω^* is the inverse order type of ω and 1 minores ω^*. Then, if $G = \bigoplus_{\beta \in \alpha} \mathbb{Z}_\beta$ and $H = \bigoplus_{\beta \in \omega^*} \mathbb{Z}_\beta$, for every $n \geq 2$ $H = A_n(1_1)$ (here the subindex 1 stands for the last element of α) and $\bigcap\{B_n(g) \mid g \in G$ and $H \subseteq B_n(g)\} = G$.

4.- But we can prove that if $\bigcup\{A_n(g) \mid g \in G \text{ and } A_n(g) \subseteq H\} \subset H$ for every $n \geq 2$ then $H = B_n(h)$ for some h and n. Let n as given by Theorem 4.1 and $h \in H \setminus \bigcup\{A_n(g) \mid g \in G \text{ and } A_n(g) \subseteq H\}$. By remark 2, $B_n(h) \subseteq H$. If there exists $h' \in H \setminus B_n(h)$, then $h \in A_n(h') \subseteq H$, a contradiction. In a similar way we can prove that if $H \subset \bigcap\{B_n(g) \mid g \in G \text{ and } H \subseteq A_n(g)\}$ for every n then $H = A_n(h)$ for some h and n.

COROLLARY 4.2. *Let G be an o.a.g. and H a convex subgroup of G. If H is definable in G by a LOG-formula with parameters, then there exist $n \in \mathbb{N}$ and an initial segment Δ of $Sp_n(G)$ such that*

$$x \in H \qquad \text{iff} \qquad A_n(x) \in \Delta,$$

Δ definable in $Sp_n(G)$ by an LSP-formula with parameters. Moreover, if H is definable without parameters so is Δ. The correspondence which, to a defining formula of H, associates a defining formula of Δ is uniform in the parameters.

Proof: Let n as in Theorem 4.1. Then

$$x \in H \qquad \text{iff} \qquad A_n(x) \subset H.$$

This proves the first assertion.

For $H = \{0\}$ or $H = G$, Δ is trivially defined in $Sp_n(G)$ without parameters. Thus, from now to the end of the proof we are going to suppose that H is a proper non-trivial convex subgroup of G. Let ψ_0 and ψ_1 be as in the proof of Theorem 4.1. The first step in the proof that Δ is definable consists in modifying ψ_0 in such a way that it only depends on $A_n(x)$. For every $C_i(x) = A_n(rx + b)$ occuring in ψ_0 with $r \neq 0$, if $H \subsetneq A_n(b)$, we can replace ψ_0 by

$$A_n(x) \subset A_n(b) \wedge \psi_0\big(A_n(b)/C_i(x)\big),$$

where $\psi_0\big(A_n(b)/C_i(x)\big)$ is obtained by replacing any occurrence of $C_i(x)$ by $A_n(b)$ in ψ_0. If $A_n(b) \subset H$, we can replace ψ_0 by

$$\big(A_n(b) \subset A_n(x) \wedge \psi_0\big(A_n(x)/C_i(x)\big)\big) \vee A_n(x) \subseteq A_n(b).$$

Repeating this procedure for any $C_i(x)$ occuring in ψ_0 we arrive at an LSP-formula $\psi_0'(x)$ with parameters in $Sp_n(G)$ such that

$$x \in H \qquad \text{iff} \qquad G \models \psi_1(\overline{a}, x) \text{ and } Sp_n(G) \models \psi_0'(A_n(x)).$$

If G/H is not discrete, let α, β and M as in the claim of Theorem 4.1. Since $\psi_1(\overline{a}, Mx)$ is true in (α, β) ($H \neq \{0\}$ implies $(\alpha, \beta) \cap H \neq \emptyset$), we have

$$\begin{aligned} x \in H \quad \text{iff} \quad |x| \in H \quad & \text{iff} \quad G \models \psi_1(\overline{a}, M|x|) \text{ and } Sp_n(G) \models \psi_0'(A_n(x)) \\ & \text{iff} \quad G \models |x| < \beta \text{ and } Sp_n(G) \models \psi_0'(A_n(x)) \\ & \text{iff} \quad Sp_n(G) \models A_n(x) \subset A_n(\beta) \wedge \psi_0'(A_n(x)), \end{aligned}$$

by Theorem 4.1.

If G/H is discrete, then $H = A_n(g)$ for some $g \in G$ and thus

$$\begin{aligned} x \in H \quad & \text{iff} \quad G \models \psi_1(\overline{a}, M|x|) \text{ and } Sp_n \models \psi_0'(A_n(x)) \\ & \text{iff} \quad Sp_n(G) \models A_n(x) \subset A_n(b) \wedge \psi_0'(A_n(x)). \end{aligned}$$

In the case where $\varphi(x)$, the formula defining H, is without parameters, it is easy to see that , for a suitable $M \in \mathbb{N}\setminus\{0\}$, $\psi_1(Mx)$ remains constant, hence true, in $\{g \in G \mid g > 0\}$. Thus,

$$x \in H \quad \text{iff} \quad Sp_n(G) \models \psi_0'(A_n(x)),$$

where the formula $\psi_0'(A_n(x))$ has at most the parameter $A_n(0) = F_n(0) = \emptyset$, which is definable in $Sp_n(G)$.

The correspondence which, to a defining formula $\varphi(\overline{a}, x)$ of H, associates a defining formula $\psi(\overline{b}, y)$ of Δ can be chosen uniform in $\overline{a}$: depending on $\overline{a}$, finitely many formulas $\varphi_i''(\overline{b}_i, y)$ occurred in the proof, such that

$$G \models \varphi(\overline{a}, x) \qquad \text{iff} \qquad Sp_n(G) \models \varphi_i''(\overline{b}_i, A_n(x)).$$

The combination $\bigvee \varphi_i''(\overline{b}_i, y) \wedge c_i = d_i$ is equivalent, for a suitable choice of $\overline{c}$ and $\overline{d}$, to the formula $\varphi_i''(\overline{b}_i, y)$ we want. □

COROLLARY 4.3. *If $H \neq \{0\}$ is a definable convex subgroup of G then $G_p \subseteq H$ for some $p \in \mathbb{P}$, where G_p denotes the maximum convex p-divisible subgroup of G.*

Proof: By Proposition A vi) of the appendix we can impose in Theorem 4.1 n to be a prime number. 4.3 is then a consequence of 4.1 keeping in mind that if $g \in G_p$ then $A_p(g) \subseteq \{0\}$ and if $g \notin G_p$ then $G_p \subseteq A_p(g)$. □

If k is an ARC field, we recall that $v_1 = \max V(k)$ and for $v \in W(k)$, we are going to denote $G_v = v_1(A_v \setminus M_v)$. It is a convex subgroup of $v_1(k)$.

THEOREM 4.4. *Let k be an ARC field and $v \in W(k)$. Then A_v is definable in k iff G_v is definable in $v_1(k)$ and $v \leq v_p$ for some $p \in \mathbb{P}$.*

Proof: Let us denote $G_p = G_{v_p}$, also equal to $\left(v_1(k)\right)_p$, using the notation of §3.

1. The only if part of the proof uses the following Delon's elimination result (see [D1], chap. 2) valid for Hensel fields of zero residual characteristic: If a subset of $v(k)$ is definable by a formula of valued fields with parameters in k it is definable by a LOG-formula with parameters in $v(k)$. For, if A_v is definable in the field structure k, G_v is definable in the valued field structure (k, v_1) ($v_1(x) \in G_v$ iff $x \in A_v$ and $x^{-1} \in A_v$), hence in the o.a.g. $v_1(k)$. If $v \neq v_1$ then $G_v \neq \{0\}$ and, from corollary 4.3, $G_p \subseteq G_v$ for a certain $p \in \mathbb{P}$, whence $v \leq v_p$. If $v = v_1$ the following claim gives the result.

Claim: If A_{v_1} is definable then $v_1 = v_p$ for some p.

Proof of the claim: If $\varphi(\overline{a}, x)$ defines A_{v_1} and $v_1 > v_p$ for every $p \in \mathbb{P}$ consider the following set of formulas, consistent in k:

$$T(x) = \{\exists yz(\varphi(\overline{a}, y) \wedge \varphi(\overline{a}, z) \wedge \varphi(\overline{a}, y^{-1}) \wedge \neg\varphi(\overline{a}, z^{-1}) \wedge x = n + y^2 + z) \mid n \in \omega\}$$

Let L an elementary extension of k realizing $T(x)$. Then L is an ARC field and $\varphi(\overline{a}, x)$ defines a henselian valuation ring of L with real-closed residue field. If we denote by v this valuation, $v \in V(L)$, $v > v_p^L$ for every prime p (since $(\exists x)(\varphi_p(x) \wedge \neg\varphi(\overline{a}, x))$ holds in k, thus in L, where $\varphi_p(x)$ is a formula defining A_p) and $v < v_1$ because the realization of $T(x)$ makes L/v non-archimedean. This contradicts the last assertion before the claim.

2. If G_v is definable in $v_1(k)$ and $v \leq v_p$ for a certain p then $G_p \subseteq G_v$ and from prop. 3.1 of [D-L] G_v/G_p is definable in $v_p(k) = v_1(k)/G_p$. It is then not difficult to deduce the definability of A_v from the definability of A_p. □

Remarks:

1.- The same may be seen replacing v_1 by any $v \in V(k)$ in Theorem 4.4.

2.- The collection $\{v_p \mid p \in \mathbb{P}\}$ is cofinal in the set of definable real valuation rings.

3.- v_0 is the only possibly definable valuation in $V(k)$ and, in this case, it is the biggest definable real valuation of k. v_0 is definable iff $v_0 = v_p$ for a certain p iff G_p is divisible for a certain p iff $v_0(k)$ has no non-trivial convex p-divisible subgroups for a certain p iff there exists a minimum element in the collection $\{G_p \mid p \in \mathbb{P}\}$ iff $\{v_p \mid p \in \mathbb{P}\}$ has a maximum element.

4.- In the case $\dim_p k < \infty$ for every p, by Theorem 4.4 and remark 1 following Theorem 4.1 we characterize all definable real valuations of k.

5.- Let us describe all possible relations of inclusion occuring in the set $\{v_p \mid p \in \mathbb{P}\}$. If we define R on $\mathbb{P}$ as

$$v_q \leq v_p \qquad \text{iff} \qquad q\,R\,p,$$

then R is clearly a linear preorder, i.e., R is a reflexive and transitive relation satisfying pRq or qRp for every $p, q \in \mathbb{P}$. Conversely, for every such R, let us consider the associated equivalence relation: $p \sim q$ iff $p\,R\,q$ and $q\,R\,p$, and the quotient order $\leq$ on $\mathbb{P}/\sim$. For $\overline{q} \in \mathbb{P}/\sim$, let $G_{\overline{q}}$ to be an archimedean and dense o.a.g., p-divisible for p with $\overline{p} \leq \overline{q}$ and not p-divisible for p with $\overline{p} > \overline{q}$ (this choice is always possible, see [Za] or [S1]). If we set

$$G = \bigoplus_{\overline{p} \in \mathbb{P}/\sim} G_{\overline{p}}$$

lexicographically ordered, the maximum convex p-divisible subgroup G_p of G is $\bigoplus_{\overline{q} \geq \overline{p}} G_{\overline{q}}$. Thus $G_p \subseteq G_q$ iff $\overline{q} \leq \overline{p}$ iff $q\,R\,p$. In $\mathbb{R}((G))$ we have $v_q \leq v_p$ iff $q\,R\,p$. Hence, by remark 3, there are many examples where v_0 is not first-order definable.

6.- The situation is different in the case of v_S for $S \neq \mathbb{P}$. It can be first-order definable even in the case $v_S \neq v_p$ for all $p \in \mathbb{P}$ as the following example shows:

Let $S = \mathbb{P} \setminus \{2\}$ and, for any $q \in S$, let H_q be an archimedean o.a.g. p-divisible for $p \leq q$, $p \neq 2$, and not p-divisible if $p > q$ or $p = 2$. Let H_2 be archimedean, p-divisible for all $p \in S$ and not 2-divisible. Let then

$$G = \left(\bigoplus_{p \in S} H_p \right) \oplus H_2$$

where $\bigoplus_{p \in S} H_p$ is lexicographically ordered by the natural order of S ($S \subseteq \mathbb{N}$). If $k = \mathbb{R}((G))$, v_1 is the canonical series valuation of k with $k/v_1 = \mathbb{R}$ and $v_1(k) = G$. Then $G_{v_S} = H_2 = B_2(g)$ for $g \in H_2 \setminus \{0\}$; hence, by 4.4, v_S is definable in k. But $G_{v_S} \neq G_p$ for every $p \in \mathbb{P}$, because $G_2 = \{0\}$ and $G_p = \bigoplus_{q \geq p} H_q$ if $p \in S$.

Appendix

The statements, definitions and notation of this section are taken from [S1] and [S2]. All the proofs may be found in [S1].

We start with some definitions. Let G be an o.a.g., $g \in G$ and $n \geq 2$.

G is called n-regular iff for every convex divisible subgroup $H \neq \{0\}$ of G, then G/H is n-divisible. It is equivalent to the following first-order sentence:

$$\forall xy\big(\exists z_1, \ldots, z_n(x \leq z_1 < z_2 < \cdots < z_n \leq y) \rightarrow \exists z(x \leq nz \leq y)\big).$$

Let $g \in G \setminus \{0\}$. We define:

$A(g)$ =the largest convex subgroup of G not containing g.
$B(g)$ =the smallest convex subgroup of G containing g.
$C(g) = B(g)/A(g)$.
$A_n(g)$ =the smallest convex subgroup C of G such that $B(g)/C$ is n- regular.
$B_n(g)$ =the largest convex subgroup C of G such that $C/A(g)$ is n- regular.
$C_n(g) = B_n(g)/A_n(g)$
For $g = 0$ we define $A_n(0) = \emptyset$, $B_n(0) = \{0\}$.

For $g \in G$ we define:
$F_n(g)$ =the largest convex subgroup C of G such that $C \cap (g + nG) = \emptyset$ (i.e., $g \notin C + nG$) if $g \notin nG$, $F_n(g) = \emptyset$ otherwise.
$\Gamma_{1,n}(g) = \{h \in G \mid F_n(h) \subset F_n(g)\}$
$\Gamma_{2,n}(g) = \{h \in G \mid F_n(h) \subseteq F_n(g)\}$
If $g \notin nG$, $\Gamma_{1,n}(g)$ and $\Gamma_{2,n}(g)$ are shown to be subgroups of G (Proposition A ii)). In this case we define:
$\Gamma_n(g) = \Gamma_{2,n}(g)/\Gamma_{1,n}(g)$, $\Gamma_n(g)$ is shown to be a torsion group with n as an exponent.

The sets $A_n(g)$, $B_n(g)$ and $F_n(g)$ are shown to be definable in the language LOG= $\{0, +, \leq\}$ by a first-order formula with the only parameter g.

PROPOSITION A. *If* $g, h \neq 0$ *then:*
i) $A_n(g+h) \subseteq A_n(g) \cup A_n(h)$ *and if* $A_n(g) \subset A_n(h)$ *then* $A_n(g+h) = A_n(h)$
ii) $F_n(g+h) \subseteq F_n(g) \cup F_n(h)$ *and if* $F_n(g) \subset F_n(h)$ *then* $F_n(g+h) = F_n(h)$
iii) $F_n(g+nh) = F_n(g)$. $F_n(g) = \emptyset$ *iff* $g \in nG$.
iv) $A_n(h) \subseteq A_n(g)$ *iff* $B_n(h) \subseteq B_n(g)$ *iff* $A_n(h) \subset B_n(g)$ *iff* $h \in B_n(g)$ *iff* $g \notin A_n(h)$.
v) $A_n(h) \subset A_n(g)$ *iff* $B_n(h) \subset B_n(g)$ *iff* $B_n(h) \subseteq A_n(g)$ *iff* $g \notin B_n(h)$ *iff* $h \in A_n(g)$
vi) $A_n(g) = \bigcup\{A_p(g) \mid p$ *a prime divisor of* $m\}$

The language LSP of spines contains as non-logical symbols a binary relation symbol $\leq$ and the following monadic relation symbols: A, F, D and $\alpha(p,k,m)$ for all $k, m \in \mathbb{N}$, $k > 0$, p prime.

The n-spine of G for $n \geq 2$ is defined as the LSP-structure with universe

$$\{A_n(g) \mid g \in G\} \bigcup \{F_n(g) \mid g \in G\}$$

and with the following interpretation of the relations:

$$\begin{aligned}
C_1 \leq C_2 &\text{ iff } C_1 \subseteq C_2 \\
A(C) &\text{ iff } C = A_n(g) \text{ for some } g \in G \\
F(C) &\text{ iff } C = F_n(g) \text{ for some } g \in G \\
D(C) &\text{ iff } C \neq \emptyset \text{ and } G/C \text{ is discrete} \\
\alpha(p,k,m)(C) &\text{ iff } C = F_n(g) \text{ for some } g \in G \setminus nG \text{ and } \alpha_{p,k}(\Gamma_n(g)) \geq m
\end{aligned}$$

where $\alpha_{p,k}(C)$ is the Szmielew invariant given by $\dim_p(p^{k-1}C[p]/p^kC[p])$. $\alpha_{p,k}(\Gamma_n(g))$ is the number of cyclic groups of order p^k in the direct sum decomposition of $\Gamma_n(g)$ [Sz].

We are going to denote this structure by $Sp_n(G)$.

Let LOG* be the definitional expansion of LOG by the following unary predicates: $M(k)$[1], $E(n,k)$ and $D(p,r,i)$ for all $n \geq 2$, $r \geq 1$, $0 < i < r$, $0 < k < n$ and p prime. For $g \neq 0$ they are defined by:

$M(k)(g)$ iff $C(g)$ is discrete with $\overline{e}$ denoting its first positive element and $\overline{g} = k\overline{e}$ in $C(g)$ iff $C_n(g)$ is discrete with $\overline{e}$ denoting its first positive element and $\overline{g} = k\overline{e}$ in $C_n(g)$.

$E(n,k)(g)$ iff there exists $h \in G$ such that $F_n(g) = A_n(h)$, $M(1)(h)$ holds and $\overline{g} = k\overline{h}$ in $\Gamma_n(g)$ iff there exists $h \in G$ such that $F_n(g) = A_n(h)$, $M(1)(h)$ holds and $F_n(g - kh) \subset F_n(g)$.

$D(p,r,i)(g)$ iff $g \in p^r G$ or $\overline{g} \in p^i \Gamma_{p^r}(g)$ iff $g \in p^r G$ or there exists $h \in G$ such that $F_{p^r}(g - p^i h) \subset F_{p^r}(p^i h) = F_{p^r}(g)$.

For $g = 0$ they are defined to be false.

THEOREM B. *For every LOG-formula $\varphi(\overline{x})$ there exist $n \geq 2$, a quantifier-free LOG*-formula $\psi_1(\overline{x})$, an LSP-formula $\psi_0(y_1, \ldots, y_m, z_1, \ldots, z_r)$, LOG-terms $t_i(\overline{x})$ for $i = 1, \ldots, m$ and $s_i(\overline{x})$ for $i = 1, \ldots, r$ such that for every o.a.g. G and every $\overline{g} \in G^\omega$*

$$G \models \varphi(\overline{g}) \quad \textit{iff} \quad \begin{cases} G \models \psi_1(\overline{g}) \\ Sp_n(G) \models \psi_0(C_1, \ldots, C_m, D_1, \ldots, D_r) \end{cases}$$

where $C_i = A_n(t_i(\overline{g}))$ and $D_i = F_n(s_i(\overline{g}))$.

Theorems C and D are consequences of Theorem B.

THEOREM C. *Let G and H be two o.a.g. Then*

$$G \equiv H \quad \textit{iff} \quad Sp_n(G) \equiv Sp_n(H) \textit{ for all } n \geq 2$$

THEOREM D. *Let G and H be two o.a.g.. $H \subseteq G$. Then $H \preceq G$ iff H is a LOG*-substructure of G and for all $n \geq 2$ the application defined by:*

$$\begin{aligned} Sp_n(H) &\longrightarrow Sp_n(G) \\ A_n^H(h) &\longmapsto A_n^G(h) \quad \textit{if } h \in H \\ F_n^H(h) &\longmapsto F_n^G(h) \quad \textit{if } h \in H \end{aligned}$$

is well defined and an elementary embedding of LSP-structures.

Remark. Theorem D is not stated in [S1] but it is equivalent to Theorem 6.1 of [S1].

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[1] In fact, in [S1] and [S2] is used the notation $M(n,k)$. We eliminate n since $M(n,k)$ does not depend on n (if $C(g)$ is discrete then $A_n(g) = A(g)$; conversely if $C_n(g)$ is discrete and $\overline{g} = k\overline{e}$ in $C_n(g)$ then $A_n(g) = A(g)$).

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HENSELIAN FIELDS WITH REAL-CLOSED RESIDUE FIELD

EBERHARD BECKER
RALPH BERR
DANIELLE GONDARD-COZETTE

INTRODUCTION.

In this paper henselian fields with real-closed residue fields are studied. To study this class of fields was stimulated by the observation that many fields of importance for the theory of formally real fields in fact belong to it : real-closed fields in the original sense of Artin, generalized real-closed fields introduced by the first named author in [Be], chain closures of fields in the sense of Schwartz, [S] and [S2], and the Rolle fields first studied by Brown, Craven and Pelling in [B-C-P], to mention the main examples. The occurence of so many examples also asked for a general treatment. To facilitate slightly the notation, the fields in question are refered to as *residually real-closed henselian fields.*

In the first section we study the valuation theory of such a field R . Among the henselian valuation rings with a real closed residue field there is a smallest one, A(R) , and a largest one, $\mathcal{J}(R)$. If Γ denote the value group of $\mathcal{J}(R)$, then $S(R) := \{ p \mid p \text{ prime}, \Gamma \neq p\Gamma \}$ is called the exact type of R .

The subsequent section deals with a first-order characterization of the class of all residually real-closed henselian fields, as well as of the subclass of fields of exact type S(R) = S . That these are elementary classes has been proved first and independently by Delon and Farré, [D-F]. However, they use other methods and their set of axioms is not as simple as ours.

Residually real-closed henselian fields enjoy sharper versions of Hensel's lemma. In addition, they satisfy Rolle's theorem for certain set of polynomials. This is proved in section 3 which is inspired by Brown, Craven and Pelling's study of Rolle fields.

In the last section we study algebraic residually real-closed henselian extensions of a formally real field. In our point of view the extensions have to be considered as a type of rather general real closures which should share the basic properties of the Artin-Schreier theory. In fact we are able to describe the conjugacy classes of residually real-closed extensions in a quite satisfactory way. Minimal extensions of this kind are well understood, and finally, all this fits nicely into Schwartz's theory of real closures relative to generalized signatures.

I- THE VALUATION THEORY OF RESIDUALLY REAL-CLOSED HENSELIAN FIELDS.

From now on *residually real-closed henselian field* will mean a field admitting a henselian valuation with real-closed residue field.

We first introduce some notations and recall known facts :

Let K be a field, a subset $T \subseteq K$ is called a *preordering* if $T + T \subseteq T$, $0, 1 \in T$, $-1 \notin T$ and $T^* = T \setminus \{0\}$ is a subgroup of K^*. Now let v be a valuation of K with valuation ring V, maximal ideal m_v. Then v (resp. V) is *fully compatible* with T if $1 + m_v \subseteq T$. In this situation T induces a preordering $\bar{T}$ on the residue field of v. If in addition $\bar{T}$ is a total order, then we say that v (resp. V) *trivializes* T. Now T is called a *valuation fan* if for any $x \notin \pm T$ we have $1 \pm x \in T$ or $1 \pm x^{-1} \in T$. From [J1] we know that a preordering $T \subseteq K$ is valuation fan if an only if there exists a valuation v of K which trivializes T.

For any preordering $T \subseteq K$ let :

$$A(T) = \{ x \in K \mid \exists n \in \mathbb{N} \quad n \pm x \in T \}.$$

It is known that $A(T)$ is a Prüfer ring, and by [Br2] lemma 1-8 , T is a valuation fan if and only if $A(T)$ is a valuation ring which trivializes T .

Proposition I.1. *Let (R,v) be a residually real-closed henselian field, then any real valuation ring of R is henselian, there exists a smallest one and the set of real valuation rings is totally ordered by inclusion .*

Proof. The rings $A(P)$, $P \in X(R)$, are the minimal real valuation rings (see [L] theorem 2.6). Hence if V is real henselian, V is compatible with all orders of R (c.f. [L] 3.14) and for each $P \in X(R)$ $A(P) \subseteq V$. Then by mapping V to the residue field $\bar{R}$, $A(P)$ is sent to $A(\bar{P})$ since if $n \pm a \in P$ then $n \pm \bar{a} \in \bar{P}$, hence we get $A(\bar{P}) = A(\bar{R}^2)$. Thus $A(\bar{P})$ is henselian since the unique real valuation ring of $\bar{R}$ is such and using [R] (proposition 10 p. 211) we deduce that $A(P)$ is henselian. So every real valuation of R is henselian . In particular, if $P \in X(R)$, then every real valuation of R is compatible with P . Hence the set of real valution rings of R is totally ordered by inclusion. Finally we get that for every P and Q in $X(R)$ the minimal real valuation rings $A(P)$ and $A(Q)$ are equal hence there exists a smallest henselian valuation ring .

Remark I.2. We also get the following properties of a residually real-closed henselian field :

for all orders P and Q $A(P) = A(Q)$, *R admits a unique $\mathbb{R}$-place* (c.f. [L] corollary 2.13) and R is real henselian in the sense of [Br2] which means that *R admits a henselian valuation with archimedean real-closed residue field.*

Let $R_\infty = \bigcap_{n \in \mathbb{N}} R^{2n}$ and let $H(R)$ denote the real holomorphy ring of R, i.e. the intersection of all real valuation rings of R. For properties of $H(R)$ we refer to [B2] .

Proposition I.3. *Let (R,v) be a residually real-closed henselian field, then the smallest henselian valuation ring is described by :*

$$H(R) = A(R^2) = A(R^{2n}) = A(R_\infty)$$

Proof. From the proof of I.1 and the definition of $H(R)$ it follows that the smallest henselian valuation ring is equal to $H(R)$. First note that by Hensel's lemma R is pythagorean at any level since $\bar{R}$ is such. Hence for every $n \in \mathbb{N}^*$ $A(\sum R^{2n}) = A(R^{2n})$. Now if Γ is the value group of $H(R)$ we know from the general theory that $R^{2n} = \{ x \in R^2 \mid v(x) \in 2n\Gamma \}$, hence we get $R_\infty = \{ x \in R^2 \mid v(x) \in \bigcap_{n\in\mathbb{N}} 2n\Gamma \} = \{ x \in R^2 \mid v(x) \in \Gamma_{div} \}$ (*) where Γ_{div} is the maximal divisible subgroup of Γ . Clearly we have $A(R_\infty) \subseteq A(R^{2n}) \subseteq A(R^2) = H(R)$. On the other hand take $x \in H(R) = A(R^2)$; then there exists $n \in \mathbb{N}$ with $n \pm x \in \sum R^2 = R^2$. Hence we get $1 + n \pm x \in H(R)^x \cap \sum R^2$ which implies by (*) that $1 + n \pm x \in R_\infty$, hence $x \in A(R_\infty)$.

Remark I.4. From the proof above it follows that a residually real-closed field R is *pythagorean at any level* and that for every n *R^{2n} is a valuation fan* .

For any preordering T let $\mathcal{J}(T)$ denote the following set introduced by Jacob in [J] : $\mathcal{J}(T) = \mathcal{O}_1(K,T) \cup \mathcal{O}_2(K,T)$ where

$$\mathcal{O}_1(K,T) = \{ x \in K \mid x \notin \pm T \text{ and } 1 + x \in T \}$$

$$\mathcal{O}_2(K,T) = \{ x \in K \mid x \in \pm T \text{ and } x\, \mathcal{O}_1(K,T) \subseteq \mathcal{O}_1(K,T) \} .$$

Proposition I.5. *Let T be a valuation fan then $\mathcal{A}(T)$ is the largest valuation ring trivializing T.*

For the proof we need preliminary lemma.

Lemma I.a. *Let T be a valuation fan of a field K, v a valuation which trivializes T and let $x \in K$. Then $v(x) \in v(T^*)$ if and only if $x \in T \cup -T$.*

Proof of lemma I.a. We have only to show that $v(x) \in v(T^*)$ implies $x \in T \cup -T$. Choose $y \in T^*$ such that $v(x) = v(y)$, then $xy^{-1} \in V^*$. Since T induces a total order on the residue field we find z in T (or $-T$) with $xy^{-1} \in z + m_v$ where m_v is the maximal ideal of V ; but then xy^{-1} is in $z^{-1}(1 + z\, m_v)$. Now $1 + m_v$ is contained in T. If $z \in T$ we get $xy^{-1} \in T$. If z is in $-T$ we get xy^{-1} in $-T$. Hence $x \in T \cup -T$, as $y \in T$.

Now given a valuation fan $T \subseteq K$ we let v_T (resp. Γ_T) denote the valuation (resp. the value group) corresponding to $A(T)$. Moreover, let $\Delta_T \subseteq \Gamma_T$ be the maximal convex subgroup of $v_T(T^*)$. With these notations we get the following lemma :

Lemma I.b. *Let $T \subseteq K$ be a valuation fan and let $V \subseteq K$ be a valuation ring with $A(T) \subseteq V$. Then V trivializes T if and only if V corresponds to a convex subgroup Δ_v* of Δ_T

Proof. Suppose that V corresponds to Δ_v. If V trivializes T, then $V^* \subseteq T \cup -T$, by lemma I.a. Hence $\Delta_v \subseteq \Delta_T$. Conversely, assume $\Delta_v \subseteq \Delta_T$ and let $x \in V^*$. Then $v(x) \in v(T^*)$ hence by lemma I.a $x \in T \cup -T$. Thus

$V^* \subseteq T \cup -T$. Moreover, V is fully compatible with T as $A(T) \subseteq V$; hence V trivializes T .

Proof of proposition I.5. It is known from Jacob ([J] theorem 1 p. 96) that $\mathcal{J}(T)$ is a valuation ring which trivializes T . Thus we have just to show that if a valuation ring W trivializes T then $W \subseteq \mathcal{J}(T)$. From lemma I.a we know that $W^* \subseteq T \cup -T$.

If $x \in W^*$ we prove that $x \in \mathcal{O}_2$. Let $y \in \mathcal{O}_1$ then $xy \in \mathcal{O}_1$, and $xy \notin T \cup -T$ because x is contained in $T \cup -T$ but y is not . First we claim that $y \in m_v$ is the only possible case : $y \in W^*$ is impossible since $y \in \mathcal{O}_1$ implies $y \notin T \cup -T$; $y \notin W$ cannot hold because in this case $1 + y^{-1}$ would belong to W^* hence to $T \cup -T$ and we would deduce $1 + y = y\,(1 + y^{-1}) \notin T \cup -T$ which is impossible since $y \in \mathcal{O}_2$. Then we get $xy \in m_v$ and since $x \in W^*$ $1 + xy \in W^*$, residually $1 + xy$ is sent to $1 \in \bar{T}$ hence $1 + xy \in T$, and as $xy \notin T \cup -T$ we get $x \in \mathcal{O}_2$.

If $x \in m_v$, if $x \notin T \cup -T$ then $1 + x \in W^*$, so $1 + x$ is sent residually to $\bar{T}$ hence $1 + x \in T$ and $x \in \mathcal{O}_1$; if $x \in T \cup -T$, take $y \in \mathcal{O}_1$ then $xy \notin T \cup -T$ and $1 + xy \in T$ because $x \in m_v$, so $xy \in \mathcal{O}_1$ and $x \in \mathcal{O}_2$.

Now let $\mathfrak{w}(R)$ denote the set of all real (henselian) valuation rings with real-closed residue field . By assumption on R , $\mathfrak{w}(R)$ is not empty. Also from remark I.2 we get that $H(R) \in \mathfrak{w}(R)$.

Proposition I.6. *Let R be a residually real-closed henselian field. Then*

(i) $\mathfrak{w}(R)$ contains a smallest and a largest valuation ring ;

(ii) $H(R)$ is the smallest valuation ring in $\mathfrak{w}(R)$;

(iii) $\mathcal{J}(R_\infty)$ is the largest valuation ring in $\mathfrak{w}(R)$.

Proof of I.6.

(ii) follows from remark I.2 which implies $H(R) \in \mathfrak{w}(R)$.

(i) follows from the general fact that a henselian field (K,v) is real-closed if and only if the value group Γ_v is divisible and the residue field is real-closed . Applying this to our situation of a residually real-closed henselian field we get that the residue field of a real valuation is real-closed if and only if the corresponding value group is (convex) divisible, hence the valuations of $\mathfrak{w}(R)$ correspond to the convex divisible subgroups of the value group of $H(R)$.

(iii) Let $\mathcal{J}(R)$ denote the largest valuation ring in $\mathfrak{w}(R)$ we have to show that $\mathcal{J}(R) = \mathcal{J}(R_\infty)$.

Note that, by lemma I.b and proposition I.5, $\mathcal{J}(R_\infty)$ is the largest valuation ring trivializing R_∞ and corresponds to the maximal convex subgroup Δ of $v(R_\infty^*)$ (where v is the valuation associated to $A(R_\infty) = H(R)$) . But since $v(R_\infty^*) = \Gamma_{div}$ by the definition (*) of R_∞ and Δ is a convex subgroup of Γ_{div} , we get that Δ is divisible. From the end of proof of (i) we know that the valuations of $\mathfrak{w}(R)$ correspond to the convex divisible subgroups of the value group of $H(R)$ hence $\mathcal{J}(R_\infty) \in \mathfrak{w}(R)$ and is contained in $\mathcal{J}(R)$.
Next suppose that $\mathcal{J}(R)$ corresponds to Δ_R , then Δ_R is divisible , as the residue field of $\mathcal{J}(R)$ is real-closed, hence $\mathcal{J}(R)$ trivializes R_∞ , thus by proposition I.5 $\mathcal{J}(R) \subseteq \mathcal{J}(R_\infty)$.

From now on $A(R)$ will denote the smallest henselian valuation ring with real-closed residue field and $\mathcal{J}(R)$ the largest one.

For any real field K and any $\mathbb{R}$-place λ we note $S_\lambda(K) = \{ p \in \mathbb{P} \mid \Gamma_\lambda \neq p\,\Gamma_\lambda \}$. If R is a residually real-closed henselian field we shall denote its unique $\mathbb{R}$-place by λ, the valuation associated with λ by v, and by Γ_v its value group. For a residually real-closed henselian field R $S_\lambda(R)$ will be denoted $S(R)$ since R has a unique $\mathbb{R}$-place.

Note that $p \notin S(R)$ is equivalent to $R^2 = R^{2p} = R^{2p^2} = \dots = R^{2p^n} = \dots$.

Definition I-7. *Let S be any set of primes, then R is called a residually real-closed henselian field of type S if R is a residually real-closed residue field and $S(R) \subseteq S$; R is said of exact type S if $S(R) = S$.*

Examples.

(i) if $S = \emptyset$ we get the real-closed fields.

(ii) if $S = \{2\}$ we obtain the Rolle fields in the sense given in [B-C-P] and [G].

(iii) let S be any set of primes, there exists a residually real-closed henselian field K of exact type S :

let $\Gamma = \{ r/s \mid r \in \mathbb{Z} ,\ s = \prod_{p \notin S} p^{\alpha_p} \}$ then the field $K = \mathbb{R}((\Gamma))$ is a residually real-closed henselian field satisfying $S(\mathbb{R}((\Gamma))) = S$

For the following sections we shall need further results.

Lemma I-8. *Let R be a residually real-closed henselian field* of *type S and let $L \supset R$ ($L \neq R$) be a finite extension. If L is real, then $[L : R] \in \Pi S$; if L is non real, then $1/2\ [L : R] \in \Pi S \cup \{1\}$.*

Proof. The statement is trivial for real-closed fields . Suppose that R is not real-closed . We know by proposition I.1 that v is an henselian valuation with real-closed residue field. Let $\bar{v}$ be the unique extension of v to L . Since v is henselian and char $R_v = 0$ we have $[L : R] = e_{\bar{v}|v} f_{\bar{v}|v}$. R is a residually real-closed henselian field of type S so $e_{\bar{v}|v} \in \Pi S$. If L is real then $f_{\bar{v}|v} = 1$ and $f_{\bar{v}|v} = 2$ if and only if L is non real . This proves the claim. Moreover if $2 \in S$ then $[L : R] \in \Pi S$ for any finite extension L of R .

Remark I.9. This lemma can also be deduced from the structure of the Galois group of R . A residually real-closed henselian field R has the following properties :

(i) R is hereditarily-pythagorean (see [Be] part III) .

(ii) $G(\bar{R}|R(i)) = \prod \mathbb{Z}_p^{\alpha_p}$ where $p \in \mathbb{P}$ and

$\alpha_p = \dim_{\mathbb{F}_p} [R^{2*} : R^{2p*}] = \dim_{\mathbb{F}_p} [\Gamma_v : p\Gamma_v]$ (see [Be] (p. 120))

II-AN AXIOMATIZATION.

The following theorem gives another characterization for residually real-closed henselian fields . It will lead to first order axiomatizations for residually real-closed henselian fields and for residually real-closed henselian fields of type S .

Theorem II-1. *Let R be a field, then the following statements are equivalent*

(i) R is a residually real-closed henselian field ;

(ii) R is a hereditarily pythagorean field and for all $n \in \mathbb{N}$ R^{2n} is a valuation fan ;

(iii) R is a hereditarily pythagorean field and $R_\infty = \cap R^{2n}$ is a valuation fan.

Proof.

(i) $\Rightarrow$ (ii) see section I .

(ii) $\Rightarrow$ (iii) : since $R_\infty = \cap R^{2n}$, R_∞ is a preordering. Thus it remains to show : (*) $x \notin \pm R_\infty \Rightarrow 1 \pm x \in R_\infty$ or $1 \pm x^{-1} \in R_\infty$.

Choose $n \in \mathbb{N}$ with $x \notin \pm R^{2n}$; we may assume without lost of generality that $1 + x \in R^{2n}$ hence (**) $1 + x^{-1} \notin R^{2n}$ because otherwise $1 + x = x (1 + x^{-1}) \in R^{2n}$ but $x \notin R^{2n}$. Now let $k \in \mathbb{N}$. In view of $R^{2nk} \subseteq R^{2n}$, (**) implies $1 \pm x \in R^{2nk} \subseteq R^{2k}$, and we deduce $1 \pm x \in R_\infty$.

(iii) $\Rightarrow$ (i) : Let w be the valuation which corresponds to the Jacob ring $\mathcal{J} = \mathcal{J}(R_\infty)$. Then the pushdown of R_∞ with respect to w is a total order. Hence the residue field R_w is euclidean and $R_w^2 = R_w^{2n}$ for all $n \in \mathbb{N}$.

Next we have to show that R_w is real-closed.

First note that R_w is also hereditarily pythagorean ; now let L be a finite real extension of R_w , since R_w is hereditarily pythagorean it follows from [Be] that $L = R_w (\sqrt[2t_1]{a_1^2}, \ldots , \sqrt[2t_k]{a_k^2})$ for some $t_1 , \ldots, t_k$ in $\mathbb{N}$ and $a_1, \ldots, a_k$ in R_w . Now $R_w^2 = \cap R_w^{2n}$ implies that $L = R_w$ which shows that R_w is real-closed.

If w is trivial then $R = R_w$ and we are done.

Therefore assume that w is not trivial. We will show that w is henselian ; to this end we prove that (R, w) does not admit any proper immediate finite algebraic extension. Let $(L , \bar{w})$ be a finite extension of (R , w) .

If L is not formally real, by theorem 1 of [Be] p. 86, we know that L contains $\sqrt{-1}$, hence $f_{\bar{w}|w} = 2$. Now assume that L is real and that $(L, \bar{w})$ is an immediate extension of (R, w) ; using theorem 13 of [Be] p. 114 we may assume that $L = R(\sqrt[n]{a})$ for some $n \in \mathbb{N}$ and $a \in R$. Since $(L, \bar{w})$ is an immediate extension of (R, w) we have $w(a) \in n\,\Gamma_w$; let now $y \in R$ be such that $w(a) = n\,w(y)$ then $a\,y^{-2n} \in \mathcal{J}^*$. Define $f = X^{2n} - a^2 y^{-2n}$ then $\bar{f}$ has a simple root α in R_w . It follows from the proof of theorem 2 of [J] that there exists an overring $\mathcal{J}_1$ of $\mathcal{J}$ which satisfies Hensel's lemma for all equations $X^{2n} - a$. Since $a\,y^{-2n}$ is a unit of $\mathcal{J}_1$ as well there exist $\beta \in R$ such that $f(\beta) = 0$. Hence $a^2 = \beta^{2n} y^{2n}$ and since L is a real field this implies $L = R$.

Corollaire II-2. *The class of residually real-closed henselian fields is an elementary class.*

Proof. In [J] it is shown that the class of hereditarily pythagorean fields is elementary ; hence the claim follows from theorem II-1.

An alternative first order characterization of hereditarily pythagorean fields can be obtained as follows :

let K be a real field and $K(X)$ a rational function field in one variable. By theorem 4 of [Be] p. 94, K is hereditarily pythagorean if and only if

(#) $\sum K(X)^2 = K(X)^2 + K(X)^2$, hence if and only if

(##) $\sum K[X]^2 \subseteq K(X)^2 + K(X)^2$. Now by Cassel's theorem (##) is equivalent to

(###) $\sum K[X]^2 = K[X]^2 + K[X]^2$. Now let $f, g, h \in K[X]$ such that $f^2 = g^2 + h^2$; then the degrees of g and h are less than or equal to the degree of f since K is formally real ; hence (###) is expressible by an infinite sequence of first order sentences in the language of fields.

Corollaire II-3. *Let* S *be any set of primes ; then the class of residually real-closed henselian fields of type* S *is elementary.*

Corollary II-3 follows from corollary I-2 since a field K is a residually real-closed henselian field of type S if and only if K is a residually real-closed henselian field and for all $p \in \mathbb{P} \setminus S$ we have $K^2 = K^{2p}$

Recall from [G] that residually real-closed henselian fields of type $\{2\}$ - which means Rolle fields - are characterized by the following first order axioms :

K is real, $K^4 + K^4 = K^4$, K^2 is a fan and K has no odd extension.

In the case of residually real-closed henselian fields of type S we obtain a similar first order axiomatization :

Corollary II-4. *Let* R *be a field, then the following statements are equivalent :*

(i) R is a residually real-closed henselian field of type S .

(ii) 1- Every extension is a radical extension ;

2- R has no real extension of degree in $\mathbb{N} \setminus \Pi S \cup \{1\}$ *;*

3- $R^4 + R^4 = R^4$ *;*

4- $\forall p \in S \cup \{1\}$, R^{2p} *is a fan ;*

(i) ⇒ (ii) is clear using theorem II-1 and lemma I.8.

(ii) ⇒ (i) R has to be formally real, e. g. by axiom 4.

From axiom 1 we get that R is hereditarily pythagorean.

Axioms 3 and 4 imply that R^{2n} is a valuation fan for every $n \in \Pi S$. To see this we make use of Becker's result (theorem 3-14 of [Be2]) and of a lemma by Jacob (lemma 6 of [J]) . From axiom 2 we get that $R^2 = R^{2p}$ for every $p \notin S$, hence these R^{2p} are fans. To finish the proof we have to show that R^{2p} is a valuation fan for every $p \notin S$. Since R^2 is a fan this follows again from lemma 6 of [J] . Now apply theorem II-1 .

Remark II-5. One might wonder whether it is enough to demand the following properties :

1- Every extension of degree $p \in S$ is an extension $R(\sqrt[p]{a})$;

2- R has no real extension of odd degree $p \in \mathbb{P} \setminus S$;

3- $R^4 + R^4 = R^4$;

4- $\forall p \in S \cup \{1\}$, R^{2p} is a fan ;

The following gives an example of a field E satisfying these axioms but admitting an extension of odd non prime degree, and E is not a residually real-closed henselian field.

We shall use the following lemma :

Lemma II-6. *For $n \geq 5$, the group A_n has a subgroup of prime index if and only if n is a prime number.*

Proof of lemma II-6. $\Rightarrow$ if n is prime then A_{n-1} is a subgroup of index n since the order of A_n is $n!/2$.

$\Leftarrow$ Let U be a subgroup of A_n and $[A_n : U] = r > 1$ and assume r to be prime but not n . Let $\Phi : A_n \longrightarrow S(A_n / U)$ defined by $h \longmapsto (gU \longmapsto hgU)$. As $n \geq 5$, A_n is a simple group .

Since Φ is a group homomorphism $\mathrm{Ker}\Phi$ is a normal subgroup of A_n . Hence $\mathrm{Ker}\Phi$ is 1 or A_n . It cannot be A_n . Thus Φ is injective , and we must have $n!/2 \leq r!$. As r is a prime r divides $n!$. Hence $r \leq n-1$, thus $n!/2 \leq (n-1)!$ which is impossible for $n > 2$.

Proof of remark II-5. Let K be an extension of $\mathbb{Q}$ realizing A_n (n not prime and $n \geq 5$) as the Galois group of the extension ; let E be a maximal algebraic real field such that $K \cap E = \mathbb{Q}$.

First , E is euclidean : by the transfer theorem of Galois theory we can argue as follows , let $a \in P \setminus E^2$ for some order P , then $E(\sqrt{a})$ is a non trivial extension of degree 2 of E but, since it is real, $E(\sqrt{a}) \cap K \neq \mathbb{Q}$ and this must also be an extension of degree 2 of $\mathbb{Q}$, but the Galois group A_n of the extension K of $\mathbb{Q}$ has no subgroup of index 2 hence $P = K^2$.

Next, E has no extension of odd prime degree p . If F were such an extension then $F \cap K$ would be an extension of degree p of $\mathbb{Q}$ which is impossible since the Galois group of K over $\mathbb{Q}$ has no subgroup of index p.

Since E is a number field then $\sum E^{2p} = \sum E^2 = E^2$, hence, because E has no extension of degree p , $E^2 = E^{2p}$.

A henselian valuation ring of a real number field has to be trivial. If E were residually real-closed henselian it had to be real-closed violating the existence of a galois group A_n , $n \geq 5$.

III-LIFTING ZEROS AND ROLLE'S THEOREM.

In this part we show that the henselian valuation rings of a residually real-closed henselian field R allow, in certain situations, to lift zeros from the residue field $\bar{R}$ back to R even if these zeros are not simple. That this stronger version of "Hensel's lemma" is also related to the validity of Rolle's theorem in R was already stated in the main theorem of Brown, Craven and Pelling [B-C-P] (theorem 2.1).

Let V be any henselian valuation ring with a real-closed residue field $\bar{R}$, and keep V fixed in the sequel. Let $\pi : V \longrightarrow \bar{R}$ be the canonical epimorphism and set $\pi(b) =: \bar{b}$ for every $b \in V$. Clearly π also induces in a natural way an epimorphism from $V[X]$ onto $\bar{R}[X]$ and we set $\pi(f) =: \bar{f}$ accordingly. If $\beta \in \bar{R}$ is a zero of $\bar{f}$ such that $f(X) = (X-\beta)^m.H(X)$ with $H(\beta) \neq 0$ then $m = \mu(\beta)$ is the multiplicity of β .

From the defining property of a henselian valuation ring it is known (see theorem 4.3 page 186 of [R]) that if $f \in V[X]$ is given with $\bar{f} \neq 0$ and β is a simple (i.e. $\mu(\beta) = 1$) root of $\bar{f}$ in $\bar{R}$, then there exists $b \in V$ such that $\bar{b} = \beta$ and $f(b) = 0$, i.e. β can be lifted to a zero of f . Note that f needs not be monic.

More generally, we say that, relative to V , <u>zeros of multiplicity m can be lifted</u>, if for every polynomial $f \in V[X]$ with $\bar{f} \neq 0$ and every zero β of $\bar{f}$ in $\bar{R}$ where $\mu(\beta) = m$ there exists $b \in V$ with $\bar{b} = \beta$ and $f(b) = 0$

We set : $S(V) =: \{ m \in \mathbb{N} \mid$ zeros of multiplicity m can be lifted $\}$

Note that $1 \in S(V)$ and that if $V = R$ then $S(V) = \mathbb{N}$ obviously.

In the next theorem we compute $S(V)$ in terms of the exact type $S(R)$ of the residually real-closed henselian field R . To this end we denote by $< 2, S(R) >$ the additive semigroup in $\mathbb{N}$ generated by 2 and the prime numbers in $S(R)$.

Theorem III.1. *If R is a residually real-closed henselian field of exact type $S(R)$ and if V is a non trivial henselian valuation ring with real-closed residue field then*

$$S(V) = \mathbb{N} \setminus < 2, S(R) > \text{ , more precisely}$$

(i) $S(V) = \{$ odd numbers $\}$ if $S(R) \subseteq \{2\}$;

(ii) $S(V) = \{ m \in N \mid 1 \leq m \leq p-2 , m$ odd $\}$ where p is the smallest odd prime in $S(R)$, if $S(R)$ is not contained in $\{2\}$.

As an obvious corollary we obtain that $S(V)$ is independent of V as long as V varies over the henselian valuation rings with a real-closed residue field.

Proof of theorem III.1. First we consider a number $m \in \mathbb{N} \setminus < 2, S(R) >$ and assume a situation $f \in V[X]$, $\bar{f} = (X-\beta)^m.H(X)$ with $\beta \in \bar{R}$, $H(\beta) \neq 0$. Since V is henselian we find polynomials h , $g \in V[X]$ with $f = h.g$, $\bar{h} = (X-\beta)^m$ $\deg h = m$, $\bar{g} = H$ (see p. 186 of [R]) . Now decompose h into R-irreducible polynomials h_1 , ..., h_r . From lemma I.8 we know that each of the degrees $\deg h_i$ is divisible either by 2 or by some of the primes in $S(V)$ unless it is equal to 1 ; hence if none of the h_i's were linear then m would belong to $< 2, S(R) >$: a contradiction. Therefore , some h_i has to be linear, i.e. h , hence f , has a zero b with $\pi(b) = b$.

We next have to show that, if $V \neq R$, no element $m \in < 2, S(R) >$ belongs to S(V). Denote by v and Γ the valuation and the value group associated with V . Let $p_1, \ldots, p_r$ be primes in $S(R)$ and take $a_i \in R$ such that $0 < v(a_i)$ and $v(a_i) \notin p_i \Gamma$; now choose $b \neq 0$ and b in the maximal ideal of V . For any set $l, l_1, \ldots, l_r$ of natural numbers consider the polynomial :

$$f = (X^2+b^2)^l \, (X^{p_1} - a_1)^{l_1} \ldots (X^{p_r} - a_r)^{l_r} \in V[X] \quad .$$

Clearly, f has no zero in R . However, $\bar{f} = X^m$ with $m = 2.l + \sum_{i=1}^{r} p_i . l_i$, and the zero $\beta = 0$ of multiplicity m cannot be lifted.

Statement (i) is obvious. To deduce (ii) one just observes that every number $m \geq p - 1$ belongs to $< 2, S(R) >$ where p is the smallest odd prime in $S(R)$.

We are now going to study Rolle's theorem for our field. Quite generally we say that an ordered field $(K, <)$ satisfies Rolle's theorem for degree n if for every polynomial $f \in K[X]$ of degree n and elements $a, b \in K$ satisfying $a < b$, $f(a) = f(b) = 0$ there is $c \in K$ with $a < c < b$ such that $f'(c) = 0$. $(K, <)$ is said to be a Rolle field if Rolle's theorem holds for every degree.

Theorem III-2. *Let R be a residually real-closed henselian field, then :*
(i) if $S(R) \subseteq \{2\}$, R is a Rolle field for any of its orders ;
(ii) if $S(R)$ is not contained in $\{2\}$ and p is the smallest odd prime number in $S(R)$, R satisfies Rolle's theorem, for any of its orders, exactly for the degrees $1, 2, \ldots, p$.

Remark. Part (i) is Brown-Craven-Pelling's result, [B-C-P].

Proof of theorem III.2. Let P be any order of R and $\lambda = \lambda_p$ be the unique real place of R. The valuation ring V of λ is henselian with real-closed residue field as shown in section I. This valuation ring will be used in the sequel.

We first prove the following claim : *let $n \in \mathbb{N}$ be such that $m \in S(V)$ for every odd number $m \leq n-1$ then (R, P) satisfies Rolle's theorem for degree n.* In fact, consider $f \in R[X]$ of degree n with two zeros a, b with $a < b$ $f(a) = f(b) = 0$. As in the proof of theorem 2.1 of [B-C-P] we may assume that $a = 0$, $b = 1$ and $f \in V[X]$, $\bar{f} \neq 0$. Passing to the residue field $\bar{R}$ we find that $\bar{f}' = \overline{f'}$ has a root $\beta \in \mathbb{R}$ where $0 < \beta < 1$ By studying the Taylor expansion of $\bar{f}$ in a local extremum we even find a root $\beta \in \mathbb{R}$ of $\bar{f}'$ of odd multiplicity m, $0 < \beta < 1$. Clearly $m \leq n-1$. Since $\bar{R}$ is algebraically closed in $\mathbb{R}$ we deduce that $\beta \in \bar{R}$. By assumption f' has a root b with $\bar{b} = \beta$. From $0 < \beta < 1$ we deduce $0 <_p b <_p 1$.

This already yields the statement (i) by using theorem III.1 .

Next consider the case where $S(R)$ is not contained in $\{2\}$ and let p denote the smallest odd prime in $S(R)$. From (ii) in theorem III-1 we derive that Rolle's theorem holds for every degree $n \le p$. It remains to prove that Rolle's theorem does not hold for degrees $n > p$.

To prove this we assume that (R, P) satisfies Rolle's theorem for degree n and we will show that this implies that every odd number $m \le n-1$ belongs to $S(V)$, hence $n > p$ is impossible in view of (ii), theorem II.1.

Thus, let $f \in V[X]$ be given such that $\bar{f} \neq 0$ admits a zero β of odd multiplicity $m \le n-1$. We have in $\bar{R}[X]$: $\bar{f} = (X - \beta)^m . H$, $H(\beta) \neq 0$. As above this leads to a decomposition $f = g.h$ with g monic, $\bar{g} = (X - \beta)^m$ and $\bar{h} = H$. Choose any $c \in V$ with $\bar{c} = \beta$. Since $\bar{R}$ is Archimedean we can find a natural number $\ell < \beta$. Then consider the polynomial $F \in R[X]$ defined by $F(X) = \int_c^x g(t).(t-\ell)^{n-1-m}\, dt$ where the integration is carried out symbolically. F has degree n and

$$\bar{F}(X) = \int_\beta^x (t-\beta)^m.(t-\ell)^{n-1-m}\, dt \quad .$$

In a suitable neighbourhood of β $\bar{F}$ looks like the parabola $(1/(m+1))(t-\beta)^{m+1}$. Choosing a sufficiently small rational number ε we obtain that $\bar{F}(X) - \bar{F}(\beta+\varepsilon)$ has (at least) two distinct solutions α , γ , satisfying $\ell < \alpha < \beta < \gamma$ since $m + 1$ is even . Applying Hensel's lemma we see that $F(X) - F(c+\varepsilon)$ has two zeros a and d satisfying $\bar{a} = \alpha$, $\bar{d} = \gamma$ which yields $\ell < a < c < d$. By the assumption that Rolle's theorem holds for degree n we are led to the existence of a root b of $(F(X) - F(c+\varepsilon))'$, hence of $g(X).(X-\ell)^{n-1-m}$, in (a,d) . Clearly $g(b)$ has to be zero , and we get $f(b) = 0$ with $\bar{b} = \beta$. Hence m is in $S(V)$.

Brown, Craven, and Pelling have shown that an ordered field is a Rolle field iff it is a residually real-closed henselian field with $S(R) \subseteq \{2\}$. One may wonder whether, quite generally, residually real-closed henselian fields can be characterized by requiring the validity of Rolle's theorem for some degrees. This is not the case as we are going to show. For example, assume $3 \in S(R)$ then statement (ii) of the theorem III.2 says that R satisfies Rolle's theorem exactly for the degrees less or equal to 3. As we will see there are many fields sharing this property which are not residually real-closed henselian fields. We first prove :

Proposition III.3. *The followings statements hold :*

(i) An ordered field $(K, <)$ satisfies Rolle's theorem for the degrees less or equal to 3 iff K is pythagorean ;

(ii) a pythagorean field satisfies Rolle's theorem for the degrees less or equal to 3 relatively to any of its orders.

Proof of Proposition III.3. Clearly (i) implies (ii). To prove (i) first note that, trivially, Rolle's theorem for degrees 1 and 2 is satisfied in any ordered field. In considering polynomials of degree 3 meeting the hypothesis of Rolle's theorem we may restrict ourselves to polynomials of the type

$$f_a = X(X-1)(X-a), \quad a \in K, \text{ and the zeros } 0, 1 \text{ of } f_a .$$

Therefore we have to investigate the property that for every $a \in K$ f'_a has a root in $(0,1)$. Since $f'_a = 3X^2 + 2(a-1)X - a$ we get that f'_a has a root in K iff for every a the element $a^2 + a + 1$ is a square in K. The later property is equivalent to K being pythagorean. If K is pythagorean one finds that one of the roots lies in $(0,1)$, for every ordering in K.

Corollary III.4. *The pythagorean closure of* $\mathbb{Q}$ *satisfies Rolle's theorem exactly for the degrees* 1,2,3 *. It is not a residually real-closed henselian field.*

Remark. For the pythagorean closure we refer to [R2] .

Proof of corollary III.4. $\mathbb{Q}_{pyth}$ the pythagorean closure of $\mathbb{Q}$ is clearly not a residually real-closed henselian field. We have to show that it does not satisfies Rolle's theorem for degrees $n \geq 4$. In view of the following considerations (remark III.5 (ii)) it would be enough to show that it does not meet this property for $n = 4$. However, we want to give a direct proof. Consider the polynomial $f = X^n - X^{n-3}$ where $n \geq 4$. It has the zeros 0 and 1 . Therefore, $\mathbb{Q}_{pyth}$ would contain the zero $\alpha = \sqrt[3]{n-3/n}$ of f' if it satisfies Rolle's theorem for degree n . However, one checks that $X^3 - (n-1)/n$ is irreducible over $\mathbb{Q}$. Hence, the 2-extension $\mathbb{Q}_{pyth} \mid \mathbb{Q}$ cannot contain α .

We conclude this section by some remarks concerning ordered fields satisfying Rolle's theorem for some degree n .

Remark III.5. By adjusting the proof of the theorem (2.1) of Brown, Craven and Pelling in [B-C-P], especially of the implication (a) ⇒ (c), one can derive the following facts :

(i) if the ordered field (K,<) satisfies Rolle's theorem for degree n then, considering the natural place $\lambda : K \longrightarrow \mathbb{R} \cup \{\infty\}$ and its valuation ring V, every root of odd multiplicity of $\bar{f} \neq 0$ in $\mathbb{R}$, where f is any polynomial in V[X] of degree less or equal to $n - 2$, lies in $\lambda(V)$ and can be lifted to K . This in turn implies :

(ii) if (K,V) satifies Rolle's theorem for degree n it also satisfies this theorem for degree $n - 1$, hence for each of the degrees 1, 2, ..., n .

(iii) if (K,V) satisfies Rolle's theorem for some $n \geq 4$ the place λ has to be a 2-henselian valuation ring with an euclidean residue field.

Remark III.6. Fields satisfying Rolle's theorem up to a certain degree which are not residually real-closed henselian fields exist in abundance. Take any hereditarily euclidean number field K which is not real-closed. K has a Galois group $G(\bar{K}|K) = \langle\sigma\rangle \propto \prod_p \mathbb{Z}_p^{\alpha_p}$, where the set of p's with $\alpha_p \neq 0$ can be described arbitrarily ([Be] , p. 118 ff.) .
If p_0 is the smallest occuring p then the smallest degree of a subextension of K in a fixed real closure R of K is just p_0 . Therefore K satisfies at least Rolle's theorem for degree p_0 but not, in view of remark III.5 (i) , this theorem for degree $p_0 + 2$. An open question is whether such a field K satisfies Rolle's theorem for degree $p_0 + 1$?

IV-ALGEBRAIC EXTENSIONS AND RELATIONSHIP WITH SIGNATURES.

Let K be a given real field and R a residually real-closed henselian field which is an algebraic extension of K . We denote by λ_R the real place of R , by v_R the Krull-valuation attached to λ_R and by Γ_R its value group.

Using theorem 3-2 of [Brn] we can get the following result :

Theorem IV-1. *Let* R *and* R' *be two residually real-closed henselian fields which are algebraic extensions of a given field* K *, then the following are equivalent :*

(i) R *and* R' *are* K*-isomorphic.*

(ii) $\forall n$ $R^n \cap K = R'^n \cap K$.

(iii) $\forall n$ $R^{2n} \cap K = R'^{2n} \cap K$.

(iv) $R^2 \cap K = R'^2 \cap K$ *and* $\Gamma_R = \Gamma_{R'}$ *(as subgroups of the divisible hull of the value group of* $v_R|_K$*) .*

(v) some ordering of K extends to an ordering of R and to an ordering of R' and $\Gamma_R = \Gamma_{R'}$.

Proof :

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow directly from [Brn] since R and R' are real henselian fields in the sense of [Brn] (i.e. R admits an henselian valuation with archimedean real-closed residue field) and are extensions of the field K

(ii) $\Rightarrow$ (iii) is trivial and it remains to prove (iii) $\Rightarrow$ (ii) .

Let $n \in \mathbb{N}$ be odd and let $x \in R^n \cap K$. Then $x^2 \in R^{2n} \cap K = R'^{2n} \cap K$. Hence $x \in R'^n \cap K$ as n is odd , i.e. $R^n \cap K \subseteq R'^n \cap K$. By the same argument we get $R^n \cap K \supseteq R'^n \cap K$.

Remarks :

1- Since R^2 is a fan then $T = R^2 \cap K$ is also a fan and $|M_T(K)| = |\{ \lambda \in M(K) \mid \lambda(T) \geq 0 \}| = 1$. The same holds for $R'^2 \cap K$; hence $R^2 \cap K = R'^2 \cap K = T$ is equivalent to the fact that the sets $\chi(R)$ and $\chi(R')$ induce the same set of orderings $\chi_T = \mathrm{Res}(\chi(R)) = \mathrm{Res}(\chi(R'))$ of K .

2- As soon as $\mathrm{Res}(\chi(R)) \cap \mathrm{Res}(\chi(R')) \neq \emptyset$ we have $\lambda_R|_K = \lambda_{R'}|_K$.

In the following the place $\lambda_R|_K$ will be denoted by λ and for any preordering T we shall denote by $M_T(K)$ the set $\{ \xi \in M(K) \mid \xi(T) \geq 0 \}$.

The above theorem shows the importance of the sets $T_n = R^{2n} \cap K$; these sets clearly have the following properties :

(i) $\forall n$ T_n is a fan of higher level and $| M_{T_n}(K) | = 1$.

(ii) $T_n \supseteq K^{2n} . \mathcal{U}_\lambda$, whith $\mathcal{U}_\lambda = \{ \varepsilon \in V_\lambda^{\times} \mid \lambda(\varepsilon) > 0 \}$ where V_λ is the valuation ring associated of λ .

(iii) $\forall n,m$ $(T_n)^m \subseteq T_{nm}$.

Theorem IV-2. *Let* K *be a real field and* $\lambda \in M(K)$ *; for each sequence of fans* $(T_n)_{n\in\mathbb{N}}$ *satisfying the following two conditions :*

(i) $\forall n \in \mathbb{N}$ $T_n \supseteq (\sum K^{2n}) . \mathcal{U}_\lambda$

(ii) $\forall n,m \in \mathbb{N}$ $(T_n)^m \subseteq T_{nm}$

there exists an algebraic extension R | K *with R a residually real-closed henselian field , uniquely determined up to a* K*-isomorphism, such that* $\lambda_R|_K = \lambda$ *and for all* n $R^{2n} \cap K = T_n$.

Proof. Let v be the canonical valuation associated with λ and $k \subseteq \mathbb{R}$ the residue field, let $\mathcal{U}_\lambda = \{ \varepsilon \in V_\lambda^{\times} \mid \lambda(\varepsilon) > 0 \}$ where V_λ is the valuation ring of v . Now consider an henselian closure H of K valued with v and let R_λ be an unramified extension of the valued field H such that the residue field $r \subseteq \mathbb{R}$ is the real closure of k ordered by $\bar{P}_\lambda = k \cap \mathbb{R}^2$ (the order of k induced by the orders compatible with λ) ; then the value groups satisfy $\Gamma_\lambda = \Gamma_H = \Gamma_{R_\lambda}$.

We shall also denote by λ the extensions of λ to H and R_λ . Clearly R_λ is a residually real-closed henselian field.

By hypothesis T_n is a fan in K ; now we define

$T_n^H = T_n . H^{2n} . \mathcal{U}_\lambda^H$ and $T_n^{R_\lambda} = T_n . H^{2n} . \mathcal{U}_\lambda^{R_\lambda}$ where

$\mathcal{U}_\lambda^H = \{ \varepsilon \in V_\lambda^{H\times} \mid \lambda(\varepsilon) > 0 \}$ with V_λ^H the valuation ring associated to λ

$\mathcal{U}_\lambda^{R_\lambda} = \{ \varepsilon \in V_\lambda^{R_\lambda^x} \mid \lambda(\varepsilon) > 0 \}$ where V^{R_λ} the valuation ring associated to λ.
First the proofs of 3-2 and 3-3 of [L] p. 22 can be adapted in order to prove that T_n^H and $T_n^{R_\lambda}$ are fans :
if we use the notation of [L] we have : $T_n^H = (T_n . H^{2n}) \wedge \bar{P}_\lambda$ and $T_n^{R_\lambda} = (T_n . H^{2n}) \wedge \bar{P}_\lambda^r$ and we get that T_n^H and $T_n^{R_\lambda}$ are preorderings of level n in H and R_λ respectively which are fully compatible with the valuation associated with λ in H or R_λ and such that $\bar{T}_n^H = \bar{P}_\lambda$ and $\bar{T}_n^{R_\lambda} = \bar{P}_\lambda^r$. Then we also get by the analogous of 3-4 in [L] p. 23 that $T_n^H = \cap P$, where P is any ordering of level n in H such that $\lambda_p = \lambda$ and $T_n \subseteq P$. This comes from the fact that since it is a wedge product all the orderings above T_n^H are compatible with the valuation of λ and have the same archimedean pushdown $\bar{P}_\lambda$, so every ordering induces the place λ. Conversely, if $P \supseteq T_n$ and $\lambda_P = \lambda$, then $T_n^H \subseteq P$, as P induces a total order on the residue field.
In the same way $T_n^{R_\lambda} = \cap P$ where P is any ordering of level n in R_λ which contains T_n. Hence $T_n^H \cap K = T_n$ and $T_n^{R_\lambda} \cap H = T_n^H$.
Now fix a real closure of K for some order above T_0 and consider the following extensions of R_λ $F_n = R_\lambda(\sqrt[2n]{T_n})$; since $T_n^m \subseteq T_{nm}$ we have $F_n \subseteq F_{nm}$ hence we can define a field R by $R = \cup_n F_n$; R is a residually real-closed henselian field.
From [Be] p. 116 one derives $F_n^{2n} \cap R_\lambda = T_n R_\lambda^{2n}$; since $\mathcal{U}_\lambda \subseteq R_\lambda^{2n}$ we get $F_n^{2n} \cap R_\lambda = T_n^{R_\lambda}$ and using again [Be] p. 116 this will yield $R^{2n} \cap R_\lambda = T_n^{R_\lambda}$; finally we obtain $R^{2n} \cap K = T_n$ and by theorem IV-1 such a residually real-closed henselian field R is unique up to a K-isomorphism.

We next study algebraic extensions of a field K which are residually real-closed henselian fields of type S with fixed S.
If R is such an extension the value group Γ_R corresponding to the unique

place of R is p-divisible for any $p \notin S$ thus Γ_R contains Γ_{divS^c} the $(\mathbb{P}\setminus S)$-divisible hull of Γ . Conversely, if R is any residually real-closed henselian field and extension of K satisfying $\Gamma_R \supseteq \Gamma_{divS^c}$ then $\Gamma_R = p\,\Gamma_R$ for every $p \notin S$, and R is a residually real-closed henselian field of type S extension of K . Hence we get :

Theorem IV-3. *Let* K *be a real field and* $\lambda \in M(K)$ *; then for any* $S \subseteq S_\lambda$ K *admits minimal residually real-closed henselian extensions of type* S *. They correspond to* Γ_{divS^c} *and are determined up to* K*-conjugacy by the sequence :*

$$T_n = T_{n_S \cdot n_{S^c}} = (\sum K^{2n_S}) \cdot \mathcal{U}_\lambda \quad \textit{where} \quad n_S = \prod_{p\in S} p^{\alpha_p} \quad \textit{and} \quad n_{S^c} = \prod_{p\notin S} p^{\alpha_p} .$$

Next we will consider order spaces of higher level of K . Let p be a prime. Recall that a homomorphism $f : K^* \longrightarrow \{ 1, -1 \} \times \hat{Z}_p$ is called a chain signature of type p if ker f is a valuation fan (for details see [S]).

Given a chain signature $f : K^* \longrightarrow \{ 1, -1 \} \times \hat{Z}_p$ and $n \in \mathbb{N}_0$ we set :

$$\alpha_n(f) = f^{-1}(1 \times p^n \hat{Z}_p) \cup \{0\} , \quad \alpha(f) = (\alpha_n(f))_{n\in\mathbb{N}_0} .$$

Let us call a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}_0}$ of preorderings $\alpha_n \subseteq K$ an order chain of type p if there exists a chain signature $f : K^* \longrightarrow \{ 1, -1 \} \times \hat{Z}_p$ with $\alpha(f) = \alpha$. Let $X_p(K)$ be the set of order chains of type p of K .

A subbasis for a toppology of $X_p(K)$ is given by the sets :

$$D_n(a) = \{ \alpha \mid a \in \alpha_n \setminus \{0\} \} , \; a \in K , \; n \in \mathbb{N}_0 .$$

With this topology $X_p(K)$ becomes a quasi compact space (for details see [B-J]). Now let $R \supseteq K$ be a field extension and let $\alpha = (\alpha_n) \in X_p(R)$. Then $\alpha \cap K = (\alpha_n \cap K)_{n\in\mathbb{N}}$ is an order chain of type p of K . Hence we have a canonical map :

$res^R_K : X_p(R) \longrightarrow X_p(K)$ defined by $\alpha \longmapsto \alpha \cap K$

and as in the case of spaces of total orders, res^R_K is continuous .

Proposition IV-4. *Let R and R' be two residually real-closed henselian extensions of K ; then R and R' are K-isomorphic if and only if for any p $res^R_K(X_p(R)) = res^{R'}_K(X_p(R'))$.*

Proof.

$\Rightarrow$ is clear

$\Leftarrow$ Since $R^{2n} = \bigcap_{\alpha \in X_p(R)} \alpha_n$ we get $R^{2n} \cap K = \bigcap_{\alpha \in X_p(R)} \alpha_n \cap K = \bigcap_{\beta \in res^R_K(X_p(R))} \beta_n$ and since the same holds for R' theorem IV-1 gives the result.

The residually real-closed henselian extensions can also be understood as real closures for generalized signatures in the sense of [S2] .

By definition, a homomorphism $f : K^* \longrightarrow C$ where C is some abelian group, is called a generalized signature if $\ker f$ is a valuation fan.

Furthermore, K is called real-closed with respect to f if for any proper algebraic extension $L \supsetneq K$, f cannot be extended to a generalized signature $g : L^* \longrightarrow C$.

Theorem IV-5. *Let R be a residually real-closed henselian extension of K such that $S = S(R)$. Then there exists a signature $f : R \longrightarrow \mathbb{Z}^* \times \prod_{p \in S} \mathbb{Z}_p^I$ for some I such that (R, f) is real-closed.*

Proof. Let Γ be any torsion free abelian group , and $I = \dim_{\mathbb{F}_p} \Gamma/p\Gamma$, then there exist isomorphisms $\varphi_n : \Gamma/p^n\Gamma \longrightarrow (\mathbb{Z}/p^n\mathbb{Z})^I$ satisfying $\pi_{n,m} \circ \varphi_m = \varphi_n$ where $\pi_{n,m} : (\mathbb{Z}/p^m\mathbb{Z})^I \longrightarrow (\mathbb{Z}/p^n\mathbb{Z})^I$ is the obvious map.

To prove this let $\{\alpha_i + p\Gamma\}_{i\in I}$ be a $\mathbb{F}_p$-basis of $\Gamma/p\Gamma$ then $\{\alpha_i + p^n\Gamma\}_{i\in I}$ is a $\mathbb{Z}/p^n\mathbb{Z}$-basis of the free module $\Gamma/p^n\Gamma$:

first they form a set of generators : if $\alpha \in \Gamma$, let $\alpha = \sum n_i\alpha_i + p\beta$ and let now $\beta = \sum r_i\alpha_i + p\gamma$ then $\alpha = \sum (n_i + pr_i)\alpha_i + p^2\gamma$ and so on.

second they are free : let $\sum n_i\alpha_i \in p^n \Gamma$, this is contained in $p\Gamma$ hence for all i $p|n_i$ and $n_i = pm_i$; thus we get $p \sum m_i\alpha_i \in p^n\Gamma$ and deduce that $\sum m_i\alpha_i \in p^{n-1}\Gamma$, iterating we get $\sum t_i\alpha_i \in p\,\Gamma$, where the t_i divide the n_i , which is impossible since the $\{\alpha_i + p\Gamma\}_{i\in I}$ are a $\mathbb{F}_p$-basis of $\Gamma/p\Gamma$

So $\varphi_n\ (\sum r_i(\alpha_i + p^n\Gamma) = (r_i)_{i\in I} \in (\mathbb{Z}/p^n\mathbb{Z})^I$.

The sequence (φ_n) gives rise to a homomorphism φ from Γ to $\prod_I \mathbb{Z}_p$:

$\Gamma \longrightarrow \varprojlim \Gamma/p^n\Gamma \longrightarrow \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^{(I)} \subseteq \prod_I \mathbb{Z}_p$ with the two following properties :

(i) ker φ is the largest p-divisible subgroup of Γ and is equal to $\cap\, p^n\Gamma$.

(ii) imφ is a p-pure subgroup of $\prod_I \mathbb{Z}_p$ since in all factors imφ is not contained in $p\mathbb{Z}_p$.

Applying all this in the case $\Gamma = \Gamma_R$ yields a signature

$f : R \longrightarrow \mathbb{Z} \times \prod_{p\in S} \mathbb{Z}_p^{\ I}$.

Since all extensions of R inside the algebraic closure are known to be tamely and completely ramified one gets as in [S2] that (R,f) is real-closed hence (R,f) is the unique real closure of $(K,f|_k)$.

Corollary IV-6. *For a field R the following statements are equivalent :*

(i) R is a residually real-closed henselian field ;

(ii) R is real-closed with respect to a signature $f : R \longrightarrow \mathbb{Z} \times \hat{\mathbb{Z}}^I$;

(iii) R is real-closed with respect to a signature $f : R \longrightarrow G$, *where G is any abelian group.*

Proof. *(i)* ⇒ *(ii)* is given by theorem IV-5 .

(ii) ⇒ *(iii)* is clear.

(iii) ⇒ *(i)* is given by theorem 8 of N. Schwartz in [S2].

Note.

The model theory of "residually real-closed henselian fields" have been studied independently by F. Delon and R. Farre in [D-F]. There residually real-closed henselian fields are called "almost real-closed fields" ; these are also the "real henselian fields" of [Brn].

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Eberhard BECKER
Ralph BERR
Fachbereich Mathematik
Universität Dortmund
D-44221-Dortmund

Danielle GONDARD-COZETTE
Mathématiques, U.F.R. 920
Université Paris 6
4, place Jussieu
F-75252 Paris cedex 05

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