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Séminaire de Structures Algébriques Ordonnées

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Delon – Dickmann – Gondard
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Courbes nodales gauches à points singuliers réels.

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Integer–valued rational functions on valued fields.

26.11.91 – **D. Gondard** (Paris 6).
Places réelles et corps chaînables.

17.12.91 – **H. Lombardi** (Besançon).
Théorème des zéros réels effectif.

7.01.92 – **M.A. Dickmann** (Paris 7).
Théorie des formes quadratiques sur un corps.

14.01.92 – **M.A. Dickmann** (Paris 7).
Théories abstraites des formes quadratiques.

4.02.92 – **M. Dickmann** (Paris 7).
Théories abstraites des formes quadratiques (suite et fin).

7.02.92 – **O. Belegradek** (Kemerovo State University, Russie)
The logical strength of theories associated with infinite–dimensional vector spaces.

11.02.92 – **A. Lira** (Paris 7)
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18.02.92 – **A. Lira** (Paris 7)
Dualité entre espaces d'ordres et groupes spéciaux.

25.02.92 – **F. Miraglia** (Oxford, Grande–Bretagne et Sào–Paulo, Brésil).
Le théorème de Lowenheim–Skolem pour les faisceaux.

17.03.92 – **M. Dickmann** (Paris 7).
Théorie des modèles des groupes spéciaux.

24.03.92 – **D. Gluschankof et F. Lucas** (Angers)
Théorie des modèles des lex–groupes.
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31.03.92 – **N. Mnev** (Saint–Pétersbourg, Russie).
On the local structure of semialgebraic varieties of some classical series (non–empty polyhedral cones, closures of cones of elliptic polynomials).

07.04.92 – **D. Gluschankof et F. Lucas** (Angers).
Théorie des modèles des lex–groupes,
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21.04.92 – **C. Andradas** (Madrid, Espagne).
Characterizing basicness by subvarieties of small dimension.

12.05.92 – **F. Cucker** (Univ. Polytechnique de Catalogne – Barcelone, Espagne).
La hiérarchie arithmétique sur les réels.

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23.06.92 – **V. Powers** (Regensburg (Allemagne), Emory (Georgia)).
Valuations and orderings in commutative rings.

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Liste des contributions.

D. Pecker (Paris VI).
Un théorème de Harnack dans l'espace.

A. Prestel (Konstanz, Allemagne).
Integer–valued rational functions on valued fields (exposé correspondant au texte publié aux Manuscripta Mathematica 73 (1991), 437–452.)

E. Becker (Dortmund, Allemagne) et **D. Gondard** (Paris VI).
Sur l'espace des ℝ–places d'un corps chaînable.

H. Lombardi (Besançon).
Théorème des zéros réel effectif.

M. Dickmann (Paris VII).
Anneaux de Witt abstraits et groupes spéciaux.

O. Belegradek (Kemerovo, Russie).
Theories associated with a vector space.

A. Lira (Paris VII).
Espaces d'ordres abstraits.

A. Lira (Paris VII).
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F. Miraglia (Oxford, Grande–Bretagne et Sào Paulo, Brésil).
The downward Löwenheim–Skolem theorem.

D. Gluschankof (Angers).
Back and forth for systems of antichains.

F. Lucas (Angers).
Elementary equivalence of Hahn–powers of divisible totally ordered groups on a root system.

C. Andradas et J. Ruiz (Madrid, Espagne).
Low dimensional sections of basic semialgebraic sets.

F. Cucker (Barcelone, Espagne).
La hiérarchie arithmétique sur les réels (exposé correspondant au texte publié au Journal of Logic and Computation 2 (1992), 375–395 sous le titre "The arithmetical hierarchy over the reals").

J. Väänänen (Helsinki, Finlande).
Measuring similarities of models.

A. Gonzàlez–Corbalàn et F. Santos (Santander, Espagne).
Representation of curves in the real plane and construction of curves with given topology.

C. Delzell (Baton Rouge, Louisiane, Etats–Unis).
Continuous sums of squares of rational functions.

J. Rachûnek (Le Mans et Olomouc, République Tchèque).
Groupes faiblement réticulés.

C.U. Jensen (Copenhague, Danemark).
Sur le groupe de Galois de l'extension abélienne maximale d'un corps.

V. Powers (Regensburg, Allemagne et Atlanta, Georgie, Etats–Unis).
Orderings and valuations in commutative rings.

Un théorème de Harnack dans l'espace.

Daniel Pecker. (Paris 6)

A la mémoire de Mario Raimondo .

Le théorème de Harnack affirme qu'une courbe algébrique réelle de genre g a au plus $g+1$ composantes connexes, et que pour tout degré d ,on peut construire une courbe plane lisse ayant $g(d)+1$ composantes connexes. (où $g(d) = (d-1)(d-2)/2$ est le genre d'une courbe plane de degré d).

Dans l'espace de dimension trois, une courbe gauche de degré d a un genre $g \le \left[(d-2)^2/4\right]$ (borne de Halphen) et Hilbert a montré l'existence de courbes gauches ayant $\left[(d-2)^2/4\right] + 1$ composantes connexes.

Castelnuovo a etendu le résultat d'Halphen à l'espace P_n et trouvé une borne effective $C(d,n)$ pour le genre d'une courbe de degré d non dégénérée dans P_n (i.e. qui n'est contenue dans aucun hyperplan).

On montre ici que pour $c \le C(d,n)$ il existe une courbe lisse de degré d , non dégénérée dans $P_n(\mathbb{R})$, ayant $c + 1$ composantes connexes. Notre démonstration consiste à simplifier les points doubles de courbes qui ont $C(d,n)$ points doubles réels isolés, et qui sont situés sur des surfaces rationnelles réglées lisses. Pour illustrer cette méthode, on traite d'abord le cas plan, car même dans ce cas, notre construction diffère des constructions classiques de Harnack et Hilbert qui utilisent une récurrence sur le degré. (cf [B.C.R.] page 246, [A.] ou [G]).

Commençons par une remarque:

Lemme: *Si* A, E, B *sont des polynômes de degrés* a, e, b *la courbe* *<u>affine</u>*

$\mathcal{C}(t) = (B(t), A(t)/E(t))$ *a au plus* $(b-1)(a-1)/2$ *points singuliers.*

Démonstration: Les valeurs du paramètre pouvant donner des points doubles ou singuliers sont obtenues en trouvant l'intersection des deux courbes :

$$\begin{cases} \big(B(t)-B(s)\big)/(t-s) = 0 \\ \big(A(t)E(s)-A(s)E(t)\big)/(t-s) = 0 \end{cases}$$

Par le théorème de Bézout il y a au plus $(b-1)(a+e-1)$ solutions .Par symétrie,la multiplicité d'intersection d'un point situé sur la diagonale est au moins 2 , et par conséquent $\mathcal{C}(t)$ a au plus $(b-1)(a-e-1)/2$ points singuliers. □

Proposition 1: *Soient* a *et* b *deux entiers premiers entre eux.*

La courbe affine $\mathcal{C}(t) = (t^a, (t-1)^b)$ *possède* $(a-1)(b-1)/2$ *points singuliers qui sont réels et isolés.*

Démonstration: Considérons l'ensemble $A = \{z \in \mathbb{C} \mid (z^a-\bar{z}^a)/(z-\bar{z}) = 0\}$

Si $z \in A$, $z^a = \bar{z}^a$ d'où $(z/|z|)^{2a} = 1$, $z/|z| \neq \mp 1$. L'ensemble A se compose donc des $(a-1)$ droites de vecteurs directeurs ξ , $\xi^{2a} = 1$, $\xi \neq \mp 1$.

De même, l'ensemble $B = \{z \in \mathbb{C} \mid \big((z-1)^b - (\overline{z-1})^b\big)/(z-\bar{z}) = 0\}$ se compose de $(b-1)$ droites de vecteurs directeurs η , $\eta^{2b} = 1$, $\eta \neq \pm 1$.

Comme a et b sont premiers entre eux, aucune des droites de B n'est parallèle à une droite de A , et par conséquent l'intersection de A et de B se compose de $(a-1)(b-1)$ points distincts. En ces points on a: $\mathcal{C}(z) = \mathcal{C}(\bar{z}) = \overline{\mathcal{C}(z)}$. Ce qui fait que $\mathcal{C}$ possède $(a-1)(b-1)/2$ points doubles isolés, et par le lemme, $\mathcal{C}$ ne peut avoir d'autre point singulier dans le plan affine. □

Corollaire (Harnack) : *Pour tout* $c \leq (d-1)(d-2)/2$ *il existe une courbe lisse de degré* d *dans* $P_2(\mathbb{R})$ *ayant* $(c+1)$ *composantes connexes.*

Démonstration: La courbe plane $\mathcal{C}(t) = \left(t^d, (t-1)^{d-1}\right)$ possède $(d-1)(d-2)/2$ points doubles réels isolés et une branche infinie. On peut simplifier ces points doubles ,chaque point isolé pouvant au choix soit devenir un ovale soit disparaître (cf.[B.R.],[P_2]). On obtient ainsi une courbe ayant $c+1$ composantes connexes. □

Pour le cas général on modifiera la construction de la manière suivante:

Proposition 2: *Si* $a > e$ *et* b *sont des entiers,* $(a-e,b)=1$ *. Il existe des polynômes* $A(t)$, $E(t)$ *et* $B(t)$ *de degrés respectifs* a,e,b *tels que la courbe* <u>*affine*</u> $\mathcal{C}(t)=\left(B(t), A(t)/E(t)\right)$ *ait* $(b-1)(a+e-1)/2$ *points doubles réels isolés, et aucun autre point singulier.*

Demonstration: Si f est une fraction rationnelle, définissons un polynôme

$\Delta_f(x,y) = \Delta_f(z) = f(z)-f(\bar{z})/(z-\bar{z})$ (où $z = x+iy$) .

Si f est définie au point $a \in \mathbb{R}$, regardons Δ_g avec $g(z) = f(z) + \eta/(z-a)$, où η est un "petit" nombre réel.

On a: $\Delta_g(z) = \Delta_f(z) - \eta/|z-a|^2 = (|z-a|^2\Delta_f(z) - \eta)/|z-a|^2$.

Si η est assez petit, on voit que $\{\Delta_g=0\}$ est une "petite variation" de $\{\Delta f = 0\} \cup \{z = a\}$, c'est donc la réunion d'une "petite variation" de $\{\Delta_f = 0\}$ et d'un petit ovale autour de a (si η est du signe adéquat).

En itérant cette construction, on voit que par un choix successif des a_i et des η_i, pour $f = z^{a-e} + \left(\eta_1/(z-a_1)\right) + \dots + \left(\eta_e/(z-a_e)\right)$, $\{\Delta_f = 0\}$ se compose de e ovales emboîtés autour de a_e et d'une figure qui est une "petite variation" des $(a-e-1)$ droites $\{(z^{a-e-1} - \bar{z}^{a-e-1})/(z-\bar{z}) = 0\}$ (cf. figure avec $a-e=5$, $e=2$, $b=4$).

Si la variation est assez petite, cet ensemble va rencontrer

$B = \{z \in \mathbb{C} \mid \left((z-a_e)^b - (\overline{z-a_e})^b\right)/(z-\bar{z}) = 0\}$ en $(b-1)(a-e-1) + 2e(b-1) = (b-1)(a+e-1)$ points distincts qui sont des valeurs du paramètre donnant $(b-1)(a+e-1)/2$ points doubles pour la courbe $\mathcal{C}(z) = \left((z-a_e)^b, f(z)\right)$. Par le lemme, $\mathcal{C}$ ne peut avoir d'autre point singulier dans le plan affine. $\square$

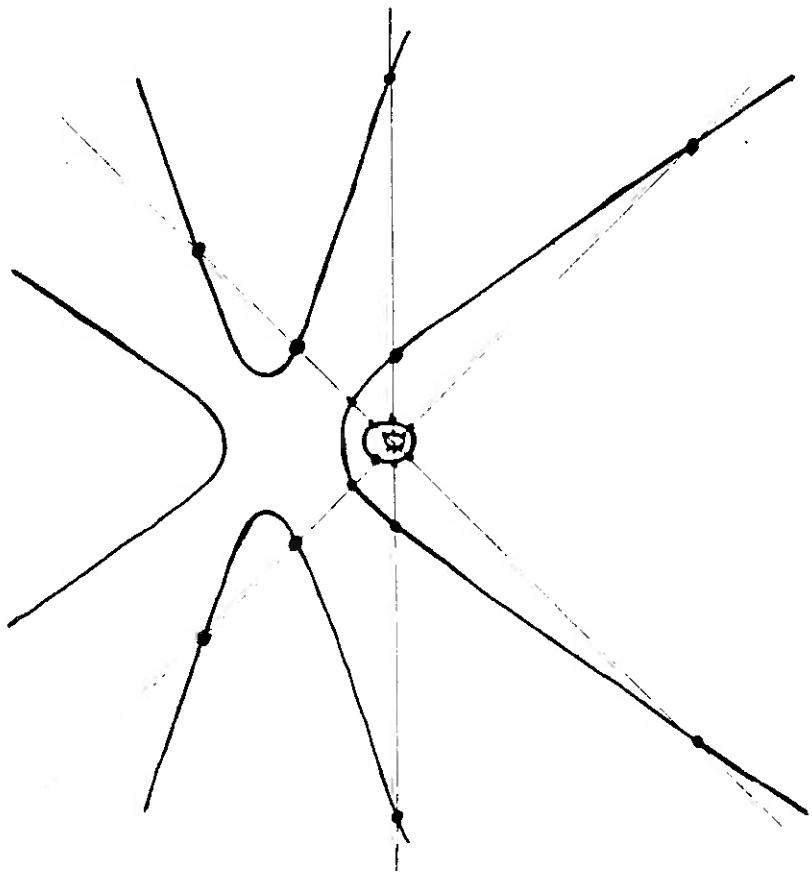

Figure:

L'intersection de $(f(z)-f(\bar{z}))/(z-\bar{z}) = 0$ et $(g(z)-g(\bar{z}))/(z-\bar{z}) = 0$ où $g(z)=(z-1)^4$, $f(z)=z^5+(\varepsilon/(z-1))-(\varepsilon'/(z-a_1))$ $a_1 \gtrsim 1$, $\varepsilon \approx 0.1$, $\varepsilon' \approx 0{,}00025$ est composée de 24 points. Cela permet la construction d'une courbe lisse de degré 11 non dégénérée dans P_4 ayant 13 composantes connexes.

Rappelons la définition de la borne de Castelnuovo.

Si $d \geq n \geq 2$, $d-1=m(n-1)+\varepsilon$ avec $0 \leq \varepsilon < n-1$ la borne de Castelnuovo $C(d,n)$ est donnée par: $C(d,n)=m((n-1)(m-1)+2\varepsilon)/2$.

Théorème 1: *Il existe une courbe irréductible de degré d , non dégénérée dans Pn ($d \geq n \geq 2$) , possèdant exactement $C(d,n)$ points doubles isolés.*

Démonstration: On a $C(d,2)=(d-1)(d-2)/2$ et par conséquent le résultat est déjà démontré si $n=2$. Supposons donc $n \geq 3$. On a $C(d,n) \geq 0$ et $C(d,n)=0$ si et seulement si $d=n$. Posons $d-1=(m+1)(n-k)+e$ avec $0 \leq e < m+1$ et $\lambda=n-2k+1$.

On a: $C(d,n)=m((m+1)\lambda+2e)/2 \geq 0$ et par conséquent $\lambda \geq -1$. Si $\lambda=-1$ alors $k=(n+2)/2$. n est pair et $n \geq 4$. Comme $(m+1)(n-k+1)>d-1 \geq m(n-1)$,on a: $m<n/n-2 \leq 2$ c'est à dire $m=1$. On en déduit que $C(d,n)=0$ et dans ce cas le théorème est évident, on peut donc supposer que $\lambda \geq 0$.

Soit $b=m+1$, $a=\lambda b+1+e=d-(k-1)(m+1)$.

Par la proposition 2 on peut trouver une courbe affine $\mathcal{C}(t)=(B(t),A(t)/E(t))$ où $B(t)$, $A(t)$ et $E(t)$ sont des polynômes de degrés b,a,e, et qui a $(b-1)(a+e-1)/2 = C(d,n)$ points singuliers qui sont réels et isolés.

La courbe $\mathcal{L}(t)=\left(B, B^2, \ldots, B^{n-k}, A/E, AB/E, \ldots, AB^{k-1}/E\right)$ est une courbe de degré d non dégénérée dans $\mathbb{C}^n$, ayant exactement $C(d,n)$ points doubles qui sont réels et isolés. Enfin comme $C(d,n)$ est une borne pour le nombre de points singuliers d'une courbe projective dans P_n , on voit que la complétée projective de $\mathcal{L}$, admet aussi $C(d,n)$ points doubles réels isolés dans $P_n(\mathbb{R})$. Cela achève la démonstration du théorème 1. □

Comme la courbe $\mathcal{L}$ est située sur une surface réglée rationnelle lisse, on peut appliquer le théorème de Tannenbaum, et simplifier ses points doubles de manière indépendante (cf. [P_2] , [T_1]).

On obtient ainsi le "théorème de Harnack dans l'espace".

Théorème 2: *Soit c un entier $c \leq C(d,n)$, il existe une courbe lisse de degré d , irréductible, et non dégénérée dans Pn qui possède exactement $c+1$ composantes connexes.*

On peut aussi formuler un résultat qui englobe des théorèmes de Tannenbaum (cf. [P_2] , [T_1] ,[T_2]).

Théorème 3: *Soit k et c deux entiers tels que $k+c \leq C(d,n)$. Il existe une courbe irréductible et non dégénérée dans Pn ayant exactement k points singuliers isolés, $c+1$ composantes connexes homéomorphes à des cercles, et de genre géométrique $C(d,n)-k$.*

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SUR L'ESPACE DES $\mathbb{R}$-PLACES D'UN CORPS CHAINABLE

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A la mémoire de Mario Raimondo

ABSTRACT.

First we remark that two orders of a real field K have the same image in the space of $\mathbb{R}$-places if and only if there exists a 2-primary chain of orderings of higher level beginning with these two orders. Using that fact we we give a criterion for the separation of connected components in M(K) and relate the number of connected components with $|\ (K^2 \cap \sum K^4)^* \ /\ ((\sum K^2)^2)^*\ |$.

I-INTRODUCTION ET NOTATIONS.

Soit K un corps ordonnable, on désigne par $\chi(K)$ l'espace des ordres sur K . Si P est le cône d'un ordre $\leq_P$ sur K alors $A(P) = \{\ a \in K \mid \exists\ r \in \mathbb{Q}\quad -r \leq_P a \leq_P r\ \}$ est un anneau de valuation dont on note v la valuation, k_v le corps résiduel et $\bar{P}$ l'ordre (archimédien) induit sur k_v . $\sum K^{2^n}$ représente l'ensemble de toutes les sommes finies de puissances 2^n-ièmes d'éléments de K .

On note M(K) l'espace des $\mathbb{R}$-places ξ de K (muni de la topologie définie comme étant la plus grossière rendant continues les applications e_a de M(K) dans $\mathbb{R} \cup \{\infty\}$ (le compactifié de $\mathbb{R}$) qui pour chaque $a \in K$ sont définies par $\xi \longmapsto \xi(a)$).

Rappelons qu'une telle $\mathbb{R}$-place $\xi : K \longrightarrow \mathbb{R} \cup \{\infty\}$ est déterminée par une paire $(v, \bar{P})$, on note $\xi = \lambda(P)$ et ξ est explicitement donnée par :

si $a \notin A(P)$ $\xi(a) = \infty$,

si $a \in A(P)$ $\xi(a) = \inf \{ r \in \mathbb{Q} \mid a \leq_p r \} = \sup \{ r' \in \mathbb{Q} \mid r' \leq_p a \} \in \mathbb{R}$.

Des travaux de Brown [Brn] et Dubois [Du] on déduit :

Proposition I-1 : [L] *Soit K un corps ordonnable :*

(a) $M(K)$ est un espace compact séparé ;

(b) L'application $\lambda : \chi(K) \longmapsto M(K)$ est surjective, continue et fermée.

Dans [Be1] Becker a défini la notion de préordre de niveau supérieur 2^n comme étant une partie P d'un corps K telle que :

$P + P \subseteq P$, $P . P \subseteq P$, $-1 \notin P$, $\sum K^{2^n} \subseteq P$. Un ordre de niveau 2^n est alors un préordre maximal de niveau 2^n .

Pour la notion d'ordre de niveau exact 2^n (i.e. $\sum K^{2^{n-1}}$ non contenu dans P) on ne peut définir de notion de clôture par extension algébrique qui permette d'obtenir une unicité à K-isomorphisme près ; mais la notion de chaîne d'ordre de niveau supérieur (de niveaux 2^n) introduite par Harman [H] permet d'obtenir une telle unicité :

(P_i) *est une chaîne d'ordres de niveau supérieur* si :

P_0 et P_1 sont des ordres (au sens usuel)

$\forall i \geq 2$ P_i est un ordre de niveau exact 2^i

$\forall i \geq 2$ $P_i \cup -P_i = (P_0 \cap P_{i-1}) \cup -(P_0 \cap P_{i-1})$.

Les rôles des deux premiers ordres étant symétriques nous dirons que P_0 et P_1 forment un *couple d'ordres chaînables* (voir [Di]).

Les caractérisations suivantes des corps qui admettent au moins une chaîne d'ordres de niveau supérieur (dans le sens ci-dessus c'est à dire chaîne 2-primaire) ont été obtenues dans [Be1] ((i) et (iii)) et [G2] ((ii)) :

Proposition I-2 : *Un corps ordonnable K est chaînable si et seulement si il satisfait l'une des propositions équivalentes suivantes :*

(i) $\sum K^2 \neq \sum K^4 \neq \sum K^8 \neq \dots \neq \sum K^{2^n} \neq \dots$

(ii) il existe $\alpha \in K$ tel que α^2 ne soit pas somme de puissances quatrièmes .

(iii) il existe sur K une valuation réelle de groupe des valeurs non 2-divisible.

Le but de cet article est d'obtenir des résultats qui mettent en valeur l'importance des relations entre la théorie des corps chaînables et l'espace des $\mathbb{R}$-places de ces corps. Leur origine vient du théorème III-1 et du corollaire III-2 qui entraînent que la relation sur $\chi(K)$ définie par " $P\mathcal{R}Q \Leftrightarrow P = Q$ ou P et Q forment un couple d'ordres chaînables (i.e. il existe une chaîne d'ordres de niveau supérieur commençant par ces deux ordres P et Q)" est une relation d'équivalence sur $\chi(K)$ dont l'espace quotient peut être identifié à $M(K)$. Les résultats principaux sont les théorèmes III-1, III-4, IV-1 et IV-4 .

II-RESULTATS PRELIMINAIRES.

Proposition II-1 : [H] *Soit* K *un corps ordonnable, deux ordres* P *et* Q *sont un couple d'ordres chaînables si et seulement si il existe une valuation* v *compatible avec les deux ordres telle que dans le corps résiduel* k_v *on ait* $\bar{P} = \bar{Q}$.

Proposition II-2 : ([L] 2.13) *Soient* P *et* Q *dans* $\chi(K)$ *, alors* $\lambda(P) = \lambda(Q)$ *si et seulement si* $A(P) = A(Q)$ *et* P *et* Q *induisent le même ordre dans le corps résiduel pour la valuation d'anneau* $A(P) = A(Q)$.

Proposition II-3 : ([L] 12.1) *Soit* P_i *,* $i \in I$ *, des éléments de* $\chi(K)$ *,* v *une valuation sur* K *d'anneau* A *telle que* $\forall i \in I \quad A \supseteq A(P_i)$.
Soit $\pi : K \longrightarrow k \cup \{\infty\}$ *la place associée à* v ($\pi(x) = \infty$ *si* $x \notin A$ *et* $\pi(x) = \bar{x}$ *si* $x \in A$) *, alors chaque* $\mathbb{R}$*-place* $\lambda(P_i)$ *se factorise uniquement via* π *et on a le diagramme commutatif suivant :*

$$\begin{array}{ccc} K & \xrightarrow{\lambda(P_i)} & \mathbb{R} \cup \{\infty\} \\ & {\scriptstyle\pi}\searrow \quad \nearrow {\scriptstyle\lambda(\bar{P}_i) \cup \{\infty\}} & \\ & k \cup \{\infty\} & \end{array}$$

En particulier si $\lambda(P_i) \neq \lambda(P_j)$ *dans* $M(K)$ *alors* $\lambda(\bar{P}_i) \neq \lambda(\bar{P}_j)$ *dans* $M(k)$

Définition II-4 : [Br] *Soit* K *un corps ordonnable, un fan* T *est un préordre satisfaisant l'une des propriétés équivalentes suivantes :*
(i) pour tout $S \supseteq T$ *vérifiant* $-1 \notin S$ *et* S^* *sous-groupe d'indice 2 dans* K^* *,* S *est un ordre sur* K .
(ii) $\forall a \notin -T \quad T + aT = T \cup aT$.

Les ordres et les intersections de deux ordres sont des fans dits triviaux.

Rappelons qu'on dit qu'une valuation v est compatible avec un préordre T si elle est est compatible avec un ordre $P \geq T$ c'est à dire si $1 + \mathfrak{m} \subseteq P$ où $\mathfrak{m}$ est l'idéal maximal de l'anneau de la valuation A_v ; v est pleinement compatible avec le préordre T si elle est compatible avec tous les ordres tels que $P \geq T$.

Proposition II-5 : [L] (p.43) *Soit K un corps ordonnable v une valuation sur K et T un préordre de K , alors :*

(a) si v est compatible avec T : T est un fan ⇒ $\bar{T}$ est un fan ;

(b) si v est pleinement compatible avec T : T est un fan ⇔ $\bar{T}$ est un fan .

Rappelons le théorème de trivialisation d'un fan de Bröcker [Br] : *Soit T un fan d'un corps ordonnable K alors il existe une valuation v pleinement compatible avec T telle que T induit sur k_v un fan trivial .*

Si K est un corps ordonnable on appelle ensemble d'Harrison les parties de $\chi(K)$ définies comme suit :

$H(a) = \{ P \in \chi(K) \mid a \in P \}$; ces ensembles sont des fermés-ouverts de $\chi(K)$.

Proposition II-6 : [H] *Soit K un corps ordonnable et $\chi(K)$ son espace d'ordres, on désigne par H(a) un ensemble d'Harrison et par λ la surjection $\chi(K) \longrightarrow M(K)$;*

(i) $\lambda^{-1}(\lambda(H(a))) = H(a)$ si et seulement si $a^2 \in \sum K^4$.

(ii) si $\chi(K) = X_1 \cup X_2$ où X_i sont des fermés-ouverts tels que $\lambda^{-1}(\lambda(X_i)) = X_i$, alors il existe a tel que $X_1 = H(a)$.

III-SUR LA SURJECTION DE X(K) DANS M(K).

Nous allons donner une interprétation de certains faits connus.

Dans [L] est défini pages 21-22 un "produit extérieur" : soit v une valuation sur K d'anneau A et de corps résiduel k ; soit π de A dans k la projection ; soit T un préordre de K et S un préordre de k tel que $S \geq \bar{T}$; soit $T \wedge S = \pi^{-1}(S^*)$ alors $T \wedge S$ est un préordre de K

totalement compatible avec v et on a $\overline{T \wedge S} = S$.
Si on note ξ une place réelle d'anneau V_ξ et de groupe des unités E_ξ , $\bar{P}$ un ordre compatible soit $T_\xi = \{ \varepsilon \in E_\xi \mid \xi(\varepsilon) > 0 \} . K^2 = \sum K^2 \wedge \bar{P}$.
Soit λ de $\chi(K) \longmapsto M(K)$ définie par $P \longmapsto \lambda(P)$, alors pour $\xi \in M(K)$ $\lambda^{-1}(\xi) = \chi(T_\xi)$ est un sous-espace de $\chi(K)$ et l'intersection des ordres correspondants est un préordre qui est un fan ([L], p. 44) ; on a aussi $[K^* : T_\xi] = [\Gamma_\xi / 2\Gamma_\xi] . 2 = 2 \mid \chi(T_\xi) \mid$ ([L] P. 26) .
Rappelons enfin que tout fan T est presque local, c'est à dire que $\mid \{ \lambda(P) \mid P \geq T \} \mid \leq 2$ ([L] 10-12 p.82).

Les théorèmes III-4 et III-5 qui suivent sont conséquenses des résultats ci-dessus et de la définition des couples d'ordres chaînables.
Nous donnons néanmoins des preuves indépendantes utilisant la théorie des corps chaînables et des résultats élémentaires sur les fans.

Théorème III-1 : *Soient P et Q deux ordres de K , $\lambda(P) = \lambda(Q)$ si et seulement si P et Q forment un couple d'ordres chaînables.*

Démonstration :
$\Rightarrow$ la propositions II-2 donne l'existence d'une valuation v associée à l'anneau $A(P) = A(Q)$ telle que dans le corps résiduel k_v $\bar{P} = \bar{Q}$; la proposition II-1 montre qu'alors il existe une chaîne d'ordres de niveau supérieur commençant par le couple d'ordres (P,Q).
$\Leftarrow$ si P et Q forment un couple d'ordres chaînables alors par la proposition I-1 il existe une valuation v compatible avec P et Q telle que $\bar{P} = \bar{Q}$ dans k_v . On a alors $A_v \geq A(P)$ et $A_v \geq A(Q)$, $\lambda(\bar{P}) = \lambda(\bar{Q})$ dans $M(k_v)$ et en appliquant la proposition II-3, on déduit que $\lambda(P) = \lambda(Q)$ dans $M(K)$.

Corollaire III-2 : *Si (P,Q) et (Q,R) sont deux couples d'ordres chaînables alors (P,R) est un couple d'ordres chaînables.*

Démonstration :
le théorème III-1 montre l'existence d'une valuation v d'anneau $A(P) = A(Q) = A(R)$ telle que $\bar{P} = \bar{Q} = \bar{R}$ dans k_v donc P et R sont un couple d'ordres chaînables (par la proposition II-1).

Corollaire III-3 : *Soit K un corps ordonnable, les propriétés suivantes sont équivalentes :*

(i) l'application λ de $\chi(K)$ dans $M(K)$ est bijective ;

(ii) K est non chaînable ;

(iii) $\forall \alpha \in K$ α^2 est une somme de puissances quatrièmes d'éléments de K .

(iv) pour toute valuation réelle v le groupe des valeurs satisfait $\Gamma = 2\Gamma$.

Démonstration : les équivalence (ii) $\Leftrightarrow$ (iii) et (ii) $\Leftrightarrow$ (iv) résultent de la proposition I-2 ; par I-1 on sait que λ est surjective ; le théorème III-1 donne alors (i) $\Leftrightarrow$ (ii) .

Théorème III-4 : *Soient P_i les ordres de K tels que $\forall i\ \forall j\ i \neq j$ (P_i, P_j) soit un couple d'ordres chaînables, alors le préordre $T = \cap P_i$ est un fan ; si $[K : T] = 2^n$ alors il y a donc 2^{n-1} ordres P_i co-chainables deux à deux.*

Démonstration :

Le théorème III-1 donne v la valuation associée à $A(P_i)$ qui est compatible avec tous les P_i donc est pleinement compatible avec T (si $Q \geq T = \cap P_i$ alors v est compatible avec les P_i donne que $1 + m \subseteq \cap P_i \subseteq Q$ et la valuation v est aussi compatible avec Q) ; les ordres P_i étant deux à deux chaînables, le même théorème donne que $\forall i\ \forall j$ $\bar{P}_i = \bar{P}_j$ dans k_v est un ordre de k_v , $\bar{T}$ est donc égal à cet ordre et est donc un fan trivial de k_v , le théorème II-5 montre alors que T est un fan de K .

N.B. $\cap P_i = T_\xi$ de l'introduction de cette partie d'où le résultat.

Puisque tout préordre contenant un fan est aussi un fan on peut définir une notion de *fan minimal* : un fan T est minimal si pour tout préordre T' , $T' \subseteq T$ et $T' \neq T$ entraînent que T' n'est pas un fan.

Un fan T est un *fan de valuation* si et seulement si il existe une valuation v compatible avec le fan T et telle que T induit sur le corps résiduel un ordre .

Théorème III-5 : *Un fan minimal T est un fan de valuation ou est égal à l'intersection de deux fans de valuation . Si T n'est pas un fan de valuation alors T est égal à l'intersection de deux fans de valuation de même cardinalité.*

Démonstration :

Par le théorème de trivialisation d'un fan dû à Bröcker, il existe une valuation v pleinement compatible avec T telle que dans k_v T induit un fan trivial $\bar{T}$. Si $\bar{T}$ est un ordre de k_v alors tous les ordres contenant T sont co-chaînables deux à deux et T est un fan d valuation. Si $\bar{T}$ est un fan trivial intersection de deux ordres P et Q de k_v alors les deux seuls ordres contenant $\bar{T}$ sont P et Q. Les ordres contenant T induisent sur k_v un ordre P ou Q, puisque v est pleinement compatible avec T, et correspondent donc à deux paquets d'ordres $\{P_i\}$ et $\{Q_j\}$ co-chainables deux à deux de K. T est alors l'intersection des deux fans de valuation $T_p = \cap P_i$ et $T_q = \cap Q_j$. Enfin dans le cas où le nombre d'ordres au dessus du fan est fini celui-ci est une puissance de 2 et l'égalité $2^{n_p} + 2^{n_q} = 2^n$ entraîne $n_p = n_q = n - 1$.

Puisque tout fan est presque local on savait déjà que tout fan T contient $T_\xi \cap T_\zeta$ deux fans correspondant à deux places réelles ξ et ζ ; $V = V_\xi V_\zeta$ est un anneau de valuation compatible avec T, et T induit un fan trivial dans le corps résiduel ; si T n'est pas de valuation ce fan est une intersection de deux ordres qui se relèvent en deux fans $T_1 \supseteq T_\xi$ et $T_2 \supseteq T_\zeta$ de même cardinalité car $\chi(T_1) = \{ P \mid P \sim V, \bar{P} = \bar{P}_0 \}$ est tel que $|\chi(T_1)| = |(\Gamma_v / 2\Gamma_v)^\wedge|$ et il en est de même pour $\chi(T_2)$.

IV-COMPOSANTES CONNEXES DE M(K).

Le carré d'une somme de carrés étant une somme de puissances quatrièmes d'après [Be1], il est clair que si α^2 n'est pas une somme de puissances quatrièmes alors α n'appartient pas à $\pm \sum K^2$. Par contre la réciproque n'est pas vraie dans tout corps chaînable.

Harman a étudié le problème de la réciproque de la propriété " $a \in \pm \sum K^2 \Rightarrow a^2 \in \sum K^4$ " et a montré que cela était équivalent à la connexité de l'espace $M(K)$ des $\mathbb{R}$-places de K ; plus précisément on a :

Proposition IV-1-a : *Soit* K *un corps ordonnable,*

a) sont équivalentes les propriétés :

(i) $\forall a \in K \quad a^2 \in \sum K^4 \Rightarrow a \in \pm \sum K^2$;

(ii) M(K) est connexe .

b) sont équivalentes les propriétés :

(j) K est pythagoricien et satisfait a) ;

(jj) K est pythagoricien au niveau 4 ;

(jjj) K est pythagoricien au niveau 2^n *pour tout* $n \geq 2$.

Du travail de Harman on peut déduire des exemples de corps tels que M(K) est connexe : $\mathbb{Q}(X)$, $\mathbb{Q}(X,Y)$, $\mathbb{Q}((t))$, $\mathbb{Q}((t_1))((t_2))$, $\mathbb{R}(X_1,\ldots,X_n)$, $\mathbb{R}((t))$ et bien sûr les corps chaîne-clos ou plus généralement d'après [G3] et [G4] les corps de Rolle .

Harman a en fait montré le résultat suivant :

Proposition IV-1-b : *Soit* K *un corps ordonnable, les propriétés suivantes sont équivalentes :*

(i) M(K) est connexe ;

(ii) M(K(X)) est connexe ;

(iii) M(K((X))) est connexe .

Les résultats de la partie III permettent d'obtenir des théorèmes sur les composantes connexes de M(K) :

Théorème IV-2 : *Soit* K *un corps ordonnable,* P *et* Q *deux ordres de* K , *alors* λ(P) *et* λ(Q) *sont dans la même composante connexe de* M(K) *si et seulement si il n'existe pas* β *séparant* P *et* Q *tel que* $\beta \notin \pm \sum K^2$ *et* $\beta^2 \in \sum K^4$.

Démonstration :

$\Rightarrow$ remarquons d'abord que si les ensembles $H(a) = \{ P \mid a \in P \}$ et $H(-a)$ forment une partition de $\chi(K)$, les ensembles $\lambda(H(a))$ et $\lambda(H(-a))$ ne sont pas forcément d'intersection vide ; cependant si β est tel que $\beta \notin \pm \sum K^2$ et $\beta^2 \in \sum K^4$ il ne peut exister $P \in H(\beta)$ et $Q \in H(-\beta)$ tels que $\lambda(P) = \lambda(Q)$: car sinon $\beta \notin (P \cap Q) \cup -(P \cap Q)$ donc β n'appartient pas à $P_2 \cup -P_2$ (où P_2 ordre de niveau 2 d'une chaîne de début le couple (P,Q)) et $\beta^2 \notin P_2$ d'où $\beta^2 \notin \sum K^4$ ce qui est impossible : d'après [Bel] en effet $\sum K^4 = \sum K^2 \cap P_{2i}$ où P_{2i} parcourt l'ensemble des ordres de niveau 4 .

Supposons alors $P \neq Q$ dans $\chi(K)$ tels que $\lambda(P)$ et $\lambda(Q)$ soient dans la même composante connexe C de $M(K)$; si il existait un β séparant P et Q tel que $\beta \notin \pm \sum K^2$ et $\beta^2 \in \sum K^4$, alors puisque les ensembles $H(a)$ sont toujours des fermés et que l'application λ est fermée , $C \cap \lambda(H(\beta))$ et $C \cap \lambda(H(-\beta))$ réaliseraient une partition de C en deux fermés non vides ce qui est impossible.

$\Leftarrow$ nous allons montrer que si $\lambda(P)$ et $\lambda(Q)$ sont dans deux composantes distinctes de $M(K)$, C et C' , alors il existe β séparant P et Q tel que $\beta \notin \pm \sum K^2$ et $\beta^2 \in \sum K^4$.

Si $M(K)$ a un nombre fini de composantes connexes on a :

les composantes connexes sont des ouverts-fermés qui réalisent une partition de $M(K)$; considérons les deux ensembles, non vides, complémentaires C et $D = \cup C_i$ où les C_i sont les composantes connexes de $M(K)$ à l'exception de C , C et D forment une partition en deux ouverts fermés de $M(K)$. Soient $X = \lambda^{-1}(C)$ et $Y = \lambda^{-1}(D)$, alors X et Y sont deux ouverts fermés non vides de $\chi(K)$ qui réalisent une partition de $\chi(K)$; puisque $\lambda^{-1}(\lambda(X)) = X$ et $\lambda^{-1}(\lambda(Y)) = Y$ la proposition II-6 permet d'obtenir l'existence de β tel que $X = H(\beta)$ et $Y = H(-\beta)$ avec $\beta \notin \pm \sum K^2$, $\beta^2 \in \sum K^4$ et bien sûr β sépare P et Q puisque $P \in X$ et $Q \in Y$.

Si $M(K)$ a une infinité de composantes connexes on peut donner la preuve générale suivante :

On sait que $M(K)$ est un espace compact séparé donc il existe un ouvert-fermé U séparant les deux composantes connexes C et C' , $C \subseteq U$ et $C' \subseteq U^c$. On reprend alors la preuve ci-dessus en remplaçant C et D par U et U^c .

Le théorème V-I montre l'importance des éléments β tels que $\beta^2 \in \sum K^4 \setminus (\sum K^2)^2$ dans la détermination des composantes connexes de $M(K)$.

Dans [Be2] (1-4) se trouve un théorème qui donne le nombre de composantes connexes de $M(K)$:

Proposition IV-3 : *Soit K un corps ordonnable, le nombre de composantes connexes de $M(K)$ est fini si et seulement si $\mathbb{E}^+$ est d'indice fini dans $\mathbb{E}$ où $\mathbb{E}$ désigne le groupe des unités de l'anneau d'holomorphie de K . S'il y a s composantes alors $[\mathbb{E} : \mathbb{E}^+] = 2^s$.*

On peut relier les deux résultats pour obtenir :

Théorème IV-4 : *Si le groupe* $|\,(K^2 \cap \sum K^4)^* \,/\, ((\sum K^2)^2)^*\,|$ *a* 2^n *éléments alors le nombre de composantes connexes de* M(K) *est* $n + 1$.

Démonstration :

Soit ϕ $\mathbb{E} \longrightarrow (K^2 \cap \sum K^4)^* \,/\, ((\sum K^2)^2)^*$ définie par $\varepsilon \longrightarrow \bar{\varepsilon}^2$;

ϕ est surjective : soit $x \in (K^2 \cap \sum K^4)^*$; $x \in (\sum K^4)^* \Rightarrow x = \varepsilon q^2$ avec $\varepsilon \in \mathbb{E}$ et $q \in \sum K^2$ (cf.[Be2] 1-9) ; comme $x \in \sum K^4$ on a $\varepsilon \in \sum K^2$ et donc $\varepsilon \in \mathbb{E}^+$; $x \in K^2$ entraîne aussi $x = y^2$ d'où $\varepsilon = (y/q)^2$; comme l'anneau d'holomorphie H est intégralement clos (cf. [Be3] p. 884), on obtient $\varepsilon \in \mathbb{E}^2$, x s'écrit donc $\varepsilon'^2(\sum q_i^2)^2$.

Le noyau de ϕ est l'ensemble des unités ε telles que $\varepsilon^2 \in (\sum K^2)^2$ donc $\ker \phi = \mathbb{E}^+ \cup \mathbb{E}^-$.

On a donc $\mathbb{E} \,/\, \mathbb{E}^+ \cup \mathbb{E}^- \cong (K^2 \cap \sum K^4)^* \,/\, ((\sum K^2)^2)^*$.

Si $|\,\mathbb{E} \,/\, \mathbb{E}^+ \cup \mathbb{E}^-\,| = 2^n$ alors $|\,\mathbb{E} \,/\, \mathbb{E}^+\,| = 2^{n+1}$ d'où le résultat en utilisant IV-3.

Le théorème IV-2 permet d'obtenir d'autres résultats :

Théorème IV-5 : *Soit* L *une extension de* K *telle que tous les ordres de* K *s'étendent à* L *alors le nombre de composantes connexes de* M(K) *est inférieur ou égal au nombre de composantes connexes de* M(L) .

Démonstration :

Soit $a \in K$ et séparant deux composantes connexes C_1 et C_2 , $a \notin \sum K^2$ et $a \in \sum K^4$, $a \in \cap T_{1i}$ (où $\lambda(T_{1i}) \in C_1$) et $-a \in \cap T_{2j}$ (où $\lambda(T_{2j}) \in C_2$) ; tous les ordres s'étendent donc $a \in \cap \bar{T}_{1i}$ (non vide) et $-a \in \cap \bar{T}_{2j}$ et donc puisque $\sum L^2 \cap K = \sum K^2$ (tout ordre s'étend) $a \notin \sum L^2$ et sépare dans L deux composantes connexes ; le nombre de composantes connexes ne peut donc que croître de K à L .

N.B. une autre preuve consiste à dire que la restriction res. de M(L) à M(K) est continue et que donc si $M(L) = \cup Z_i$, où les Z_i sont les composantes connexes, alors $M(K) = \cup \,\text{res.}(Z_i)$ et les $\text{res.}(Z_i)$ sont des connexes éventuellement non disjoints.

Schulting [S] a montré que M(K) et M(K(X)) ont le même nombre de composantes connexes. On peut aussi prouver :

Théorème IV-6 : *M(K) et M(K((X))) ont le même nombre de composantes connexes.*

Démonstration :
Soit $L = K((X))$, d'après IV-5 le nombre de composantes connexes ne peut que croître de K à $K((X))$. Supposons $x = \sum_{i=m}^{\infty} a_i t^i$ $(a_m \neq 0)$ tel que $x^2 \in \sum L^4$ et $x \notin \sum L^2$, on en déduit $2m \equiv 0 \ (4)$, $a_m \in \sum K^4$ et $a_m \notin \sum K^2$ (car $m \equiv 0 \ (2)$) donc a_m sépare aussi deux composantes connexes. Plusieurs tels x issus du même a_m étant dans les mêmes ordres de L , le nombre de composantes connexes de L est finalement égal à celui de K .
N.B. Une variante consiste à dire que si un corps K admet une valuation henselienne à corps des restes k alors $M(K) \cong M(k)$ et qu'on on a donc $M(K((X))) \cong M(K)$.

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THÉORÈME DES ZÉROS RÉEL EFFECTIF

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Résumé Nous donnons les idées et résultats essentiels d'un calcul d'une majoration des degrés pour le théorème des zéros réels effectif.

Abstract We give the main ideas and results concerning a computation of a degree majoration for the real nullstellensatz.

Introduction

Nous rendons compte ici d'une preuve constructive du Positivstellensatz réel et de ses variantes (voir les références [**Lom** x]). Nous reprenons ici les notations de [**Lom e**].

Une formulation générale du théorème des zéros réel et de ses variantes peut être la suivante (cf [**BCR**] théorème 4.4.2) : on considère un système d'égalités et inégalités portant sur des polynomes de $\mathbf{K}[X] = \mathbf{K}[X_1,X_2,\ldots,X_n]$, où **K** est un corps ordonné de clôture réelle **R** ; ce système définit une partie S semialgébrique de $\mathbf{R}^n$; le théorème affirme que S est vide (fait géométrique) si et seulement si il y a une certaine identité algébrique construite à partir des polynomes donnés, identité qui donne une preuve de ce fait géométrique.

L'idée générale de notre preuve constructive est la suivante. Pour un corps ordonné **K** il y a un algorithme de conception très simple pour tester si un système de csg (conditions de signes généralisées) portant sur ces polynomes en plusieurs variables est possible ou impossible dans la clôture réelle de **K** . C'est l'algorithme de Hörmander (cf. la preuve du principe de Tarski-Seidenberg dans [**BCR**] chap. 1), appliqué de manière itérative pour diminuer par étapes le nombre de variables sur lesquelles portent les csg. Si on regarde les arguments sur lesquels est basée la preuve d'impossibilité (en cas d'impossibilité), on voit qu'il y a essentiellement des identités algébriques (traduisant la division euclidienne), le théorème des accroissements finis et l'existence d'une racine pour un polynome sur un intervalle où il change de signe.
Les ...-stellensatz réels effectifs doivent donc pouvoir être obtenus si on arrive à "algébriser" les arguments de base de la preuve d'incompatibilité et les méthodes de déduction impliquées.
Un pas important a déjà été réalisé avec la version algébrique du théorème des accroissements finis pour les polynomes (cf. [**LR**]), qui a été à l'origine des formules de Taylor mixtes et généralisées.
Un autre pas a consisté à traduire sous forme de *constructions d'identités algébriques* certains raisonnements élémentaires (du genre si $A \Rightarrow B$ et $B \Rightarrow C$ alors $A \Rightarrow C$).
Il fallait en outre trouver une version "identité algébrique" des axiomes d'existence dans la théorie des corps réels clos. C'est ce qui est fait à travers la notion d'*existence potentielle.*

Calculer une borne sur les degrés pour le théorème des zéros réels consiste à calculer une majoration sur les degrés des polynomes intervenant dans le résultat final (l'identité algébrique construite) à partir de la taille de l'entrée (le système de conditions de signes portant sur la liste de polynomes donnée au départ). Les paramètres qui controlent la majoration des degrés dans le résultat sont en fait : le nombre k de polynomes dans l'entrée, le degré d des polynomes dans

l'entrée, et le nombre n de variables.
Le calcul de majoration est obtenu en suivant pas à pas la preuve constructive d'existence de l'identité algébrique et en explicitant les majorations à chaque étape de la preuve.
C'est une majoration primitive récursive, donnée par une tour d'exponentielles : le nombre d'étages dans la tour est $n+4$ et en haut de la tour on trouve :

$$d.\log(d) + \log\log(k) + \text{cte}.$$

Ce résultat n'est pas trop mauvais, dans la mesure où la principale responsabilité de l'explosion est supportée par l'algorithme de Hörmander, à la base de la preuve effective. On peut espérer baser une autre preuve effective sur des algorithmes plus performants et néanmoins de conception très simple, et obtenir en conséquence une majoration où le paramètre n interviendrait de manière moins catastrophique, sans tour d'exponentielles.

Bien que nous nous placions a priori dans un cadre constructif "à la Bishop", tel que développé dans **[MRR]** pour ce qui concerne la théorie des corps discrets, comme nous ne précisons pas le sens du mot effectif ni celui du mot décidable, toutes les preuves peuvent être lues avec des lunettes adaptées à la philosophie ou au cadre de travail de chaque lecteur particulier.
En fait les preuves données fournissent des algorithmes uniformément primitifs récursifs, "uniformément" s'entendant par rapport à un oracle qui donne la structure du corps des coefficients du système de csg considéré :

si $(c_i)_{i=1,\dots,m}$ est la famille des coefficients et si $P \in \mathbb{Z}[(C_i)_{i=1,\dots,m}]$ l'oracle répond à la question « Quel est le signe de $P((c_i)_{i=1,\dots,m})$? ».

Incompatibilités fortes

Nous considérons un corps ordonné $\mathbf{K}$, et une liste de variables $X_1, X_2, \dots, X_n$ désignée par $\mathbf{X}$.
Nous notons donc $\mathbf{K}[\mathbf{X}]$ l'anneau des polynomes $\mathbf{K}[X_1,X_2,\dots,X_n]$.
Etant donnée une partie finie F de $\mathbf{K}[\mathbf{X}]$:

nous notons F^{*2} l'ensemble des carrés d'éléments de F.

le *monoïde multiplicatif engendré* par F est l'ensemble des produits d'éléments de $F \cup \{1\}$, nous le noterons $\mathcal{M}(F)$.

le *cône positif engendré* par F est l'ensemble des sommes d'éléments du type $p.P.Q^2$ où p est positif dans $\mathbf{K}$, P est dans $\mathcal{M}(F)$, Q est dans $\mathbf{K}[\mathbf{X}]$. Nous le noterons $\mathcal{Cp}(F)$.

enfin nous noterons $\mathcal{I}(F)$ l'idéal engendré par F.

Définition et notation 1 : Etant donnés 4 parties finies de $\mathbf{K}[\mathbf{X}]$: $F_>$, $F_\geq$, $F_=$, $F_\neq$, contenant des polynomes auxquels on souhaite imposer respectivement les conditions de signes >0, ≥ 0, $=0$, $\neq 0$, on dira que $F = [F_> ; F_\geq ; F_= ; F_\neq]$ est *fortement incompatible* dans $\mathbf{K}$ si on a une égalité dans $\mathbf{K}[\mathbf{X}]$ du type suivant :

$$S + P + Z = 0 \quad \text{avec} \quad S \in \mathcal{M}(F_> \cup F_\neq^{*2}),\ P \in \mathcal{Cp}(F_\geq \cup F_>),\ Z \in \mathcal{I}(F_=)$$

Nous utiliserons la notation suivante pour une incompatibilité forte:

$$\downarrow [\, S_1 > 0, \dots, S_i > 0, P_1 \geq 0, \dots, P_j \geq 0, Z_1 = 0, \dots, Z_k = 0, N_1 \neq 0, \dots, N_h \neq 0 \,] \downarrow$$

Il est clair qu'une incompatibilité forte est une forme très forte d'incompatibilité. En particulier, elle implique l'impossibilité d'attribuer les signes indiqués aux polynomes souhaités, dans *n'importe quelle* extension ordonnée de $\mathbf{K}$.
Si on considère la clôture réelle $\mathbf{R}$ de $\mathbf{K}$, l'impossibilité ci-dessus est testable par l'algorithme de Hörmander, par exemple.

Le théorème des zéros réels et ses variantes

Les différentes variantes du théorème des zéros dans le cas réel sont conséquence du théorème général suivant :

Théorème : Soit **K** un corps ordonné et **R** une extension réelle close de **K**. Les trois faits suivants, concernant un système de csg portant sur des polynomes de **K[X]**, sont équivalents :

l'incompatibilité forte dans **K**

l'impossibilité dans **R**

l'impossibilité dans toutes les extensions ordonnées de **K**

Ce théorème des zéros réels remonte à 1974 ([**Ste**]). Des variantes plus faibles ont été établies par Krivine ([**Kri**]), Dubois ([**Du**]), Risler ([**Ris**]), Efroymson ([**Efr**]). Toutes les preuves jusqu'à ([**Lom a**]) utilisaient l'axiome du choix et d'autres méthodes non constructives.

Degré d'une incompatibilité forte

Si nous voulons préciser les majorations de degré fournis par notre preuve du théorème des zéros réel, nous devons préciser la terminologie.
Nous manipulons des incompatibilités fortes écrites *sous forme paire,* c.-à-d.:

$$S + P + Z = 0 \quad \text{avec} \quad S \in \mathcal{M}(F_>^{*2} \cup F_{\neq}^{*2}), \quad P \in \mathcal{C}p(F_\geq \cup F_>), \quad Z \in \mathcal{I}(F_=)$$

(la considération des formes paires d'implications fortes a pour unique utilité de faciliter un peu le calcul de majoration des degrés).
Quand nous parlons de degré, sauf précision contraire, il s'agit du degré total maximum.
Le degré d'une incompatibilité forte est par convention au moins égal à 1, c'est le degré maximum des polynomes qui «composent» l'incompatibilité forte.
Par exemple, si nous avons une incompatibilité forte :

$$\downarrow [A > 0, B > 0, C \geqslant 0, D \geqslant 0, E = 0, F = 0] \downarrow$$

explicitée sous forme d'une identité algébrique :

$$A^2.B^6 + C.\sum_{i=1}^{h} p_i.P_i^2 + A.B.D.\sum_{j=1}^{k} q_j.Q_j^2 + E.U + F.V = 0$$

le degré de l'incompatibilité forte est :

$$\sup \{ d(A^2.B^6), d(C.P_i^2) \ (i = 1,\ldots,h), d(A.B.D.Q_j^2) \ (j = 1,\ldots,k), d(E.U), d(F.V) \}.$$

Constructions d'incompatibilités fortes

Définition 2 : Nous parlerons de construction d'une incompatibilité forte à partir d'autres incompatibilités fortes, lorsque nous avons un algorithme qui permet de construire la première à partir des autres.

Il s'agit donc d'une implication logique, au sens constructif, liant des incompatibilités fortes.

Notation 3 : Nous noterons cette implication logique (au sens constructif) par un signe de déduction "constructif" . La notation

$$(\downarrow \mathbb{H}_1 \downarrow \text{ et } \downarrow \mathbb{H}_2 \downarrow) \vdash_{\text{cons}} \downarrow \mathbb{H}_3 \downarrow$$

signifie donc qu'on a un algorithme de construction d'une incompatibilité forte de type $\mathbb{H}_3$ à partir d'incompatibilités fortes de types $\mathbb{H}_1$ et $\mathbb{H}_2$

Le raisonnement par séparation des cas (selon le signe d'un polynome)

Nous donnons ici un énoncé détaillé des «raisonnements cas par cas», incluant la propagation des majorations de degrés.

Proposition 4 : Soit $\mathbb{H}$ un système de csg portant sur des polynomes de **K[X]** , Q un élément de **K[X]** , alors:

$$[\downarrow(\mathbb{H}, Q<0)\downarrow \text{ et } \downarrow(\mathbb{H}, Q>0)\downarrow] \vdash_{cons} \downarrow(\mathbb{H}, Q\neq 0)\downarrow \quad \text{(a)}$$

$$[\downarrow(\mathbb{H}, Q\leqslant 0)\downarrow \text{ et } \downarrow(\mathbb{H}, Q\geqslant 0)\downarrow] \vdash_{cons} \downarrow \mathbb{H} \downarrow \quad \text{(a')}$$

De même :

$$[\downarrow(\mathbb{H}, Q>0)\downarrow \text{ et } \downarrow(\mathbb{H}, Q=0)\downarrow] \vdash_{cons} \downarrow(\mathbb{H}, Q\geqslant 0)\downarrow \quad \text{(b)}$$

$$[\downarrow(\mathbb{H}, Q\neq 0)\downarrow \text{ et } \downarrow(\mathbb{H}, Q=0)\downarrow] \vdash_{cons} \downarrow \mathbb{H} \downarrow \quad \text{(c)}$$

$$[\downarrow(\mathbb{H}, Q>0)\downarrow \text{ et } \downarrow(\mathbb{H}, Q\leqslant 0)\downarrow] \vdash_{cons} \downarrow \mathbb{H} \downarrow \quad \text{(d)}$$

Dans chacun de ces cas, notons d_1 et d_2 les degrés des deux incompatibilités fortes données dans l'hypothèse, le degré de l'incompatibilité forte construite est respectivement majoré par :

$$\varphi_a(d_1,d_2) = \varphi_{a'}(d_1,d_2) = d_1 + d_2$$
$$\varphi_b(d_1,d_2) = d_1.d_2$$
$$\varphi_c(d_1,d_2) = d_1.d_2$$
$$\varphi_d(d_1,d_2) = d_1.d_2 + d_2$$

Ces 4 fonctions sont majorées par $\varphi(d_1,d_2) = d_1.d_2 + d_1 + d_2$

Enfin, pour démontrer que $\mathbb{H}$ est fortement incompatible, on peut raisonner en séparant selon les 3 cas $Q>0$, $Q<0$, $Q=0$. Ce qui renvient à affirmer :

$$[\downarrow(\mathbb{H}, Q>0)\downarrow \text{ et } \downarrow(\mathbb{H}, Q<0) \text{ et } \downarrow(\mathbb{H}, Q=0)\downarrow] \vdash_{cons} \downarrow \mathbb{H} \downarrow \quad \text{(e)}$$

La version algébrique dynamique de l'implication

Définition et notation 5 :

Soient $\mathbb{H}_1$ et $\mathbb{H}_2$ deux systèmes de csg portant sur des polynomes de **K[X]** . Nous dirons que *le système* $\mathbb{H}_1$ *implique dynamiquement* $\mathbb{H}_2$ lorsque, pour tout système de csg $\mathbb{H}$ portant sur des polynomes de **K[X,Y]** , on a la construction d'incompatibilité forte :

$$\downarrow [\mathbb{H}_2(\mathbf{X}), \mathbb{H}(\mathbf{X},\mathbf{Y})] \downarrow \vdash_{cons} \downarrow [\mathbb{H}_1(\mathbf{X}), \mathbb{H}(\mathbf{X},\mathbf{Y})] \downarrow$$

Nous noterons cette implication dynamique par :

$$^\bullet(\mathbb{H}_1(\mathbf{X}) \Rightarrow \mathbb{H}_2(\mathbf{X}))^\bullet.$$

Lorsque le système $\mathbb{H}_1$ est vide, nous utilisons la notation $^\bullet(\mathbb{H}_2(\mathbf{X}))^\bullet$.

Remarque :

On a trivialement l'équivalence des affirmations :

$$\downarrow \mathbb{H}_1 \downarrow \quad \text{et} \quad ^\bullet(\mathbb{H}_1 \Rightarrow (1=0))^\bullet$$

Fonction-degré d'une implication dynamique

Une implication dynamique $^\bullet(\mathbb{H}_1 \Rightarrow \mathbb{H}_2)^\bullet$ signifie par définition un algorithme fournissant la construction :

$$\downarrow [\mathbb{H}_2, \mathbb{H}] \downarrow \vdash_{cons} \downarrow [\mathbb{H}_1, \mathbb{H}] \downarrow$$

Chaque fois que nous établissons une implication dynamique particulière, nous devons établir des 'majorations primitives récursives de degré' pour cette construction d'incompatibilités fortes : le degré de l'incompatibilité forte construite est majoré par une fonction $\Delta(d,..;k,...)$ où d est le degré de l'incompatibilité forte initiale, k le nombre de csg dans $\mathbb{H}_2$ etc.... (le point-virgule isole les 'variables', qui dépendent de l'incompatibilité forte initiale, des 'paramètres', qui ne dépendent que de $\mathbb{H}_1$ et $\mathbb{H}_2$).
Nous disons qu'il s'agit d'une fonction-degré acceptable pour l'implication dynamique considérée, ou encore, (par abus) nous parlons de *la* fonction-degré attachée à l'implication dynamique.

La transitivité des implications dynamiques

La proposition suivante est immédiate : il suffit d'enchainer les deux algorithmes de constructions d'incompatibilités fortes.

Proposition 6 : Soient $\mathbb{H}_1$, $\mathbb{H}_2$, $\mathbb{H}_3$ trois systèmes de csg portant sur des polynomes de **K**[X] . Alors:

$[\ ^\bullet(\mathbb{H}_1 \Rightarrow \mathbb{H}_2)^\bullet$ et $^\bullet([\mathbb{H}_1, \mathbb{H}_2] \Rightarrow \mathbb{H}_3)^\bullet\ ]$ impliquent $^\bullet(\mathbb{H}_1 \Rightarrow \mathbb{H}_3)^\bullet$

Supposons que la première implication dynamique admette comme fonction-degré acceptable $\Delta^1(d;\mathbf{p})$ où d est le degré de $\downarrow[\mathbb{H}_2, \mathbb{H}]\downarrow$ et $\mathbf{p}$ représente certains paramètres dépendant de $\mathbb{H}_1$ et $\mathbb{H}_2$, supposons de même une fonction-degré acceptable $\Delta^2(d;\mathbf{q})$ pour la deuxième implication dynamique, alors une fonction-degré pour l'implication dynamique construite est obtenue en composant les deux fonctions-degré précédentes :

$$\Delta(d;\mathbf{p},\mathbf{q}) = \Delta^1(\Delta^2(d;\mathbf{q});\mathbf{p})$$

La version algébrique dynamique de la disjonction

Définition et notation 7 :

Soient $\mathbb{H}_1$, $\mathbb{H}_2$, ... , $\mathbb{H}_k$ et $\mathbb{K}_1$, $\mathbb{K}_2$, ... , $\mathbb{K}_m$ des systèmes de csg portant sur des polynomes de **K**[X] .

Nous disons que *le système* $\mathbb{H}_1$ *implique dynamiquement la disjonction* $\mathbb{K}_1 \vee \mathbb{K}_2 \vee \ldots \vee \mathbb{K}_m$ lorsque, pour tout système de csg $\mathbb{H}$ portant sur des polynomes de **K**[X,Y] , on a la construction d'incompatibilité forte :

$$\{\downarrow[\mathbb{K}_1(\mathbf{X}), \mathbb{H}(\mathbf{X},\mathbf{Y})]\downarrow \text{ et ... et } \downarrow[\mathbb{K}_m(\mathbf{X}), \mathbb{H}(\mathbf{X},\mathbf{Y})]\downarrow\} \vdash_{\text{cons}} \downarrow[\mathbb{H}_1(\mathbf{X}), \mathbb{H}(\mathbf{X},\mathbf{Y})]\downarrow$$

Nous noterons cette implication-disjonction dynamique par :

$$^\bullet(\mathbb{H}_1(\mathbf{X}) \Rightarrow [\mathbb{K}_1(\mathbf{X}) \vee \mathbb{K}_2(\mathbf{X}) \vee \ldots \vee \mathbb{K}_m(\mathbf{X})])^\bullet.$$

Lorsque le système $\mathbb{H}_1$ est vide, nous utilisons la notation

$$^\bullet(\mathbb{K}_1(\mathbf{X}) \vee \mathbb{K}_2(\mathbf{X}) \vee \ldots \vee \mathbb{K}_m(\mathbf{X}))^\bullet.$$

Enfin, la notation :

$$^\bullet([\mathbb{H}_1 \vee \mathbb{H}_2 \vee \ldots \vee \mathbb{H}_k] \Rightarrow [\mathbb{K}_1 \vee \mathbb{K}_2 \vee \ldots \vee \mathbb{K}_m])^\bullet$$

signifie que chacune des implications-disjonctions dynamiques

$$^\bullet(\mathbb{H}_i(\mathbf{X}) \Rightarrow [\mathbb{K}_1(\mathbf{X}) \vee \mathbb{K}_2(\mathbf{X}) \vee \ldots \vee \mathbb{K}_m(\mathbf{X})])^\bullet \quad (i = 1, ..., k)$$

est vérifiée

Remarques : Toute formule sans quantificateur de la théorie du premier ordre des anneaux totalement ordonnés dicrets à paramètres dans **K** est équivalente à une formule en forme normale disjonctive et donc à une formule du type

$$\mathbb{K}_1(\mathbf{X}) \lor \mathbb{K}_2(\mathbf{X}) \lor \ldots \lor \mathbb{K}_m(\mathbf{X})$$

où les $\mathbb{K}_i(\mathbf{X})$ sont des systèmes de csg portant sur des polynomes de $\mathbf{K}[\mathbf{X}]$.
Les implications-disjonctions dynamiques consituent une forme de raisonnement purement «identité algébrique» concernant les formules sans quantificateur, où la logique a été évacuée au profit d'algorithmes de constructions d'identités algébriques.

La fonction-degré d'une implication-disjonction dynamique se définit comme pour les implications dynamiques

La proposition 4 peut être relue comme affirmant des disjonctions ou implications-disjonctions dynamiques :

Proposition 4 bis : On a les implications-disjonctions dynamiques suivantes :

$$^\bullet(\ Q \neq 0 \ \Rightarrow [\ Q > 0 \ \lor \ Q < 0 \ ]\ )^\bullet \qquad \text{(a)}$$

$$^\bullet(\ Q \leqslant 0 \ \lor \ Q \geqslant 0 \ )^\bullet \qquad \text{(a')}$$

$$^\bullet(\ Q \geqslant 0 \Rightarrow [\ Q > 0 \ \lor \ Q = 0 \ ]\ )^\bullet \qquad \text{(b)}$$

$$^\bullet(\ Q \neq 0 \ \lor \ Q = 0 \ )^\bullet \qquad \text{(c)}$$

$$^\bullet(\ Q > 0 \ \lor \ Q \leqslant 0)^\bullet \qquad \text{(d)}$$

$$^\bullet(\ Q = 0 \ \lor \ Q > 0 \ \lor \ Q < 0 \ )^\bullet \qquad \text{(e)}$$

La transitivité des implications-disjonctions dynamiques

L'énoncé le plus général est le suivant :
Les implications-disjonctions dynamiques

$$^\bullet(\ [\ \mathbb{H}_1 \lor \mathbb{H}_2 \lor \ldots \lor \mathbb{H}_k\ ] \Rightarrow [\ \mathbb{K}_1 \lor \mathbb{K}_2 \lor \ldots \lor \mathbb{K}_m\ ]\ )^\bullet$$

et

$$^\bullet(\ [\ \mathbb{K}_1 \lor \mathbb{K}_2 \lor \ldots \lor \mathbb{K}_m\ ] \Rightarrow [\ \mathbb{L}_1 \lor \mathbb{L}_2 \lor \ldots \lor \mathbb{L}_n\ ]\ )^\bullet$$

impliquent :

$$^\bullet(\ [\ \mathbb{H}_1 \lor \mathbb{H}_2 \lor \ldots \lor \mathbb{H}_k\ ] \Rightarrow [\ \mathbb{L}_1 \lor \mathbb{L}_2 \lor \ldots \lor \mathbb{L}_n\ ]\ )^\bullet$$

Cette transitivité s'obtient en enchainant les algorithmes de constructions d'incompatibilités fortes. Les fonctions-degré résultantes s'obtiennent donc par composition convenable des fonctions-degré initiales.

Implications dynamiques faciles

Définition 7 : (implications triviales)

Une implication $\mathbb{H}_1(\mathbf{X}) \Rightarrow \mathbb{H}_2(\mathbf{X})$ est dite triviale lorsque toute incompatibilité forte

$$\downarrow [\ \mathbb{H}_2(\mathbf{X})\ ,\ \mathbb{H}(\mathbf{X},\mathbf{Y})\ ] \downarrow$$

fournit par simple relecture l'incompatibilité forte

$$\downarrow [\ \mathbb{H}_1(\mathbf{X}),\ \mathbb{H}(\mathbf{X},\mathbf{Y})\ ] \downarrow .$$

L'implication dynamique $^\bullet(\mathbb{H}_1(\mathbf{X}) \Rightarrow \mathbb{H}_2(\mathbf{X}))^\bullet$ accepte alors pour fonction-degré :

$\Delta_0(d) = d$.

Exemple : L'implication $[\ A > 0\ , B > 0\ ] \Rightarrow AB > 0$ est triviale : dans l'incompatibilité forte

$$\downarrow [\ AB > 0\ ,\ \mathbb{H}\ ] \downarrow$$

on relit chaque constituant AB (dans la partie «monoïde» ou dans la partie «cone») sous forme du produit de A par B pour obtenir l'incompatibilité forte

$$\downarrow [\ A > 0\ ,\ B > 0,\ \mathbb{H}\ ]\ \downarrow$$

Notez que l'implication 'contrapposée' $[\ A > 0\ ,\ A.B \leq 0\ ] \Rightarrow B \leq 0$ n'est pas une implication simple.

De nombreuses implications, sans être des implications triviales, sont d'un traitement "rapide" en tant qu'implications dynamiques :

Par exemple :

L'implication $[\ A > 0\ ,\ A.B \geq 0\ ] \Rightarrow B \geq 0$ accepte pour fonction-degré :

$(d;\delta) \longmapsto d + 2.\delta$ où $\delta = d(A)$.

Preuve : on multiplie, terme à terme, l'implication forte par A^2, en prenant soin de remplacer les $B.A^2$ par $(BA).A$.

Le principe de substitution

Proposition 8 :

On considère des variables $X_1,X_2,...,X_n$, $U_1,U_2,...,U_h$, $Z_1,Z_2,...,Z_k$, et des polynomes $P_1,P_2,...,P_n$ de $\mathbf{K}[\mathbf{Z}]$. Notons $\mathbf{P(Z)}$ pour $P_1(\mathbf{Z}), ..., P_n(\mathbf{Z})$.

Si on a $\quad {}^\bullet(\ \mathbb{H}_1(\mathbf{X,U}) \Rightarrow \mathbb{H}_2(\mathbf{X,U})\ )^\bullet$ (a)

alors on a aussi $\quad {}^\bullet(\ \mathbb{H}_1(\mathbf{P(Z),U}) \Rightarrow \mathbb{H}_2(\mathbf{P(Z),U})\ )^\bullet$ (b)

Formules de Taylor mixtes

On considère deux variables U et V et on pose $\Delta := U - V$. On considère un polynome P à coefficients dans un corps ordonné **K** ou *plus généralement dans un anneau commutatif* **A** *qui est une* $\mathbb{Q}$ *-algèbre.*

Si $\deg(P) \leqslant 4$, on a les 8 formules de Taylor mixtes suivantes:

$$P(U) - P(V) = \Delta.P'(V) + (1/2).\Delta^2.P''(V) + (1/6).\Delta^3.P^{(3)}(V) + (1/24).\Delta^4.P^{(4)}$$
$$P(U) - P(V) = \Delta.P'(V) + (1/2).\Delta^2.P''(V) + (1/6).\Delta^3.P^{(3)}(U) - (1/8).\Delta^4.P^{(4)}$$
$$P(U) - P(V) = \Delta.P'(V) + (1/2).\Delta^2.P''(U) - (1/3).\Delta^3.P^{(3)}(V) - (5/24).\Delta^4.P^{(4)}$$
$$P(U) - P(V) = \Delta.P'(V) + (1/2).\Delta^2.P''(U) - (1/3).\Delta^3.P^{(3)}(U) + (1/8).\Delta^4.P^{(4)}$$
$$P(U) - P(V) = \Delta.P'(U) - (1/2).\Delta^2.P''(V) - (1/3).\Delta^3.P^{(3)}(V) - (1/8).\Delta^4.P^{(4)}$$
$$P(U) - P(V) = \Delta.P'(U) - (1/2).\Delta^2.P''(V) - (1/3).\Delta^3.P^{(3)}(U) + (5/24).\Delta^4.P^{(4)}$$
$$P(U) - P(V) = \Delta.P'(U) - (1/2).\Delta^2.P''(U) + (1/6).\Delta^3.P^{(3)}(V) + (1/8).\Delta^4.P^{(4)}$$
$$P(U) - P(V) = \Delta.P'(U) - (1/2).\Delta^2.P''(U) + (1/6).\Delta^3.P^{(3)}(U) - (1/24).\Delta^4.P^{(4)}$$

Comme toutes les combinaisons de signes possibles se présentent, on obtient :

– supposons que u et v attribuent la même suite de signes (au sens large) pour les dérivées successives d'un polynome P non constant de degré ≤ 4 , notons $\varepsilon_1 = 1$ ou -1 selon que $P'(u)$ et $P'(v)$ sont tous deux ≥ 0 ou tous deux ≤ 0 , alors le fait que $P(u) - P(v)$ a même signe que $\varepsilon_1.(u - v)$ est rendu évidnet par l'une des formules ci-dessus, ce qui donne l'implication sous forme d'une implication simple (u et v peuvent être des éléments de **K** mais aussi des variables, ou des polynomes)

– si u et v n'attribuent pas la même suite de signes pour un polynome P de degré ≤ 4 et ses dérivées successives, alors on a une identité algébrique qui donne le signe de $u - v$ à partir des signes des $P^{(i)}(u)$ et des $P^{(i)}(v)$: la formule de Taylor mixte à utiliser est avec $P^{(i)}$ ($i = 0, 1, 2$, ou 3) où i est le plus grand indice pour lequel les deux signes ne sont pas identiques

Plus généralement on a :

Proposition 9 : (formules de Taylor mixte)

Pour chaque degré s , il y a 2^{s-1} formules de Taylor mixtes et toutes les combinaisons de signes possibles apparaissent.

Formules de Taylor généralisées (le lemme de Thom sous forme d'identités algébriques)

Le lemme de Thom affirme (entre autres) que l'ensemble des points où un polynome et ses dérivées successives ont chacun un signe fixé, est un intervalle. Une preuve facile, par récurrence sur le degré du polynome, est basée sur le théorème des accroissements finis. Nous pouvons, grâce aux formules de Taylor mixtes, traduire ce fait géométrique sous forme d'identités algébriques, que nous appellerons des **formules de Taylor généralisées**. Plutôt que de risquer un énoncé, nous donnons un exemple.

Un exemple : Considérons le polynome générique de degré 4

$$P(X) = c_0 X^4 + c_1 X^3 + c_2 X^3 + c_3 X^2 + c_4 X^4 + c_5$$

Considérons le système de conditions de signe portant sur le polynome P et ses dérivées successives par rapport à la variable X :

$$\mathbb{H}(U) : \ P(U) > 0, \ P'(U) < 0, \ P^{(2)}(U) < 0, P^{(3)}(U) < 0, P^{(4)}(U) > 0.$$

Considérons également le système de conditions de signe généralisées obtenues en relachant toutes les inégalités, sauf la dernière :

$$\mathbb{H}'(U) : \ P(U) \geq 0, \ P'(U) \leq 0, \ P^{(2)}(U) \leq 0, P^{(3)}(U) \leq 0, P^{(4)}(U) > 0.$$

Le lemme de Thom affirme (entre autres) :

$$[\ \mathbb{H}'(U)\ , \mathbb{H}'(V)\ ,\ U < Z < V\ ] \quad \Rightarrow \quad \mathbb{H}(Z) \qquad (1)$$

Nous allons voir que ce fait géométrique est rendu évident par des identités algébriques.

On écrit les formules de Taylor mixtes suivantes :

α) $P^{(3)}(Z) = P^{(3)}(V) + P^{(4)}.(Z - V)$

β) $P^{(2)}(Z) = P^{(2)}(U) + P^{(3)}(Z).(Z - U) - 1/2\ P^{(4)}.(Z - U)^2$

γ) $P'(Z) = P'(U) + P^{(2)}(U).(Z - U) + 1/2\ P^{(3)}(Z).(Z - U)^2 - 1/3\ P^{(4)}.(Z - U)^3$

δ) $P(Z) = P(V) + P'(Z).(Z - V) - 1/2P^{(2)}(Z).(Z - V)^2 + 1/6\ P^{(3)}(V).(Z - V)^3 + \ldots$
$1/8\ P^{(4)}.(Z - V)^4$

Posons $\boxed{\Delta_1 = Z - U\ ,\ \Delta_2 = V - Z}$

Dans β) on remplace $P^{(3)}(Z)$ par son expression donnée dans α) et on obtient :

$$\beta') \quad P^{(2)}(Z) = P^{(2)}(U) + P^{(3)}(V).\Delta_1 - P^{(4)}\ [\Delta_1.\Delta_2 + 1/2\ \Delta_1^2]$$

On obtient de la même manière, par substitutions :

$$\gamma') \quad P'(Z) = P'(U) + P^{(2)}(U).\Delta_1 + 1/2\ P^{(3)}(V).\Delta_1^2 - P^{(4)}.[\Delta_1^2.\Delta_2/2 + \Delta_1^3/3]$$

et enfin

$$\begin{aligned}\delta')\ P(Z) \ = \ & P(V) - P'(U).\Delta_2 - P^{(2)}(U).[\Delta_1.\Delta_2 + 1/2\Delta_2^2] \\ & - P^{(3)}(V).[\ \Delta_1^2.\Delta_2/2 + \Delta_1.\Delta_2^2/2 + \Delta_2^3/6] \\ & + P^{(4)}.[\Delta_1^3.\Delta_2/3 + \Delta_1^2.\Delta_2^2/2 + \Delta_1.\Delta_2^3/2 + \Delta_2^4/8]\end{aligned}$$

Les égalités α), β'), γ'), δ') donnent l'implication (1) sous forme d'une implication simple. La première égalité est une formule de Taylor ordinaire portant sur le polynome $P^{(3)}$. Les trois dernières peuvent être vues comme des formules de Taylor généralisées portant sur les polynomes $P^{(2)}$, P' et P .

Plus généralement, on obtient:

Théorème 10 : (évidence forte du lemme de Thom)

Soit T une variable distincte des C_i . Soient $P \in \mathbf{K}[C][T]$, de degré s en T , $\sigma_1, \sigma_2, ..., \sigma_s$ une liste formée de $<$ ou $>$. On note $\mathbb{H}(C,T)$ ou $\mathbb{H}(T)$ le système de csg : $P'(C,T)\ \sigma_1\ 0, ..., P^{(i)}(C,T)\ \sigma_i\ 0, ..., P^{(s)}(C,T)\ \sigma_s\ 0$ (les dérivées sont par rapport à T) .

Soit $\mathbb{H}'(T)$ le système de csg obtenu à partir de $\mathbb{H}(T)$ en relachant toutes les conditions de signe sauf celle relative à $P^{(s)}$.

Soit $\mathbb{H}_1(T)$ le système de csg : $P^{(s)}(C,T) > 0, P^{(i)}(C,T) \geqslant 0, i = 1, ..., s-1$.

Soient enfin trois variables U , V , Z distincte des C_i .

On a alors les implications dynamiques suivantes :

$$^\bullet(\ [\ \mathbb{H}'(U),\ \mathbb{H}'(V)\ ,\ U\ \sigma_1\ V\ ]\ \Rightarrow\ P(U) > P(V)\ )^\bullet \qquad \text{(a)}$$

$$^\bullet(\ [\ \mathbb{H}_1(U),\ V > U\ ]\ \Rightarrow\ P(V) > P(U)\ )^\bullet \qquad \text{(b)}$$

$$^\bullet(\ [\ \mathbb{H}'(U)\ ,\ \mathbb{H}'(V)\ ,\ U < Z < V\ ]\ \Rightarrow\ \mathbb{H}(Z)\ )^\bullet \qquad \text{(c)}$$

Ce sont des implications dynamiques qui ne coûtent rien ($d \longmapsto d$ est une fonction degré acceptable).

Existences potentielles

Notations et définitions

Elles sont tout à fait analogues à celles données pour les implications-disjonctions dynamiques.

Définition et notation 11 :

Soient $\mathbb{H}_1$ un système de csg portant sur des polynomes de **K[X]** , $\mathbb{H}_2$ un système de csg portant sur des polynomes de $\mathbf{K}[X,T_1,T_2,...,T_m] = \mathbf{K}[X,T]$.

Nous dirons que *les hypothèses* $\mathbb{H}_1$ *autorisent l'existence des* T_i *vérifiant* $\mathbb{H}_2$ lorsque, pour tout système de csg $\mathbb{H}$ portant sur des polynomes de **K[X,Y]** , les variables Y_i et T_j étant deux à deux distinctes, on a la construction d'implication forte :

$$\downarrow\ [\ \mathbb{H}_2(X,T)\ ,\ \mathbb{H}(X,Y)\ ]\downarrow\ \ \vdash_{cons} \downarrow\ [\ \mathbb{H}_1(X)\ ,\ \mathbb{H}(X,Y)\ ]\downarrow .$$

Nous parlerons également *d'existence potentielle des* T_i *vérifiant* $\mathbb{H}_2$ *sous les hypothèses* $\mathbb{H}_1$

Nous noterons cette existence potentielle par :

$$^\bullet(\ \ \mathbb{H}_1(X)\ \Rightarrow\ \exists\ T\ \ \mathbb{H}_2(X,T)\ )^\bullet.$$

Lorsque le système $\mathbb{H}_1$ est vide, nous utilisons la notation $^\bullet(\ \exists\ T\ \ \mathbb{H}_2(X,T)\ )^\bullet$.

La notion de fonction-degré acceptable pour une existence potentielle peut être elle aussi directement recopiée du cas des implications dynamiques.

Remarques :

1) La notion d'existence potentielle est une notion d'existence faible. L'existence potentielle signifie qu'il n'est pas grave de faire comme si les T_i existaient vraiment, parce que cela n'introduit pas de contradiction: on peut paraphraser la définition en disant :

pour construire l'incompatibilité forte $\downarrow [\ \mathbb{H}_1(\mathbf{X})\ ,\ \mathbb{H}(\mathbf{X},\mathbf{Y})\ ] \downarrow$

il suffit d'avoir construit $\downarrow [\ \mathbb{H}_2(\mathbf{X},\mathbf{T})\ ,\ \mathbb{H}(\mathbf{X},\mathbf{Y})\ ] \downarrow$

2) On pourrait étendre la définition de l'existence potentielle en remplaçant le système de csg $\mathbb{H}_2(\mathbf{X},\mathbf{T})$ par une disjonction de systèmes de csg, comme on a fait avec la notion d'implication-disjonction dynamique.

Transitivité, principe de substitution, preuves cas par cas

La transitivité des existences potentielles est immédiate, comme dans le cas des implications dynamiques.
Le principe de substitution pour les existences potentielles s'énonce et se démontre comme pour les implications dynamiques.
Voici maintenant un énoncé correspondant aux preuves cas par cas d'une existence potentielle, conséquence immédiate de la proposition 4 .

Proposition 4 ter : (raisonnement cas par cas pour les existences potentielles)

Soit Q un polynome de $\mathbf{K}[\mathbf{X}]$.

a) Pour démontrer une existence potentielle

$${}^{\bullet}(\ [\ \mathbb{H}_1(\mathbf{X})\ ,\ Q \neq 0\ )\ \Rightarrow\ \exists\ \mathbf{T}\ \ \mathbb{H}_2(\mathbf{X},\mathbf{T})\ )^{\bullet}$$

il suffit de démontrer chacune des existences potentielles

${}^{\bullet}(\ [\ \mathbb{H}_1(\mathbf{X}),\ Q > 0]\ \Rightarrow\ \exists\ \mathbf{T}\ \ \mathbb{H}_2(\mathbf{X},\mathbf{T})\ )^{\bullet}$ et ${}^{\bullet}(\ [\ \mathbb{H}_1(\mathbf{X}),\ Q < 0]\ \Rightarrow\ \exists\ \mathbf{T}\ \ \mathbb{H}_2(\mathbf{X},\mathbf{T})\ )^{\bullet}$

Si Δ^i $(i = 1,2)$ sont les deux fonctions-degré des existences potentielles supposées, une fonction-degré pour l'existence potentielle déduite est donnée par : $\Delta^1 + \Delta^2$

a'), b), c), d), e) : énoncés analogues décalqués de la proposition 4

L'existence implique l'existence potentielle

Un autre principe utile est le fait que l'existence implique l'existence potentielle. Il s'obtient facilement : on remplace les variables T_i «existentielles» par les polynomes concrets P_i qui réalisent l'existence. On reconnait là une analogie formelle avec la règle d'introduction du quantificateur existentiel en calcul naturel par exemple (cf. [**Pra**]).

Proposition 12 : (l'existence implique l'existence potentielle)

Soient $P_1,P_2,\ldots,P_m \in \mathbf{K}[\mathbf{X}]$ et notons $\mathbf{P}(\mathbf{X})$ pour $P_1(\mathbf{X}), \ldots, P_m(\mathbf{X})$. On a l'existence potentielle : ${}^{\bullet}(\ \ \mathbb{H}_2(\mathbf{X},\mathbf{P}(\mathbf{X}))\ \Rightarrow\ \exists\ \mathbf{T}\ \ \mathbb{H}_2(\mathbf{X},\mathbf{T})\ )^{\bullet}$.

Si δ majore les degrés des P_i , l'existence potentielle accepte pour fonction-degré :

$(d;\delta) \longmapsto d.\sup(1,\delta)$

Existences potentielles fondamentales

On sait démontrer les existences potentielles correspondant aux axiomes existentiels de la théorie des corps réels clos.

Théorème 13 : (autorisation de rajouter l'inverse d'un non nul)

On a l'existence potentielle de l'inverse d'un non nul. Ce qui s'écrit:

$${}^{\bullet}(\ U \neq 0\ \Rightarrow\ \exists\ T\ \ \ 1 = U.T\ )^{\bullet}$$

Soit δ le degré de U, une fonction-degré acceptable pour l'existence potentielle est

$(d;\delta) \longmapsto d + d.\delta + \delta$

Remarque: La preuve de cette existence potentielle recopie ce qu'on fait, dans la preuve du théorème des zéros de Hilbert, pour passer du théorème des zéros faible au théorème des zéros général (c'est le «Rabinovitch trick», par exemple dans l'exposé classique de van der Waerden). La notion d'existence potentielle de l'inverse d'un non nul est donc en filigrane dans les classiques.

Théorème 14 : (autorisation de rajouter une racine sur un intervalle où un polynome change de signe)

Soit $P(\mathbf{C},X)$ un polynome de degré s en X et de degré global δ.

On a l'existence potentielle d'une racine sur un intervalle où ce polynome change de signe. Ce qui s'écrit, en notant $P(X)$ pour $P(\mathbf{C},X)$:

$$^{\bullet}(\ [\,P(X).P(Y) < 0\ ,\ X < Y\ ]\ \Rightarrow\ \exists\, Z\ [\,P(Z) = 0\ ,\ X < Z < Y\,]\ )^{\bullet}$$

et, si X , Y , Z désignent des variables, une fonction-degré acceptable est donnée par :

$$(d;\delta,s) \longmapsto (\,(2d+7)\,(\delta+1)\,)^{\gamma'(s)} \quad \text{où} \quad \gamma'(s) = 2^{(s+2)^2/2}$$

Remarque : La preuve du théorème précédent "recopie" la preuve classique, par récurrence sur le degré du polynome P , du théorème «si un corps est ordonné et si $P(u).P(v) < 0$ avec P irréductible, alors le corps $\mathbf{K}[W]/P(W)$ est réel». Ceci donne l'existence potentielle d'une racine. Pour avoir la racine sur l'intervalle, il y a de nouveau une récurrence à faire. Tout ceci conduit à une relativement mauvaise fonction-degré. Le problème semble difficile à contourner. Dans le cas complexe (théorème des zéros de Hilbert), l'existence potentielle d'une racine d'un polynome non constant est au contraire extrêmement simple et conduit à une fonction-degré tout à fait raisonnable : par exemple si $P(\mathbf{X},Y)$ est un polynome unitaire en Y de degré s en Y et de degré δ en $\mathbf{X}$, il suffit de tout réduire modulo P et une fonction-degré acceptable pour l'existence potentielle $^{\bullet}(\ \exists\, Y\ P(\mathbf{X},Y) = 0\ )^{\bullet}$ est donnée par : $(d;\delta) \longmapsto d.(\delta+1)$

Tableaux de Hörmander

Nous donnons ici quelques majorations directement liées à l'algorithme de Hörmander lui-même (cf. [**Hör**] annexe, ou [**BCR**] chap. 1).
L'algorithme de Hörmander traite des polynomes en n variables, en éliminant chaque variable l'une après l'autre. A chaque élimination d'une variable, le nombre de polynomes à considérer et leurs degrés croissent de manière impressionnante. Ceci est précisé dans la proposition suivante :

Proposition 15 : (Tableau de Hörmander paramétré)

Soit $\mathbf{K}$ un corps ordonné, sous-corps d'un corps réel clos $\mathbf{R}$.

Soit $L = [Q_1\ , Q_2\ , ..., Q_k]$ une liste de polynomes de $\mathbf{K}[X_1, X_2, ..., X_n][Y]$.

On peut construire une famille finie $\mathcal{F}$ de polynomes de $\mathbf{K}[X_1, X_2, ..., X_n]$ telle que, pour tous $x_1, x_2, ..., x_n$ dans $\mathbf{R}$, en posant $P_i(Y) = Q_i(x_1, x_2, ..., x_n;Y)$, le tableau complet des signes pour $L = [P_1\ , P_2\ , ..., P_k]$ est calculable à partir des signes des $S(x_1, x_2, ..., x_n)$ pour $S \in \mathcal{F}$.

Supposons que la liste L possède k éléments de degré en $\mathbf{X}$ majoré par δ et de degré en Y majoré par s . Considérons la famille $\mathcal{G}$, formée de tous les coefficients de tous les polynomes de tous les tableaux de Hörmander possibles, construits sur L , en remplaçant l'opération "reste" par l'opération "pseudo-reste". Une famille $\mathcal{F}$ convenable peut être extraite de $\mathcal{G}$. Alors :

le degré de chaque polynome de $\mathcal{G}$ et de chaque pseudo-division est majoré par :

$\delta.(s+1)!$, (sauf si $n = 0$, donc $\delta = 0$, et les degrés sont majorés par s).

le nombre d'éléments de la famille $\mathcal{G}$ est majoré par : $(k+1)^{2^s}$

Mené jusqu'au bout, cet algorithme produit donc une explosion de degrés obtenue en itérant n–1 fois (n étant le nombre de variables) la fonction $s \longmapsto s\,!$. Ceci conduit à la majoration finale.

Nullstellensatz, positivestellensatz et nichtnegativestellensatz réels effectifs

Théorème 16 : Soit **K** un corps ordonné, sous-corps d'un corps réel clos **R** .

Soit $\mathbb{H}(X_1,X_2,...,X_n)$ un système de csg portant sur une famille finie de polynomes de $\mathbf{K}[X_1,X_2,...,X_n]$. Ce système est impossible dans **R** si et seulement si il est fortement incompatible dans **K** . En termes plus formalisés :

Si $\downarrow \mathbb{H}(X_1,X_2,...,X_n) \downarrow$ (dans **K**) ,

alors les csg $\mathbb{H}$ sont impossibles à réaliser dans n'importe quelle extension ordonnée de **K** .

Si $\forall\, x_1,x_2,...,x_n \in \mathbf{R}$ $\mathbb{H}(x_1,x_2,...,x_n)$ est absurde,

alors : $\downarrow \mathbb{H}(X_1,X_2,...,X_n) \downarrow$ (dans **K**).

Précisément, si k est le nombre de csg dans $\mathbb{H}(X_1,X_2,...,X_n)$ et d le degré maximum, on peut calculer une implication forte

$\downarrow \mathbb{H}(X_1,X_2,...,X_n) \downarrow$ (dans **K**) de degré majoré par le nombre $\mu_{26}(d,k,n)$ donné par la tour d'exponentielle à n+4 étages

$$2^{2^{\cdot^{\cdot^{\cdot^{d.\lg(d)+\lg\lg(k)+cte}}}}}$$

Remarque : La principale cause d'explosion des degrés dans la majoration finale actuelle réside dans l'utilisation de l'algorithme de Hörmander.

On peut donc espérer améliorer sensiblement ces majorations en se basant sur d'autres preuves, élémentaires mais moins longues, d'incompatibilité.

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Anneaux de Witt abstraits et groupes spéciaux

M. A. Dickmann

Introduction.

Cet article rend compte de manière abrégée d'une partie d'une série d'exposés dont le but était de:
(1) Rassembler les résultats principaux de la théorie des formes quadratiques sur les corps de caractéristique $\neq 2$, en mettant en relief les plus significatifs parmi eux en vue des géneralisations axiomatiques de cette théorie.
(2) Présenter ces généralisations axiomatiques, notamment la *théorie des anneaux de Witt abstraits.*
(3) Introduire une nouvelle théorie axiomatique de formes quadratiques, la *théorie des groupes spéciaux*, équivalente en fait à celle des anneaux de Witt abstraits, mais beaucoup mieux adaptée qu'elle à l'étude des formes quadratiques du point de vue des langages du premier ordre.
(4) Enfin, présenter un premier résultat modèle-théorique concernant les groupes spéciaux des corps de nombres algébriques (extensions finies du corps des nombres rationnels).

Le premier point ci-dessus sera omis dans ce compte-rendu écrit. Les résultats et les constructions exposés se trouvent dans les Chapitres I-III et X de Lam [3], et dans les Chapitres 1 et 3 de Marshall [6].

Le point (4) est également omis; il fera l'objet d'une publication séparée actuellement en préparation.

En ce qui concerne le point (2), nous nous bornons à une présentation très succincte de la notion d'anneau de Witt abstrait et de certaines de leurs propriétés élémentaires (celles utilisées dans ce texte), avec une discussion du sens des axiomes illustrée par le cas des corps. Les *structures quaternioniques*, brièvement présentées dans l'exposé oral, sont également omises ici; on renvoie le lecteur intéressé au Chapitre 2 de Marshall [6].

Les *groupes spéciaux* sont introduits au §2. Ils sont les modèles d'un système fini d'axiomes simples formulés dans un langage du premier ordre mathématiquement naturel. Certains aspects de cette théorie du premier ordre sont discutés brièvement. Ensuite nous ébauchons la construction de l'anneau de Witt d'un groupe spécial et nous démontrons:

Théorème. *La correspondance qui à chaque groupe spécial assigne son anneau de Witt est une équivalence entre la catégorie des groupes spéciaux et celle des anneaux de Witt abstraits (toutes les deux munies de leurs homomorphismes naturels).* □

L'article [4] (ce volume) contient un exposé plus détaillé de la théorie des groupes spéciaux, où l'accent est mis sur les relations entre les groupes spéciaux *réduits* et les espaces d'ordres abstraits. Nous publierons ultérieurement une étude plus complète de la théorie des groupes spéciaux, réduits et non-réduits, sous la perspective de la théorie des modèles.

1 Anneaux de Witt abstraits.

La notion d'anneau de Witt abstrait fut introduite par Knebusch, Rosenberg et Ware au début des années 70 dans le but de traiter de manière axiomatique (ou "abstraite") la théorie algébrique des formes quadratiques sur les corps. Ils réussissent à donner dans ce cadre des preuves uniformes (et simplifiées) de beaucoup de résultats connus préalablement pour le cas des corps; nous citons quelques exemples à la fin de cette section. Ultérieurement, Marshall (dans le cas réduit) et Kleinstein-Rosenberg (dans le cas non-réduit) améliorent et étendent cette théorie. L'exposé ci-dessous suit, essentiellement, celui de Marshall [6; Ch. 4].

Définition 1.1. Un *anneau de Witt abstrait* est une paire $< R, G >$ où:
(1) R est un anneau commutatif avec unité;
(2) G est un sous-groupe d'exposant 2 du groupe multiplicatif $R^\times$ des éléments inversibles de R contenant -1;
vérifiant les axiomes (W_1)-(W_3) ci-dessous:
(W_1) G engendre R additivement, i.e. tout $r \in R$ peut être écrit (de manière non-unique) dans la forme $r = g_1 + \cdots + g_n$, avec $g_1, \ldots, g_n \in G$, $n \geq 1$.

On désigne par I_R l'idéal de R engendré par les éléments de la forme $a + b$ avec $a, b \in G$; I_R est appelé l'*idéal fondamental* de R.

(W_2) i) $G \cap I_R = \emptyset$.
ii) $(G + G) \cap I_R^2 = \{0\}$.

(W_3) Pour $n \geq 3$: si $g_1 + \cdots + g_n = h_1 + \cdots + h_n$ avec $g_1, \ldots, g_n, h_1, \ldots, h_n \in G$, il existe $g, h, z_3, \ldots, z_n \in G$ tels que $g_1 + g = h_1 + h$ et $g_2 + \cdots + g_n = g + z_3 + \cdots + z_n$ (d'où $h_2 + \cdots + h_n = h + z_3 + \cdots + z_n$). □

1.2. Explication des axiomes; exemples.
Le modèle qui motive ces axiomes est celui de l'anneau de Witt d'un corps, F, commutatif et de caractéristique $\neq 2$. Dans ce cas on pose:

$$G(F) = F^\times / F^{\times 2},$$

(ou $G(F) = F^\times / \sum F^{\times 2}$ si l'on veut faire la théorie *reduite*), et

$$W(F) = \text{l'anneau de Witt du corps } F.$$

$W(F)$ est l'ensemble des formes quadratiques à coefficients dans F modulo l'équivalence de Witt,"$\sim$", muni des opérations de somme et produit tensoriel; voir Lam [3; Ch.II, §1] et 2.3 plus bas. On identifie $G(F)$ à un sous-ensemble de $W(F)$ par l'application

$\overline{g} \mapsto < g >/_\sim$, $\overline{g} \in G(F)$. [**Attention.**<u>Toutes</u> les formes quadratiques considerées ici sont en forme diagonale.]
L'axiome (W_1) exprime le fait –évident d'après la définition de somme de formes–que toute forme est la somme de formes unaires:

$$< \overline{a_1}, \ldots \overline{a_n} >/_\sim = < \overline{a_1} >/_\sim \oplus \cdots \oplus < \overline{a_n} >/_\sim .$$

L'axiome (W_3) traduit la description inductive de l'isométrie des formes de dimension $n \geq 3$ en termes de l'isométrie des formes binaires:

$$< a_1, \ldots, a_n > \equiv < b_1, \ldots, b_n > \text{ ssi } \text{il existe } a,\ b,\ c_3,\ \ldots\ , c_n \text{ tels que } \begin{array}{rcl} < a_1, a > & \equiv & < b_1, b >, \\ < a_2, \ldots, a_n > & \equiv & < a, c_3, \ldots, c_n >, \\ < b_2, \ldots, b_n > & \equiv & < b, c_3, \ldots, c_n > . \end{array}$$

Dans l'exemple $< W(F), G(F) >$ cette caractérisation est un corollaire du théorème de simplification de Witt; cf. Marshall [6; Thms. 1.12 et 1.13].
En vue de la clause (2) de la Définition 1.1 l'axiome (W_2.i) dit, simplement, que l'idéal I_R est propre. L'origine de l'axiome (W_2.ii) est moins immédiat. Le théorème suivant est l'un des résultats importants de la théorie des formes quadratiques sur les corps:

Théorème. (Arason-Pfister) *Soit f une forme anisotrope sur un corps F. Si $f/_\sim \in I(F)^k$, alors $dim(f) \geq 2^k$.* □

($I(F)$ désigne l'idéal fondamental de $W(F)$.)

Corollaire. $\bigcap_{k=1}^{\infty} I(F)^k = \{0\}$. □

L'axiome (W_2.ii) exprime, donc, la propriété d'Arason-Pfister pour $k = 2$.

La preuve du théorème d'Arason-Pfister utilise des extensions transcendantes du corps de base F –opération sans analogue dans le cadre abstrait– et on ne sais pas faire autrement. Pour des preuves de ce théorème, voir Lam [3; Ch.X, §3] et Knebusch-Scharlau [1; §12]. Or, dans les cas $k = 1, 2$ il y a une preuve très simple qui n'utilise pas de méthode générique, valable également pour les théories abstraites. Nous donnons cette preuve ci-après pour le cas des corps.

Le *discriminant*, $d(f)$, et le *discriminant signé*, $d_\pm(f)$, d'une forme f sont définis par:

$$\begin{array}{rcll} d(< a_1, \ldots, a_n >) & = & \prod_{i=1}^{n} a_i & \qquad (a_1, \ldots, a_n \in G), \\ d_\pm(f) & = & (-1)^{\frac{n(n-1)}{2}} d(f). & \end{array}$$

Le discriminant est invariant par l'isométrie des formes (verifiez) mais il n'est pas invariant par l'équivalence de Witt: $< 1, 1, -1 > \sim < 1 >$, mais $d(< 1, 1, -1 >) = -1$ alors que $d(< 1 >) = 1$. Or, le discriminant signé l'est:

1.3. Faits. i) $f \sim g \Rightarrow d_\pm(f) = d_\pm(g)$.
ii) $d_\pm(f \oplus g) = (-1)^{dim(f) dim(g)} d_\pm(f) d_\pm(g)$.

Preuve. Pour (ii) utiliser: $\frac{(m+n)(m+n-1)}{2} = \frac{m(m-1)}{2} + \frac{n(n-1)}{2} + mn$.
(i) suit du fait que $d_{\pm}(< 1, -1 >) = 1$. □

Alors l'application $\partial : W(F) \longrightarrow G(F)$ donnée par:

$$\partial(f/_{\sim}) = \overline{d_{\pm}(f)},$$

est bien définie et $\partial(0) = 1$. Evidemment, ∂ n'est pas un homomorphisme de groupes entre $< W(F), \oplus >$ et $< G(F), \cdot >$ (par exemple, $\partial(< 1 >/_{\sim}) = 1$, tandis que $\partial(< 1, 1 >/_{\sim}) = -1$), mais sa restriction $\partial' = \partial\lceil< I(F), \oplus >$ l'est par 1.3(ii). En plus:

Proposition 1.4. *L'application ∂' est un homomorphisme de $< I(F), \oplus >$ sur $< G(F), \cdot >$, et $ker(\partial') = I(F)^2$. Alors le groupe additif $I(F)/I(F)^2$ est isomorphe à $< G(F), \cdot >$.*

Preuve. La surjectivité suit de $d_{\pm}(< 1, -a >/_{\sim}) = a$ et $< 1, -a >/_{\sim} \in I(F)$, où $a \in F^{\times}$. Par ailleurs, $I(F)$ est engendré par les formes $< 1, g >/_{\sim}$, $g \in G(F)$, car $< g_1, g_2 >/_{\sim} = < 1, g_1 >/_{\sim} \oplus < -1, g_2 >/_{\sim}$. Donc, $I(F)^2$ est additivement engendré par les formes du type $< 1, g_1 >/_{\sim} \otimes < 1, g_2 >/_{\sim} = < 1, g_1, g_2, g_1g_2 >/_{\sim}$ (appelées *formes de Pfister* de degré 2). Comme

$$d_{\pm}(< 1, a_1, a_2, a_1a_2 >) = 1 \qquad (a_1, a_2 \in F^{\times}),$$

on a $ker(\partial') \supseteq I(F)^2$ (1.3(ii)). En particulier, ∂' induit un homomorphisme surjectif $\partial'' : I(F)/I(F)^2 \longrightarrow G(F)$.

Réciproquement, pour prouver que ∂'' est injectif –et alors que $ker(\partial') = I(F)^2$ – il suffit de vérifier que ∂'' a une inverse à droite surjective; celle-ci est donnée par:

$$\gamma(\overline{a}) = < 1, -a >/_{\sim}/I(F)^2.$$

Cette application est bien définie, car $< 1, -1 >/_{\sim}$ est l'unité additive de $W(F)$. Elle est surjective, car $I(F)$ est engendré par les formes $< 1, a >/_{\sim}$. Finalement, $d_{\pm}(< 1, -a >) = a$ entraîne que $\partial'' \circ \gamma =$ identité. □

Corollaire 1.5. *Soit f une forme sur F. Alors:*
(a) $f/_{\sim} \in I(F)^2$ ssi $4|dim(f)$ et $d_{\pm}(f) \in F^{\times^2}$.
(b) Si $f/_{\sim} \in I(F)$, alors il existe une forme g sur F telle que $g/_{\sim} \in I(F)^2$ et $f \sim g \oplus < 1, d_{\pm}(f) >$.

Démonstration. (a) traduit le fait que $ker(\partial') = I(F)^2$ (notez que $I(F)$ consiste des classes de toutes les formes de dimension paire).
(b) La preuve de la Proposition 1.4 montre que γ est aussi inverse à gauche de ∂'', i.e. $\gamma \circ \partial'' =$ identité. L'énoncé (b) traduit précisement cette propriété. □

Maintenant nous sommes prêts à prouver:

Fait 1.6. $< W(F), G(F) > \models W_2$.

Preuve. (W_2.i) D'après la définition de l'équivalence de Witt (voir 2.3(9)), il est clair que

$f \sim g$ implique $dim(f) \equiv dim(g) \pmod{2}$. Comme les éléments de $I(F)$ correspondent aux formes de dimension paire et ceux de $G(F)$ aux formes de dimension un, il s'ensuit que $G(F) \cap I(F) = \emptyset$.
(W_2.ii) Soit $f/_\sim \in I(F)^2 \cap (G(F) + G(F))$; alors $f \sim < \overline{a}, \overline{b} >$, avec $a, b \in F^\times$. Par la Proposition 1.4, $\partial'(f/_\sim) = 1$, d'où $d_\pm(f) = -ab \in F^{\times^2}$, et $b = -a \pmod{F^{\times^2}}$. On a donc $f \sim < \overline{a}, \overline{-a} > \equiv < 1, -1 >$, i.e. $f/_\sim = 0$. □

Signalons à titre d'information générale quelques résultats majeurs qui peuvent être prouvés dans le cadre des anneaux de Witt abstraits (les références renvoient à Marshall [6] où ces résultats sont démontrés en détail):
– Le principe local-global de Pfister (Ch.4, §4).
– La détermination de tous les idéaux premiers et du nil-radical de n'importe quel anneau de Witt abstrait en termes de son espace de signatures $X_R = Hom(R, \mathbb{Z})$ (Ch.4, §5).
– La détermination du groupe multiplicatif $R^\times$ des unités de R en termes de G et du nil-radical (Ch. 4, §6).
–Une classification partielle des anneaux de Witt abstraits de type fini (Ch. 5) ainsi qu'une classification complète des anneaux de Witt abstraits *réduits* de type fini (Ch. 6).
– Le théorème de représentation, qui caractérise R comme sous-anneau de l'anneau des fonctions continues sur X_R à valeurs entiers, $\mathcal{C}(X_R, \mathbb{Z})$ (Ch.7).
– Le principe local-global d'isotropie (Ch. 9).

2 Groupes spéciaux.

Comme on vient de le signaler, les anneaux de Witt abstraits constituent une généralisation axiomatique assez riche de la théorie des formes quadratiques sur les corps. Néanmoins, il y a dans cette théorie un point peu satisfaisant, au moins du point de vue du logicien intéressé par la perspective d'utiliser les outils de la théorie des modèles dans cette branche de l'algèbre; à savoir, l'axiome (W_1) est irrémédiablement non-premier-ordre dans n'importe quel langage adapté aux anneaux de Witt abstraits. Bien sûr, cette difficulté ne nous empêche pas, a priori, de poser des questions de type modèle-théorique sur la classe des anneaux de Witt abstraits ou sur certaines de ses sous-classes, mais elle peut être également une source de problèmes dans les investigations de cette nature.

Nous allons remédier à cet inconvénient en introduisant une théorie abstraite des formes quadratiques qui est axiomatisée par un ensemble fini d'énoncés simples d'un langage mathématiquement naturel. Il s'avère, en outre, que cette théorie est équivalente, dans un sens assez fort, à celle des anneaux de Witt abstraits. Nous appelons cette axiomatisation *théorie des groupes spéciaux.*

L'introduction de cette théorie est motivée par deux observations très simples:
(1) Au lieu de considérer l'anneau de toutes les formes quadratiques modulo l'équivalence de Witt, on peut essayer d'axiomatiser la relation d'isométrie des formes sur un groupe arbitraire de coefficients, G, d'exposant 2.
(2) En utilisant la description inductive de l'isométrie (voir p. 3 plus haut) –description donnée par des énoncés du premier ordre au-dessus du groupe G– nous sommes amenés à donner une axiomatisation de l'isométrie pour les seules formes binaires.

Remarquablement, il s'avère que ce dernier problème a une solution simple et élégante.

Définition 2.1. (a) Un *groupe spécial* est une structure $< G, -1, \equiv >$, où:
(1) G est un groupe d'exposant 2;
(2) -1 est un élément distingué de G;
(3) $\equiv$ est une relation quaternaire sur G;
(4) Les axiomes $\mathcal{SG}_0 - \mathcal{SG}_6$ donnés dans l'article [4] (ce volume) sont satisfaits.
(b) On appele *langage des groupes spéciaux* le langage $\text{L} = \{\cdot, 1, -1, \equiv\}$, où $\cdot$ est un symbole d'opération binaire, 1, -1 des constantes individuelles, et $\equiv$ un symbole de relation quaternaire. On appele **SG** la théorie du premier ordre de langage L engendrée par les axiomes $\mathcal{SG}_0 - \mathcal{SG}_6$. □

2.2. Remarques (i) On aurait pu faire une présentation alternative en prenant comme primitive la relation binaire $R(a, b)$ qui exprime la notion "a est représenté par la forme $< 1, b >$" ($a \in D(1, b)$, dans la terminologie habituelle de la théorie des formes quadratiques), au lieu de la relation quaternaire $\equiv$. La différence est minimale (plûtot psychologique), chacune de ces relations étant définissable sans quantificateurs en termes de l'autre:

$$\begin{array}{rcl} R(a,b) & ssi & < a, ab > \equiv < 1, b >, \\ < a, b > \equiv < c, d > & ssi & ab = cd \ \wedge \ R(ac, cd). \end{array}$$

Les axiomes $\mathcal{SG}_0 - \mathcal{SG}_6$ peuvent être traduits facilement en des axiomes équivalents exprimés en termes de la relation binaire R (cf. [2; pp. 183, 186]).
(ii) Remarquez que la théorie **SG** est donnée par des axiomes universels-existentiels ($\mathcal{SG}_0 - \mathcal{SG}_5$ sont universels; seul $\mathcal{SG}_6$ est $\forall\exists$). Il s'ensuit que **SG** est close par limites directs filtrants. □

Dans ce qui suit nous indiquons la construction de l'anneau de Witt d'un groupe spécial et prouvons l'équivalence entre la catégorie des anneaux de Witt abstraits et celle des groupes spéciaux.

2.3. L'anneau de Witt d'un groupe spécial.

A chaque groupe spécial $< G, -1, \equiv_G >$ on associe un anneau de Witt abstrait $W(G)$. La construction de $W(G)$ s'effectue selon une procedure bien connue; nous indiquons sans preuve les pas à suivre, en renvoyant à Marshall [6; Ch. 2] pour plus de détails.
(1) A partir de la relation $\equiv$ on définit, par récurrence sur $n \geq 3$, une relation binaire ("isométrie") entre n-uplets d'éléments de G (que nous appelerons "formes quadratiques de dimension n sur G"):

$$< a_1, \ldots, a_n > \equiv_G^n < b_1, \ldots, b_n > \text{ ssi } \ \text{il existe } a, \ b, \ c_3, \ \ldots \ , c_n \text{ tels que} \\ \begin{array}{rcl} < a_1, a > & \equiv & < b_1, b >, \\ < a_2, \ldots, a_n > & \equiv_G^{n-1} & < a, c_3, \ldots, c_n >, \\ < b_2, \ldots, b_n > & \equiv_G^{n-1} & < b, c_3, \ldots, c_n > . \end{array}$$

(2) Pour tout $n \geq 2$, $\equiv_G^n$ est réflexive et symétrique (pour $n = 2$, c'est l'axiome $\mathcal{SG}_0$; récurrence, pour $n \geq 3$).
(3) Permutabilité de $\equiv_G^n$: si π est une permutation de $\{1, \ldots, n\}$, alors $< a_1, \ldots, a_n > \equiv_G^n$

$< a_{\pi(1)}, \ldots, b_{\pi(n)} >$ (pour $n = 2$ c'est $\mathcal{SG}_1$; récurrence, pour $n \geq 3$; cf. [6; Prop. 2.1]).
(4) $f \equiv_G^n g \Longrightarrow d(f) = d(g)$;
(le discriminant est défini comme en p. 3 ci-dessus; pour $n = 2$ c'est $\mathcal{SG}_3$; récurrence pour $n \geq 3$; cf. [6; Prop. 2.2]).
(5) $f \equiv_G^n g \Longrightarrow af \equiv_G^n ag$;
($a \cdot < a_1, \ldots, a_n > = < aa_1, \ldots, aa_n >$; pour $n = 2$ c'est $\mathcal{SG}_5$; récurrence pour $n \geq 3$; cf. [6; Prop. 2.4]).
(6) Transitivité de $\equiv_G^n$ pour tout $n \geq 3$.
Pour $n = 2$ c'est $\mathcal{SG}_0$. Le cas crucial ici est $n = 3$, où la transitivité est assurée par $\mathcal{SG}_6$. Récurrence pour $n \geq 4$.
Nota. Celui-ci est le seul résultat délicat. Il est utile de comparer avec le cas des corps, où la transitivité de l'isométrie est banale, tandis que le théorème de simplification est délicat.
(7) Définition (évidente) de la somme et du produit de formes quadratiques. On prouve sans difficulté les résultats suivants par récurrence sur k pour n fixe:

i) $g \equiv_G^n g' \Longleftrightarrow f \oplus g \equiv_G^{n+k} f \oplus g'$.
ii) $f \equiv_G^k f' \;\wedge\; g \equiv_G^n g' \Longrightarrow f \oplus g \equiv_G^{n+k} f' \oplus g'$.
iii) $f \equiv_G^k f' \;\wedge\; g \equiv_G^n g' \Longrightarrow f \otimes g \equiv_G^{n \cdot k} f' \otimes g'$.

(Cf. [6; Prop. 2.7-2.9]). L'implication ($\Longleftarrow$) dans (i) est le *théorème de simplification de Witt.*
(8) Définition (habituelle) de *représentation* et de *forme isotrope*:

$$a \in D_G(< a_1, \ldots, a_n >) \quad ssi \quad \exists x_2, \ \ldots \ , x_n(< a_1, \ldots, a_n > \equiv_G^n < a, x_2, \ldots, x_n >).$$

Pour une forme f de dimension $n \geq 2$:

f est *isotrope* ssi il existe une forme g, $dim(g) = n - 2$, telle que $f \equiv_G^n g \oplus < 1, -1 >$.

On prouve sans difficulté:

i) $D(f \ \oplus \ g) = \bigcup \{D(< x, y >) \mid x \in D(f) \ et \ y \in D(g)\}$.
ii) $f \ \oplus \ g$ isotrope ssi il existe $x \in D(f)$ tel que $-x \in D(g)$.

(Récurrence sur k pour n fixe, où $dim(f) = k$, $dim(g) = n$; cf. [6; 2.10 et 2.12].)

(9) Définition (habituelle) de l'*équivalence de Witt*:

$f \sim g$ ssi il existe $k, l \in \mathbb{N}$ tels que $f \ \oplus \ k < 1, -1 > \ \equiv \ g \ \oplus \ l < 1, -1 >$,

où $\equiv$ désigne "$\equiv_G^n$" pour un n convenable (desormais nous omettons le "n" dans $\equiv_G^n$, sauf en cas de besoin).
On prouve facilement que "$\sim$" est une relation d'équivalence compatible avec les opérations $\oplus$ et $\otimes$; donc, l'ensemble $\bigcup_{n \in \mathbb{N}} G^n/_\sim$ muni des opérations induites par $\oplus$ et $\otimes$ est un anneau commutatif ayant $< 1, -1 >/_\sim$ comme unité additive et $< 1 >/_\sim$ comme unité multiplicative. On l'appelle l'*anneau de Witt de G*, noté $W(G)$.

Proposition 2.4. *$< W(G), G >$ est un anneau de Witt abstrait.*

(On identifie G à un sous-ensemble de $W(G)$ par l'application $g \mapsto < g >/_{\sim}$.)

Démonstration. (W_1) est clair car toute forme est somme de formes unaires. (W_3) est évident d'après la définition de $\equiv_G^n$ (description inductive de l'isométrie). Pour vérifier (W_2) il suffit de répeter la preuve de 1.3, 1.4 et 1.6, en utilisant $\mathcal{SG}_2$ dans le dernier pas de 1.6. □

Réciproquement, on a:

Proposition 2.5. *A tout anneau de Witt abstrait, $< R, G_R >$, on peut associer un groupe spécial, $< G_R, -1, \equiv_R >$, où -1 est le -1 de R, et pour $a,\ b,\ c,\ d\ \in G_R$:*

$$< a, b > \equiv_R < c, d > \quad \Longleftrightarrow \quad a + b = c + d \quad (dans\ R).$$

Démonstration. La vérification des axiomes $\mathcal{SG}_0 - \mathcal{SG}_6$ se fait sans difficulté sauf, peut-être, dans les cas suivants:
$\mathcal{SG}_3$) Supposons $< a_1, a_2 > \equiv_R < b_1, b_2 >$, i.e. $a_1 + a_2 = b_1 + b_2$. On calcule:

$$(a_1 - a_2)(a_1 + a_2) = a_1(a_1 + a_2) - b_1(b_1 + b_2) = a_1^2 + a_1a_2 - b_1^2 - b_1b_2 = a_1a_2 - b_1b_2.$$

Donc, $a_1a_2 - b_1b_2 \in I_R^2 \cap (G + G)$; par ($W_2$.ii), on a: $a_1a_2 = b_1b_2$.
$\mathcal{SG}_6$) En utilisant (W_3) pour l'implication ($\Longleftarrow$) et la définition inductive de $\equiv_G^n$ pour ($\Longrightarrow$), on a:

$$< a_1, \ldots, a_n > \equiv_G^n < b_1, \ldots, b_n > \quad \Longleftrightarrow \quad a_1 + \cdots + a_n = b_1 + \cdots + b_n,$$

pour $n \geq 3$; alors, $\equiv_G^n$ est transitive. □

Les notions naturelles de morphisme pour les anneaux de Witt abstraits et pour les groupes spéciaux sont les suivantes:

Définition 2.6. (a) Un *homomorphisme d'anneaux de Witt abstraits,* $\varphi :< R_1, G_1 > \longrightarrow < R_2, G_2 >$, est un homomorphisme d'anneaux unitaires $\varphi : R_1 \longrightarrow R_2$ tel que $\varphi[G_1] \subseteq G_2$.
(b) Un *homomorphisme de groupes spéciaux,* $\psi :< G_1, -1, \equiv_{G_1} > \longrightarrow < G_2, -1, \equiv_{G_2} >$, est un homomorphisme de groupes $\psi : G_1 \longrightarrow G_2$ tel que $\psi(-1) = -1$, et pour $a,\ b,\ c,\ d \in G_1$:

$$< a, b > \equiv_{G_1} < c, d > \quad \Longrightarrow \quad < \psi(a), \psi(b) > \equiv_{G_2} < \psi(c), \psi(d) > . \qquad \square$$

Les classes des anneaux de Witt abstraits et des groupes spéciaux, munies des notions de morphismes que nous venons de définir, constituent des catégories que nous appelerons **AWA** et **SG**, respectivement. Nous allons prouver ensuite que les constructions de 2.3 et 2.5 donnent lieu à une équivalence entre ces catégories.

2.7. Notations. (a) On désigne par Ψ : **AWA** $\longrightarrow$ **SG** le morphisme qui à chaque anneau de Witt abstrait associe le groupe spécial construit dans la Proposition 2.5:

$$\Psi(< R, G_R >) = < G_R, -1, \equiv_R >,$$

et qui à chaque homomorphisme d'anneaux de Witt $\varphi :< R_1, G_1 > \longrightarrow < R_2, G_2 >$, fait correspondre l'homomorphisme de groupes spéciaux $\Psi(\varphi) :< G_1, -1, \equiv_{R_1}> \longrightarrow < G_2, -1, \equiv_{R_2}>$ défini par:

$$\Psi(\varphi)(g) = \varphi(g) \qquad (g \in G_1).$$

(b) Réciproquement, on désigne par Φ : **SG** $\longrightarrow$ **AWA** le morphisme qui à chaque groupe spécial associe son anneau de Witt:

$$\Phi(< G, -1, \equiv_G>) = < W(G), G >,$$

et qui à chaque homomorphisme $\psi :< G_1, -1, \equiv_{G_1}> \longrightarrow < G_2, -1, \equiv_{G_2}>$ de groupes spéciaux fait correspondre l'homomorphisme d'anneaux de Witt abstraits $\Phi(\psi) : < W(G_1), G_1 > \longrightarrow < W(G_2), G_2 >$ donné par:

$$\Phi(\psi)(< g_1, \ldots, g_n >/_\sim) = < \psi(g_1), \ldots \psi(g_n) >/_\sim \qquad \text{pour } g_1, \ldots, g_n \in G_1.$$

[$\Phi(\psi)$ est bien défini: ψ respecte la relation $\equiv$; par récurrence, il respecte aussi la relation $\equiv^n$ pour tout $n \geq 2$; comme il respecte 1 et -1, il respecte aussi l'équivalence de Witt. $\Phi(\psi)$ est également un homomorphisme d'anneaux de Witt abstraits.] □

Théorème 2.8. *Le foncteur Ψ établit une équivalence entre les catégories* **AWA** *et* **SG**. *Le foncteur Φ est son adjoint.*

Avant de faire la preuve de ce théorème démontrons que la construction donnée dans la Proposition 2.5 suivie de celle de 2.3 aboutit à un anneau canoniquement isomorphe à celui de départ.

Lemme 2.9. *Soit $< R, G_R >$ un anneau de Witt abstrait. Alors $< W(G_R), G_R > \cong < R, G_R >$ canoniquement, par l'application $\rho : W(G_R) \longrightarrow R$ définie par:*

$$\rho(< g_1, \ldots, g_n >/_\sim) = g_1 + \cdots + g_n \ \ (\textit{dans } R), \qquad (g_1, \ldots, g_n \in G_R).$$

Avec la notation de 2.7: $\Phi(\Psi(< R, G >)) \cong_\rho < R, G >$.

Démonstration. (i) ρ est bien définie, i.e.

$$(*) \qquad < g_1, \ldots, g_n > \sim_R < g_1', \ldots, g_m' > \Longrightarrow g_1 + \cdots + g_n = g_1' + \cdots + g_m' \quad (\textit{dans } R).$$

D'après la définition du côté gauche, voir 2.3(9), il suffit de prouver cette implication lorsque $m = n$ et $< g_1, \ldots, g_n > \equiv_R < g_1', \ldots, g_n' >$. Ceci se fait par récurrence sur n; pour $n = 1$ il n'y a rien à prouver; pour $n = 2$ c'est la définition de $\equiv_R$; pour $n \geq 3$ utilisez la définition inductive de $\equiv_R^n$.
(ii) ρ est un homomorphisme surjectif d'anneaux de Witt abstraits:
par $\rho(< g >/_\sim) = g$ pour $g \in G_R$, et l'axiome (W_1).
(iii) ρ est injective, i.e. la réciproque de (*) est vraie.
D'abord on observe:

$$(**) \qquad g_1 + \cdots + g_n = 0 \ \text{ avec } \ g_1, \ldots, g_n \in G_R \quad \Longrightarrow \quad n \ \text{pair}.$$

Donc, $g_1 + \cdots + g_n = g'_1 + \cdots + g'_m$ implique $n \equiv m$ (*mod.* 2). Si, par exemple, $n < m$, on a l'égalité

$$g_1 + \cdots + g_n + (1 + -1) + \cdots + (1 + -1) = g'_1 + \cdots + g'_n,$$

avec $(m-n)/2$ termes $(1+-1)$ et le même nombre de termes des deux côtés. Alors, on peut supposer $n = m$; par récurrence sur $n \geq 1$ on montre que $< g_1, \ldots, g_n > \equiv_R^n < g'_1, \ldots, g'_n >$. Le cas $n = 1$ est gratuit, et le cas $n = 2$ est la définition de $\equiv_R$. Pour $n \geq 3$ utiliser l'axiome (W_3), le cas $n = 2$ et l'hypothèse de récurrence.

Preuve de (**). On doit prouver que $g_1 + \cdots + g_n \neq 0$ pour n impair ($n = 2k+1$, disons) et $g_1, \ldots, g_n \in G_R$. Si $k = 0$ c'est clair, car $0 \notin G_R$ ($G_R \subseteq R^\times$). Si $k \geq 1$ et $g_1 + \cdots + g_n = 0$, on a $-g_n = (g_1 + g_2) + (g_3 + g_4) + \ldots + (g_{2k-1} + g_{2k})$. Or, le côté gauche est dans G_R, tandis que le côté droit est dans I_R, ce qui contredit l'axiome (W_2.i). □

Démonstration du Théorème 2.8. D'après le Théorème 1, §4, Ch.IV de Mac Lane [5] les points (1)-(3) ci-dessous prouvent que le foncteur Ψ est une équivalence de catégories. On laisse la preuve d'adjonction en exercise.

(1) Pour tout $< G, -1, \equiv_G > \models$ **SG** il existe $< R, H > \models$ **AWA** tel que $\Psi(< R, H >) = < G, -1, \equiv_G >$.
Preuve. Prendre $R = W(G)$ et $H = G$.

(2) Ψ est plein, i.e. étant donnés $< R_i, H_i > \models$ **AWA** $(i = 1, 2)$ et un homomorphisme de groupes spéciaux $\psi : \Psi(< R_1, G_1 >) \longrightarrow \Psi(< R_2, G_2 >)$, il existe un homomorphisme d'**AWA**, $\varphi :< R_1, G_1 > \longrightarrow < R_2, G_2 >$, tel que $\Psi(\varphi) = \psi$.
Preuve. D'après 2.7(b) l'homomorphisme ψ induit un homomorphisme d'**AWA**

$$\Phi(\psi) : W(\Psi(< R_1, G_1 >)) \longrightarrow W(\Psi(< R_2, G_2 >)).$$

Soit $\rho_i : W(\Psi(< R_i, G_i >)) \longrightarrow R_i$ $(i = 1, 2)$ l'homomorphisme donné par le Lemme 2.9. On pose:

$$\varphi = \rho_2 \circ \Phi(\psi) \circ \rho_1^{-1} : < R_1, G_1 > \longrightarrow < R_2, G_2 > .$$

D'après 2.7(a), $\Psi(\varphi) = \psi$ est conséquence de $\varphi(g) = \psi(g)$ pour $g \in G_1$. Comme $\rho_i(< h >/_\sim) = h$ pour $h \in G_i$ $(i = 1, 2)$, et $\Phi(\psi)(< h >/_\sim) = < \psi(g) >/_\sim$ pour $g \in G_1$, on a:

$$\varphi(g) = (\rho_2 \circ \Phi(\psi))(\rho_1^{-1}(g)) = \rho_2(\Phi(\psi)(< g >/_\sim)) = \rho_2(< \psi(g) >/_\sim) = \psi(g).$$

(3) Ψ est fidèle, i.e. étant donnés $< R_i, H_i > \models$ **AWA** $(i = 1, 2)$ et des homomorphismes d'**AWA**, $\varphi_1, \varphi_2 :< R_1, G_1 > \longrightarrow < R_2, G_2 >$,

$$\Psi(\varphi_1) = \Psi(\varphi_2) \Longrightarrow \varphi_1 = \varphi_2.$$

Preuve. Soit $\psi_i = \Psi(\varphi_i) :< G_1, -1, \equiv_{G_1} > \longrightarrow < G_2, -1, \equiv_{G_2} >$ $(i = 1, 2)$. L'hypothèse $\psi_1 = \psi_2$ donne évidemment $\Phi(\psi_1) = \Phi(\psi_2)$. Il suffit, donc, de prouver:

(3') Etant donné un homomorphisme d'**AWA**, $\varphi :< R_1, G_1 > \longrightarrow < R_2, G_2 >$, soit $\psi =$

$\Psi(\varphi)$. Alors $\varphi = \rho_2 \circ \Phi(\psi) \circ \rho_1^{-1}$.

Preuve de (3'). D'après la définition de $\Psi(\varphi)$ (2.7(a)) on a $\varphi(g) = \psi(g)$ pour $g \in G_1$. Aussi on a

$$\Phi(\psi)(< g_1, \dots, g_n >/_\sim) = < \psi(g_1), \dots \psi(g_n) >/_\sim \qquad (2.7(b))$$

et

$$\rho_i(< g_1, \dots, g_n >/_\sim) = g_1 + \dots + g_n \qquad (g_1, \dots, g_n \in G_i) \qquad (2.9).$$

Si $r \in R_1$, alors $r = g_1 + \dots + g_n$ avec $g_1, \dots, g_n \in G_1$. Donc on a:

$$\begin{aligned}(\rho_2 \circ \Phi(\psi) \circ \rho_1^{-1})(r) &= (\rho_2 \circ \Phi(\psi))(\rho_1^{-1}(g_1 + \dots + g_n)) = \rho_2(\Phi(\psi)(< g_1, \dots, g_n >/_\sim)) = \\ &= \rho_2(< \psi(g_1), \dots \psi(g_n) >/_\sim) = \psi(g_1) + \dots + \psi(g_n) = \\ &= \varphi(g_1) + \dots + \varphi(g_n) = \varphi(r). \qquad \square\end{aligned}$$

Remarques. (a) Le point (3') de la preuve précedente montre, en fait, que le foncteur Φ est plein.

(b) Rappelons qu'un anneau est dit *réduit* si son nil-radical est $\{0\}$. Un groupe spécial est dit *réduit* s'il vérifie l'axiome:

$\mathcal{SG}_{red}$) $\quad \forall a(< a, a > \equiv < 1, 1 > \longrightarrow a = 1)$.

On déduit facilement de Marshall [6; Thm. 4.27] que:

$$< R, G > \models \textbf{AWA} \text{ réduit} \implies < G, -1, \equiv_R > \models \textbf{SG} \text{ réduit},$$

et

$$< G, -1, \equiv_G > \models \textbf{SG} \text{ réduit} \implies < W(G), G > \models \textbf{AWA} \text{ réduit}.$$

C'est-à-dire, les foncteurs Ψ, Φ, donnent aussi une équivalence entre les sous-catégories des **AWA** réduits et des **SG** réduits.

(c) Le Théorème 12 de [4] montre que la catégorie des groupes spéciaux réduits est isomorphe à la catégorie opposée de celle des espaces d'ordres abstraits. $\square$

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THEORIES ASSOCIATED WITH A VECTOR SPACE

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In 1988 at Durham conference on Model Theory and Groups Ulrich Felgner posed the following problem. Let F be a field. For which cardinals κ, κ' the groups $\mathrm{GL}(\kappa, F)$ and $\mathrm{GL}(\kappa', F')$ are elementarily equivalent? An analogous problem was posed for other classical groups.

The similar problem for symmetric groups was studied by Yu.M.Vazhenin and V.V.Rasin, R.McKenzie, A.G.Pinus and, finally, by S.Shelah who gave a criterion of elementary equivalence of the groups $\mathrm{Sym}(\kappa)$ and $\mathrm{Sym}(\kappa')$.

The aim of this talk is to report on a solution of Felgner's problem due to Vladimir Tolstykh, a research student of mine [T1,2]. Some intermediate and related results were obtained in [BT].

Let V be a vector space of infinite dimension κ over a skew field F. We consider the following structures associated with this vector space.

Denote by $\mathrm{GL}(V)$, $\mathrm{PGL}(V)$, $\Gamma\mathrm{L}(V)$, $\mathrm{P}\Gamma\mathrm{L}(V)$, $\mathrm{End}(V)$, $\mathit{End}(V)$ and $\mathrm{P}(V)$ the general linear group, the projective linear group, the collineation group, the projective collineation group, the endomorphism ring, the endomorphism semigroup and the lattice of subspaces of the vector space V, respectively.

Let $\mathcal{V}$ denote the vector space above considered as a two-sorted structure: its first sort is the set of vectors, its second sort is the set of scalars, and its basic relations are the vector addition, the field operations and the operations of multiplication of vectors by scalars.

Denote by $\langle \kappa, F \rangle$ the following two-sorted structure: its first sort is the cardinal κ, its second sort is the universe of F, and its only relations are the skew field operations on F.

For a structure $\mathfrak{A}$ and a logic L, we denote the theory of $\mathfrak{A}$ in the logic L by $Th(\mathfrak{A}, L)$. If L is the first order logic, we write simply $Th(\mathfrak{A})$.

For an infinite cardinal λ, we denote by $\mathrm{L}_2(\lambda)$ the second order logic with quantifiers over relations of power $< \lambda$; $\mathrm{Mon}(\lambda)$ is its monadic fragment.

We denote by $Th(\mathcal{V}, \mathrm{End})$ and $Th(\mathcal{V}, \mathrm{Sub})$ the theories of $\mathcal{V}$ in the second order logics with quantifiers over endomorphisms and subspaces of the vector space V, respectively.

Theorem 1. *The following theories are syntactically bi-interpretable (uniformly in κ and F):*

a) $Th(\mathrm{GL}(V))$,
b) $Th(\mathrm{PGL}(V))$,
c) $Th(\mathrm{End}(V))$,
d) $Th(\mathit{End}(V))$,
e) $Th(\mathrm{P}(V))$,

f) $Th(\mathcal{V}, \mathrm{End})$,
g) $Th(\mathcal{V}, \mathrm{Sub})$,
h) $Th(\mathcal{V}, \mathrm{Mon}(\kappa^+))$,
i) $Th(\mathcal{V}, \mathrm{L}_2(\kappa^+))$,
j) $Th(\langle\kappa, F\rangle, \mathrm{L}_2(\kappa^+))$.

Note that the bi-interpretability of $Th(\mathrm{P}(V))$ and $Th(\langle\kappa, F\rangle, \mathrm{L}_2(\kappa^+))$ improves the following result of M.Magidor, J.Rosenthal, M.Rubin and G.Srour [MRRS]: for any infinite field F, the second order theory on the cardinal $\min(|F|, \kappa)$ is syntactically interpretable in $Th(\mathrm{P}(V))$.

Let V and V' be vector spaces of infinite dimensions κ and κ' over skew fields F and F', respectively. Let H be one of the functors GL, PGL, End, *End*, P. Denote $T(\kappa, F) = Th(\langle\kappa, F\rangle, \mathrm{L}_2(\kappa^+))$. As a consequence of Theorem 1 we have

Corollary. *$H(V) = H(V')$ iff $T(\kappa, F) = T(\kappa', F')$.*

This gives a satisfactory criterion of elementary equivalence of structures of the form $H(V)$ and, in particular, a solution of Felgner's problem.

Denote the condition $T(\kappa, F) = T(\kappa', F')$ by $(\star)$. Note that it is "algebra-free" as far as possible and in many cases can be easily checked.

Clearly, $(\star)$ implies that
(1) $\kappa = |F|$ iff $\kappa' = |F'|$;
(2) $\kappa > |F|$ iff $\kappa' > |F'|$;
(3) $\kappa < |F|$ iff $\kappa' < |F'|$;
(4) κ and κ' are second order equivalent;
(5) $Th(\langle\kappa, F\rangle, \mathrm{L}_2(\kappa^+)) = Th(\langle\kappa', F'\rangle, \mathrm{L}_2(\kappa'^+))$.

If $\kappa \geq |F|$, the condition $(\star)$ is exactly the second order equivalence of the structures $\langle\kappa, F\rangle$ and $\langle\kappa', F'\rangle$.

If $\kappa = |F|$ then $(\star)$ holds iff $\kappa' = |F'|$ and F, F' are second order equivalent.

If $\kappa \leq |F|$ then $(\star)$ holds iff $\kappa' \leq |F'|$ and $Th(F, \mathrm{L}_2(\kappa^+)) = Th(F', \mathrm{L}_2(\kappa'^+))$.

If $\kappa \geq |F|$ and F can be described up to isomorphism by a single second order sentence then $(\star)$ holds iff $\kappa' \geq |F'|$, $F \simeq F'$ and κ and κ' are second order equivalent. (Examples of such F are **Q**, **R**, **C** and finite fields.) In the case of $\kappa < |F|$ the criterion does not work. For example, $T(\kappa, F) = T(\aleph_0, \mathbf{R})$ iff $\kappa = \aleph_0$ and $F \simeq \mathbf{R}$; however $T(\kappa, F) = T(\aleph_0, \mathbf{C})$ iff $\kappa = \aleph_0$ and F is an uncountable algebraically closed field of zero characteristic.

For the groups $\Gamma\mathrm{L}(V)$ and $\mathrm{P}\Gamma\mathrm{L}(V)$, the situation is more complicated.

Let $\mathfrak{T}(\kappa, F)$ be the theory of the structure $\langle\kappa, F\rangle$ in the second order logic with quantifiers over arbitrary relations of power $\leq \kappa$ and over automorphisms of F. If $\kappa \geq |F|$, the theory $\mathfrak{T}(\kappa, F)$ is in fact the same as $T(\kappa, F)$. For $\kappa < |F|$, the theory $\mathfrak{T}(\kappa, F)$ is in general stronger than $T(\kappa, F)$: for example, $\mathfrak{T}(\kappa, F) = \mathfrak{T}(\aleph_0, \mathbf{C})$ iff $\kappa = \aleph_0$ and $F \simeq \mathbf{C}$, in contrast with the remark above.

Theorem 2. *The theories $Th(\Gamma\mathrm{L}(V))$, $Th(\mathrm{P}\Gamma\mathrm{L}(V))$ and $\mathfrak{T}(\kappa, F)$ are pairwise syntactically bi-interpretable.*

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Espaces d'Ordres Abstraits

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Sommaire.

Les espaces d'ordres abstraits ont été introduits à la fin des années 70 (plus précisément en 76) par M. Marshall dans son article "A Reduced Theory of Quadratic Forms", Queens paper n°46. Depuis, toute une théorie fût développée. Elle toucha aussi bien l'étude des formes quadratiques associées aux anneaux de Witt que l'étude de la théorie des espaces d'ordres d'un corps. Les résultats les plus importants de cette théorie, obtenus essentiellement par M. Marshall, sont donnés ci-dessous. L'étude de l'axiome $\mathcal{O}_4$, ainsi que certaines démonstrations sont des résultats de l'auteur.

Introduction.

On commence par introduire la notion d'espace d'ordres abstrait. Ensuite on étudie l'axiome $\mathcal{O}_4$ qui apparaît dans la définition de ces espaces. On en donne deux formulations équivalentes fortes utiles en pratique (Proposition 6). Dans la section 'Structures Associées à un Espace d'Ordres', on introduit très brièvement quelques constructions concernant les espaces d'ordres ; nous nous bornons aux constructions principales ; d'autres peuvent être trouvées dans les références.

Définition 1: *Soient G un groupe multiplicatif d'exposant 2 ($x^2 = 1,\ \forall x \in G$) avec un élément distingué -1 et $X \subseteq \mathcal{X}(G) = Hom(G, \{\pm 1\})$. Le couple (X, G) est un* **espace d'ordres abstrait** *si les axiomes suivants sont vérifiés :*

$\mathcal{O}_1$) *X est un sous ensemble fermé de $\mathcal{X}(G)$.*

$\mathcal{O}_2$) $\sigma(-1) = -1,\ \forall \sigma \in X$.

$\mathcal{O}_3$) $\bigcap_{\sigma \in X} Ker\sigma = \{-1\}$.

$\mathcal{O}_4$) *Si f et g sont deux formes quadratiques sur G, alors :*

$$\underset{X}{D} < f \oplus g > = \{\underset{X}{D} < x, y >;\ x \in \underset{X}{D} < f > \ \text{et}\ y \in \underset{X}{D} < g > \}$$

(Voir définition de $\underset{X}{D} < f >$ dans la remarque (2,4) ci-dessous.)

Remarques 2:

1) La topologie de $\mathcal{X}(G)$ est la plus petite qui rend continues toutes les fonctions de la forme :

$$\begin{aligned} \widehat{a}\ :\ \mathcal{X}(G) &\longrightarrow \{1, -1\} \\ \sigma &\longmapsto \sigma(a) \end{aligned}$$

L'ensemble $\{1, -1\}$ est muni de la topologie discrète. Une sous-base de cette topologie est donnée par les ensembles $H(a, z) = \{\sigma \in \mathcal{X}(G),\ \sigma(a) = z\}$ où $a \in G$ et $z \in \{1, -1\}$. Avec cette topologie, $\mathcal{X}(G)$ est un groupe topologique compact et totalement discontinu.

2) Une **forme quadratique de dimension n** sur G est un n-uplet $f =< a_1, .., a_n >$, avec $a_1, .., a_n \in G$. Soit $g =< b_1, .., b_m >$; on définit les opérations de somme et produit comme suit :

$$\begin{cases} f \oplus g =< a_1, .., a_n, b_1, ..b_m > \\ f \otimes g =< a_1 b_1, ..., a_1 b_m, ..., a_n b_1, ..., a_n b_m > \\ af =< a > \otimes f =< aa_1, ..., aa_n >, \ \forall a \in G. \end{cases}$$

3) Etant données deux formes quadratiques $f =< a_1, .., a_n >$ et $g =< b_1, .., b_m >$ sur G, on définit la relation d'**isométrie** ainsi :

$$f \underset{X}{\equiv} g \ \Leftrightarrow \ n = m \quad et \quad \sum_{i=1}^{n} \sigma(a_i) = \sum_{i=1}^{n} \sigma(b_i), \ \forall \sigma \in X$$

Cette relation est clairement une relation d'équivalence.

4) Si $f =< a_1, ..., a_n >$, $a_i \in G$, on définit l'**ensemble des éléments représentés** par f comme suit :

$$\underset{X}{D} < f >= \{b \in G;\ \exists\ b_2, ..., b_n \in G;\ f \underset{X}{\equiv} < b, b_2, ..., b_n >\}$$

Exemple 3: L'exemple le plus naturel (il sert comme motivation) d'un espace d'ordres abstrait est l'espace d'ordres d'un corps formellement réèl. C'est-à-dire, si F est un corps formellement réèl, on prend :

$$G_F = \dot{F}/\Sigma\dot{F}^2 \quad \text{et} \quad \mathcal{X}(F) = \{Sgn_P \ ;\ P \text{ est ordre de } F\}, \quad \text{où}$$

$$Sgn_P : \ G_F \longmapsto \{+1, -1\} \text{ est : } Sgn_P(\dot{a}/\Sigma\dot{F}^2) = \begin{cases} 1 & \text{si } a \in P \\ -1 & \text{si } a \notin P \end{cases} \text{ pour } a \in \dot{F}.$$

Alors, $(\mathcal{X}(F),\ G_F)$ est un espace d'ordres abstrait.

La preuve des axiomes $\mathcal{O}_1 - \mathcal{O}_3$ est élémentaire, si on utilise la théorie d'Artin-Schreier. Par contre, pour prouver l'axiome $\mathcal{O}_4$ on fait appel au **principe local-global de Pfister** (voir, par exemple, Lam $[L_1]$, ch.8, §4, ou pour une version abstraite, Marshall $[M_6]$, thm. 4.12).

Exemple 4: Considérons (R, G_R) un anneau de Witt abstrait réduit (voir $[M_6]$, ch. IV) et soit $\mathcal{X}(R) = Hom(R, \mathbb{Z})$. Alors $(\mathcal{X}(R), G_R)$ est un espace d'ordres abstrait.

L'axiome $\mathcal{O}_4$

On donne ici une caractérisation de l'axiome $\mathcal{O}_4$, qui est sûrement l'axiome le moins clair de tous ceux qui apparaissent dans la définition des espaces d'ordres. Le problème, pour obtenir une forme plus "parlante" de l'axiome $\mathcal{O}_4$ avait été posé par plusieurs spécialistes en formes quadratiques. Or, je me suis apperçue en étudiant les groupes spéciaux (voir l'article suivant) que l'axiome $\mathcal{O}_4$ équivaut en fait, à identifier la relation d'**Isométrie**, voir remarque 2.3, et celle d'**Isométrie Forte**, dont la définition est donnée ensuite.

Définition 5: *Si (X,G) est un espace d'ordres abstrait, on définit par récurrence sur la dimension, la relation d'***isométrie forte** *pour les formes quadratiques sur G, comme suit :*

$$< a,b > \overset{*}{\underset{X}{\equiv}} < c,d > \iff < a,b > \underset{X}{\equiv} < c,d >$$

$$< a_1,..,a_n > \overset{*}{\underset{X}{\equiv}} < b_1,..,b_n > \iff \exists\ x,y,z_3,..z_n \in G \text{ tels que}$$

$$< a_1,x > \overset{*}{\underset{X}{\equiv}} < b_1,y >$$

$$< a_2,..,a_n > \overset{*}{\underset{X}{\equiv}} < x,z_3,..z_n >$$

$$< b_2,..,b_n > \overset{*}{\underset{X}{\equiv}} < y,z_3,..,z_n >$$

(Notez que la relation $\overset{*}{\underset{X}{\equiv}}$ n'est pas, à priori, transitive.)

Considérons l'axiome suivant :

$\mathcal{O}_5$) $\forall \sigma \in \mathcal{X}(G)$ tel que $\sigma(-1) = -1$, $\sigma \in X$ ssi $\forall\ a \in Ker\ \sigma$, $\underset{X}{D} < 1,a > \subseteq Ker\ \sigma$

Dans $[M_2]$ et $[M_3]$, Marshall a démontré que cet axiome est une conséquence de $\mathcal{O}_1$), $\mathcal{O}_2$) et $\mathcal{O}_4$).

Proposition 6: *Soit $X \subseteq \mathcal{X}(G)$ tel que $X \models \mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$; alors les propriétés suivantes sont équivalentes :*

i) $\underset{X}{\equiv} \iff \overset{*}{\underset{X}{\equiv}}$

ii) $X \models \mathcal{O}_4$

iii) $\overset{*}{\underset{X}{\equiv}}$ *est 3-transitive et* $X \models \mathcal{O}_5$

($\overset{*}{\underset{X}{\equiv}}$ 3-transitive veut dire qu'elle est transitive sur les formes de dimension 3.)

Démonstration :
(i)$\Rightarrow$(ii) On fait la récurrence sur $k = dim(f)$. Soient $f = < a_1,..,a_k >$, $g = < a_{k+1},..,a_n >$ et $b \in \underset{X}{D} < f \oplus g >$; alors :

$$\exists\ b_2,..,b_n,\ f\oplus g \underset{X}{\equiv} < b, b_2, .., b_n > \ ssi\ \exists\ x, y, z_3, .., z_n,\ \text{tels que :}$$

$$(1) < a_1, x > \overset{*}{\underset{X}{\equiv}} < b, y >$$

$$(2) < a_2, .., a_n > \overset{*}{\underset{X}{\equiv}} < x, z_3, ..z_n >$$

$$(3) < b_2, .., b_n > \overset{*}{\underset{X}{\equiv}} < y, z_3, .., z_n >$$

• $k = 1$.— Par (1), $b \in \underset{X}{D} < a_1, x >$ et par (2), $x \in \underset{X}{D} < g >$.

• $k > 1$.— Par (1), $b \in \underset{X}{D} < a_1, x >$ et par (2), $x \in \underset{X}{D}\ (< a_2, .., a_k > \oplus g)$. Donc par hypothèse de récurrence :

$$\exists\ c \in \underset{X}{D} < a_2, .., a_k > \ \text{ et } d \in \underset{X}{D} < g > \ \text{ tels que } x \in \underset{X}{D} < c, d >$$

Mais $b \in \underset{X}{D} < a_1, x > \subseteq \underset{X}{D} < a_1, c, d > = \underset{X}{D}\ (< d > \oplus < a_1, c >)$.
Donc, en utilisant le cas, $k = 1$ on a :

$$\exists\ e \in \underset{X}{D} < a_1, c >, \text{ tel que } b \in \underset{X}{D} < d, e > = \underset{X}{D} < e, d >,$$

$$\text{où } e \in \underset{X}{D} < a_1, c > \subseteq \underset{X}{D} < a_1, a_2, .., a_n > = \underset{X}{D} < f > \ \text{ et } d \in \underset{X}{D} < g >$$

(ii)⇒(i) Clairement, $\overset{*}{\underset{X}{\equiv}} \implies \underset{X}{\equiv}$. Réciproquement, si $< a_1, .., a_n > \underset{X}{\equiv} < b_1, .., b_n >$ et $n \geq 3$ alors $a_1 \in \underset{X}{D}\ (< b_1 > \oplus < b_2, .., b_n >)$. Donc, par $\mathcal{O}_4$, $\exists y \in \underset{X}{D} < b_2, .., b_n >$ tel que $a_1 \in \underset{X}{D} < b_1, y >$. Ceci équivaut à:

$$(1) \quad \exists\ x \in G,\ < a_1, x > \overset{*}{\underset{X}{\equiv}} < b_1, y >$$

Dès que $y \in \underset{X}{D} < b_2, .., b_n >$, on a:

$$(2) \quad \exists\ z_3, .., z_n \in G,\ < b_2, .., b_n > \underset{X}{\equiv} < y, z_3, .., z_n >,$$

et par hypothèse de récurrence $< b_2, .., b_n > \overset{*}{\underset{X}{\equiv}} < y, z_3, .., z_n >$. Alors:

$$\begin{array}{ll} < a_1, .., a_n > & \underset{X}{\equiv} < b_1, .., b_n > = < b_1 > \oplus < b_2, .., b_n > \\ \text{par } (2) & \underset{X}{\equiv} < b_1 > \oplus < y, z_3, .., z_n > = < b_1, y > \oplus < z_3, .., z_n > \\ \text{par } (1) & \underset{X}{\equiv} < a_1, x > \oplus < z_3, .., z_n > = < a_1 > \oplus < x, z_3, .., z_n > . \end{array}$$

Ainsi $< a_2, .., a_n > \underset{X}{\equiv} < x, z_3, .., z_n >$ et par hypothèse de récurrence, $< a_2, .., a_n > \overset{*}{\underset{X}{\equiv}} < x, z_3, .., z_n >$. Ceci prouve que $< a_1, .., a_n > \overset{*}{\underset{X}{\equiv}} < b_1, .., b_n >$.

(ii)$\Rightarrow$(iii) Si $X \models \mathcal{O}_4$, alors X est un espace d'ordres. Donc, $X \models \mathcal{O}_5$ (Marshall [M_3] lemme 4.1). Comme on vient de montrer que (i)$\Longleftrightarrow$(ii), on sait que $\overset{*}{\underset{X}{\equiv}}$ coïncide avec $\underset{X}{\equiv}$, qui est trivialement 3-transitive.

(iii)$\Rightarrow$(i) Il suffit de prouver que $\underset{X}{\equiv} \Longrightarrow \overset{*}{\underset{X}{\equiv}}$. Si $\overset{*}{\underset{X}{\equiv}}$ est 3-transitive, alors par récurrence, on prouve que $\overset{*}{\underset{X}{\equiv}}$ est transitive (voir [M_6],thm. 2.6), donc on peut construire à partir de X l'anneau de Witt associé à $\overset{*}{\underset{X}{\equiv}}$. On l'appellera $W^*(X)$. Démontrons d'abord :

Lemme 7: *Il existe une correspondance biunivoque entre X et $Hom(W^*(X), \mathbb{Z})$.*

Démonstration : Soit $X \xrightarrow{\Psi} Hom(W^*(X), \mathbb{Z})$ défini comme suit:

$$\forall \sigma \in X,\ \Psi(\sigma) = \hat{\sigma},\ \text{où } \hat{\sigma}(\overline{< a_1, .., a_n >}) = \sum_{i=1}^{n} \sigma(a_i)$$

On désigne par $\overline{< a_1, .., a_n >}$ la classe de la forme $< a_1, .., a_n >$ dans $W^*(X)$, c'est-à-dire la classe de $< a_1, .., a_n >$ modulo l'équivalence de Witt.

Etant donné $\varphi \in Hom(W^*(X), \mathbb{Z})$, soit $\sigma_\varphi : G \longrightarrow \mathbb{Z}$ l'application définie par $\sigma_\varphi(a) = \varphi(\overline{< a >})$. Comme $< a > \otimes < b > \underset{X}{\equiv} < ab >$, σ_φ est un homomorphisme et, comme $\sigma_\varphi(a^2) = \sigma_\varphi(1) = 1$, σ_φ prend ses valeurs dans $\{1, -1\}$. Avec $\mathcal{O}_5$, on peut prouver facilement que $\varphi \in X$ (exercice). Clairement, $\Psi(\sigma_\varphi) = \hat{\sigma}_\varphi = \varphi$, c'est-à-dire Ψ est surjective.

En revenant à la preuve de la proposition 6, on considère $f = < a_1, .., a_n >$ et $g = < b_1, .., b_n >$, tels que $f \underset{X}{\equiv} g$. Alors, $\sigma(f - g) = 0$, $\forall \sigma \in X$ et par le lemme 7, $f - g \in Ker\varphi$, $\forall \varphi \in Hom(W^*(X), \mathbb{Z})$, et donc $f - g \in Nil(W^*(X))$ (voir [M_6], p.85). Mais on sait que $Nil(W^*(X)) \subseteq W^*(X)_{Tor}$ (voir [K_1] lemme 3.3, p.221 ou [M_6] cor.4.20, p.83). Alors, $\exists\ n \in \mathbb{N}$ tel que $n(f - g) = 0$ dans $W^*(X)$. Si $n(f - g)$ est isotrope alors $f - g$ est isotrope (voir preuve du thm. 4.27, p.89, dans [M_6]), donc $f - g = 0$ dans $W^*(X)$, c'est-à-dire $f \overset{*}{\underset{X}{\equiv}} g$.

Structures Associées à un Espace d'Ordres.

Soit (X, G) un espace d'ordres abstrait.

Définition 8 (Sous-espace) : *Un sous-espace d'un espace d'ordres (X, G) est un couple $(Y, G/\Delta)$, où $Y \subseteq X$ et Δ est un sous-groupe de G, tels que :*

$$(1)\ Y^\perp = \Delta \qquad et \qquad (2)\ Y = X \cap \Delta^\perp\ ,$$

où, $Y^\perp = \{a \in G\ ;\ \sigma(a) = 1\ \forall\ \sigma \in Y\}$ *et* $\Delta^\perp = \{\sigma \in \mathcal{X}(G)\ ;\ \sigma(a) = 1\ \forall\ a \in \Delta\}$.

Evidemment, Y n'est pas un sous-ensemble de $\mathcal{X}(G/\Delta)$; mais ce petit inconvéniant formel est résolu au moyen de l'identification suivante:

$$\begin{array}{rcl} \pi : \mathcal{X}(G/\Delta) & \longrightarrow & \mathcal{X}(G) \\ \overline{\sigma} & \longmapsto & \sigma : G \longrightarrow \{1,-1\} \\ & & \quad a \longmapsto \overline{\sigma}(\overline{a}) \end{array}$$

Soit $\overline{Y} = \pi^{-1}(Y)$.

Théorème 9: *$(\overline{Y}, G/\Delta)$ est un espace d'ordres.*

Une démonstration de ce théorème se trouve par exemple dans $[M_3]$, thm.2.2. Mais dans l'article qui fait suite, on retrouve ce résultat en utilisant le théorème de dualité entre les espaces d'ordres et les groupes spéciaux.

Exemple 10: Soient F un corps formellement réel et $T \subseteq \dot{F}$ un pré-ordre de F. Si $\mathcal{X}(F,T) = \{Sgn_P \in \mathcal{X}(F);\ T \subseteq P\}$, alors $(\mathcal{X}(F,T), \dot{F}/\dot{T})$ est un sous-espace de $(\mathcal{X}(F), \dot{F}/\Sigma\dot{F}^2)$.

Définition 11 (Espace Quotient) : *Soient G' un sous groupe de G, tel que $-1 \in G'$ et $X' = \{\sigma_{|G'};\ \sigma \in X\}$. Si (X', G') est un espace d'ordres, alors (X', G') est appelé un* **espace quotient de** (X, G).

Remarque 12: Une condition nécessaire et suffisante pour que (X', G') soit un espace quotient n'a pas encore été trouvée. La difficulté est contenue dans l'axiome $\mathcal{O}_4$. Cependant on a une condition suffisante, à savoir :

$$\begin{cases} \forall\ f,\ g \text{ forme quadratique sur } G' \\ \underset{X}{D}< f > \cap \underset{X}{D}< g > \neq \emptyset \implies \underset{X}{D}< f > \cap \underset{X}{D}< g > \cap\, G' \neq \emptyset. \end{cases}$$

Exemple 13: Soient F un corps formellement réel, v une valuation réelle de F et F_v le corps résiduel de v. Alors $(\mathcal{X}(F_v), \dot{F}_v/\Sigma\dot{F}_v^2)$ est un espace quotient de $(\mathcal{X}(F, T_v), \dot{F}/\dot{T}_v)$ où T_v est l'intersection de tous les ordres de F compatibles avec la valuation v et $\mathcal{X}(F, T_v)$ est comme dans l'exemple 10.

Définition 14 (Fan) : *Un espace d'ordres (X, G) est un* **fan** *ssi :*

$$X = \{\sigma \in \mathcal{X}(G);\ \sigma(-1) = -1\}.$$

Evidemment, si X est un fan et $\sigma_1, \sigma_2, \sigma_3 \in X$ alors le produit $\sigma_1\sigma_2\sigma_3$ est dans X. Donc, pour tout espace d'ordres abstrait X, l'ensemble $\{\sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2\sigma_3\}$, en tant que sous-espace de X, est un fan.

Dans le cas des corps, les fans de $\mathcal{X}(F)$ correspondent exactement aux pré-ordres de F qui sont des "fans" aux sens de Lam ($[L_2]$, déf. 5.1, p.39).

Exemple 15: Soit $F = I\!R((x))((y))$ et $G = \dot{F}/\Sigma\dot{F}^2$; on peut observer que $\mathcal{X}(F) = \{P_1, P_2, P_3, P_4\}$ où:

$$\begin{array}{llll} P_1: & G \longrightarrow \{1,-1\} & P_2: & G \longrightarrow \{1,-1\} \\ & \overline{x} \longmapsto 1 & & \overline{x} \longmapsto 1 \\ & \overline{y} \longmapsto 1 & & \overline{y} \longmapsto -1 \\ P_3: & G \longrightarrow \{1,-1\} & P_4: & G \longrightarrow \{1,-1\} \\ & \overline{x} \longmapsto -1 & & \overline{x} \longmapsto -1 \\ & \overline{y} \longmapsto 1 & & \overline{y} \longmapsto -1 \end{array}$$

Donc, $P_4 = P_1P_2P_3$ et $(\mathcal{X}(F), G)$ est un fan à 4-éléments.

Théorème 16 (Somme directe) : *Soient (X_i, G_i), $i \in \{1,2\}$ deux espaces d'ordres abstraits. Considérons $G = G_1 \times G_2$ le produit direct de groupes et :*

$$Y_i = \{\sigma \in \mathcal{X}(G);\ \sigma_{|G_i} \in X_i \quad et \quad \sigma_{|G_j} = 1\},\ i,j \in \{1,2\},\ i \neq j.$$

Soit $X = Y_1 \cup Y_2$. Alors, (X,G) est un espace d'ordres.

Cette construction peut être généralisée à un nombre infini d'espaces d'ordres (voir Kula-Marshall-Sladek, $[K_2]$, pg. 392).

Remarques 17:

1) Soient $f = <(a_1,b_1),..,(a_n,b_n)>$ et $g = <(a'_1,b'_1),..,(a'_n,b'_n)>$ deux formes quadratiques sur G; on peut observer que :

$$f \underset{X}{\equiv} g \ ssi \ <a_1,..,a_n> \underset{X_1}{\equiv} <a'_1,..,a'_n> \ \text{ et } \ <b_1,...,b_n> \underset{X_2}{\equiv} <b'_1,..,b'_n>$$

2) Si l'on identifie $G_i \cong G/G_j$, $i \neq j$, alors (Y_i, G_i) est un sous-espace de (X,G).

3) Dans le cas d'un corps, on a par exemple :

$$(\mathcal{X}(I\!R), \mathbb{Z}_2) \oplus (\mathcal{X}(I\!R), \mathbb{Z}_2) \cong (\mathcal{X}(I\!R((X))), \mathbb{Z}_2 \times \mathbb{Z}_2).$$

Problème de Marshall.

Dans son article de 1976, "A Reduced Theory of Quadratic Forms", Marshall a posé la question suivante :

Un espace d'ordres abstrait est-il isomorphe à l'espace d'ordres d'un corps Pythagoricien?

Cette question est restée jusqu'aujourd'hui ouverte. Mais pour quelques cas particuliers, il existe une réponse.

a) (X,G) est un espace fini, c'est-à-dire G est fini.
En 1979, M. Marshall ($[M_2]$, thm. 4.10) a répondu par l'affirmatif.

b) (X, G) est engendré par un nombre fini de fans.
En 1980, M. Marshall ($[M_5]$, thm. 1.6) a répondu par l'affirmatif.

a) (X, G) est l'ensemble des signatures d'un anneau semi-local réduit.
Le résultat est vrai (démonstration personnelle en 1991).

Commentaires :

1) En 1984 Kula, Marshall et Sladek $[K_2]$ ont prouvé :
"Tout espace d'ordres abstrait est un espace quotient d'un espace d'ordres d'un corps".

2) Plusieurs résultats valables dans la théorie des espaces d'ordres d'un corps ont été généralisés aux espaces d'ordres abstraits. Par exemple :
i) "le **principe local-global** pour les formes anisotropes" ($[M_6]$, thm. 9.2 et $[A_1]$, thm.9.1).
ii) "le théorème de la **P-structure**"($[M_5]$, thm.3.2).

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Les Groupes Spéciaux

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Introduction.

L'étude des espaces d'ordres abstraits nous a amené à considérer certaines structures que nous appellons **groupes spéciaux** (Définition 1). Celles-ci sont des groupes d'exposant 2 avec un élément distingué -1 et une relation d'équivalence sur G^2 représentant l'isométrie des formes quadratiques binaires à coefficients dans G. Les groupes spéciaux sont caractérisés par un système d'axiomes simples, formulés dans un langage de premier ordre mathématiquement naturel. Le résultat principal de ce travail est que les groupes spéciaux avec leurs homomorphismes naturels constituent une catégorie duale à celle des espaces d'ordres abstraits de Marshall. On donnera deux démonstrations de ce résultat. Une en utilisant le Théorème de répresentation (Théorème 13) et l'autre en utilisant le théorème de dualité de Pontrjagin pour le cas des groupes compacts (Théoréme 17). Evidemment, on peut dualiser les constructions connues pour les espaces d'ordres (section 'Résultats dans le Dual'). Ainsi, par exemple, à un sous-espace d'ordres correspond un groupe quotient spécial (Définition 27), à un espace quotient correspond un sous-groupe spécial complet (Définition 23), etc. On étudie enfin les limites des systèmes (projectifs et inductifs) de groupes spéciaux réduits (Proposition 33 et 37). Mais on trouve également des constructions nouvelles même pour les espaces d'ordres. Par exemple, le dual d'un système inductif de groupes spéciaux ou le dual projectif de groupes spéciaux profinis (remarque 34). Cependant, la généralisation des résultats trouvés ici pour les groupes spéciaux réduits aux groupes spéciaux non nécessairement réduits est loin d'être accomplie (cf.'Commentaires' en fin d'article).

Cet article est la suite de $[L]$. Les résultats cités sans démonstration seront repris dans ma thèse de doctorat.

La Notion de Groupe Spécial.

Dans cette section on va définir la notion de groupe spécial et donner quelques exemples importants pour cet exposé.

Définition 1 : *Un* **groupe spécial** *est une structure* $< G, -1, \underset{G}{\equiv} >$, *où* G *est un groupe multiplicatif d'exposant 2,* -1 *un élément distingué de* G *et* $\underset{G}{\equiv}$ *est une relation binaire sur* G^2 *telle que :*

$\mathcal{SG}_0$) $\underset{G}{\equiv}$ *est une relation d'équivalence.*

$\mathcal{SG}_1$) $\forall a\ \forall b\ (a,b) \underset{G}{\equiv} (b,a)$.

$\mathcal{SG}_2$) $\forall a\ (a,-a) \underset{G}{\equiv} (1,-1)$ où $-a := a. -1$.

$\mathcal{SG}_3$) $\forall a\ \forall b\ \forall c\ \forall d\ (a,b) \underset{G}{\equiv} (c,d) \implies ab = cd$.

$\mathcal{SG}_4$) $\forall a\ \forall b\ \forall c\ \forall d\ (a,b) \underset{G}{\equiv} (c,d) \implies (a,-c) \underset{G}{\equiv} (-b,d)$.

$\mathcal{SG}_5$) $\forall\ a\ \forall b\ \forall c\ \forall d\ \forall x\ (a,b) \underset{G}{\equiv} (c,d) \implies (xa,xb) \underset{G}{\equiv} (xc,xd)$.

$\mathcal{SG}_6$) **(3-transitivité)** $\forall\ a_1\ a_2\ a_3\ \forall\ b_1\ b_2\ b_3\ \forall\ c_1\ c_2\ c_3$

$$< a_1, a_2, a_3 > \underset{G}{\equiv} < b_1, b_2, b_3 > \ \wedge\ < b_1, b_2, b_3 > \underset{G}{\equiv} < c_1, c_2, c_3 >$$
$$\implies < a_1, a_2, a_3 > \underset{G}{\equiv} < c_1, c_2, c_3 > .$$

On définit sur l'ensemble des n-uplets d'éléments de G (appelé aussi ensemble des formes quadratiques sur G) la relation suivante:

$$< a_1, a_2 > \underset{G}{\equiv} < b_1, b_2 > \ \text{ssi}\ (a_1, a_2) \underset{G}{\equiv} (b_1, b_2)$$
$$< a_1, .., a_n > \overset{n}{\underset{G}{\equiv}} < b_1, .., b_n > \ \text{ssi}\ \exists\ x\ y\ z_3 ... z_n$$
$$< a_1, x > \underset{G}{\equiv} < b_1, y > \wedge < a_2, .., a_n > \overset{n-1}{\underset{G}{\equiv}} < x, z_3, .., z_n > \wedge < b_2, .., b_n > \overset{n-1}{\underset{G}{\equiv}} < y, z_3, .., z_n >$$

Si une relation $\underset{G}{\equiv}$ sur G^2 satisfait les axiomes $\mathcal{SG}_0 - \mathcal{SG}_6$ on dira qu'elle est une **relation spéciale**. Si, en plus, $\underset{G}{\equiv}$ satisfait :
$\mathcal{SG}_{red}$) (**Axiome de réduction**) $\forall a\ (a,a) \underset{G}{\equiv} (1,1) \implies a = 1$,
alors $< G, -1, \underset{G}{\equiv} >$ est appelé un **groupe spécial réduit**.

Exemple 2 : Soit (X, G) un espace d'ordres abstrait, alors $< G, -1, \underset{X}{\equiv} >$ est un groupe spécial réduit où :

$$(a,b) \underset{X}{\equiv} (c,d) \quad ssi \quad \forall \sigma \in X \quad \sigma(a) + \sigma(b) = \sigma(c) + \sigma(d)$$

Observons que les axiomes $\mathcal{SG}_0 - \mathcal{SG}_5$ sont trivialement vérifiés et $\mathcal{SG}_6$ est une conséquence de l'axiome $\mathcal{O}_4$ de la définition d'espace d'ordres (voir Proposition 6 dans $[L]$).

Remarque 3 : Si $< G, -1, \equiv_G >$ est un groupe spécial, on peut construire l'anneau de Witt $W(G)$ (voir $[D]$, 2.3), qui sera un anneau réduit ssi $< G, -1, \underset{G}{\equiv} > \models \mathcal{SG}_{red}$.

Exemple 4 : Soit G un groupe multiplicatif d'exposant 2 avec un élément distingué -1. On peut définir sur G la relation **triviale** comme suit :

$$(a, b) \underset{T}{\equiv} (c, d) \quad ssi \quad ab = cd$$

A propos de cette relation on peut dire :

1) $\underset{T}{\equiv}$ est une relation spécial et elle est la plus grande relation spéciale que l'on peut donner sur G, c'est-à-dire, si $\underset{G}{\equiv}$ est une relation spéciale sur G, alors $\underset{G}{\equiv} \subseteq \underset{T}{\equiv}$ (voir $\mathcal{SG}_3$).

2) $< G, -1, \underset{T}{\equiv} >$ est un groupe spécial réduit ssi $G = \{1\}$.

Exemple 5 : Si G un groupe multiplicatif d'exposant 2 avec un élément distingué -1, on peut définir sur G la relation **fan** comme suit :

$$(a, b) \underset{fan}{\equiv} (c, d) \; ssi \begin{cases} a \neq -b \;\wedge [(a = c \;\wedge\; b = d) \vee (a = d \;\wedge\; b = c)] \\ ou \\ a = -b \;\wedge\; c = -d \end{cases}$$

1) $\underset{fan}{\equiv}$ est une relation spécial (la preuve de ce fait ne pas immédiate) et elle est la plus petite relation spéciale sur G, c'est-à-dire, si $\underset{G}{\equiv}$ est une relation spéciale sur G alors, $\underset{fan}{\equiv} \subseteq \underset{G}{\equiv}$.
2) $< G, -1, \underset{fan}{\equiv} > \models \mathcal{SG}_{red} \;\wedge G \neq \{1\}$ ssi $1 \neq -1$.

Dualité entre les espaces d'ordres abstraits et les groupes spéciaux réduits.

Dans cette section on va définir un foncteur contravariant entre la catégorie des espaces d'ordres et la catégorie des groupes spéciaux et on va montrer que ce foncteur est un isomorphisme de catégories.

Théorème 6 : *Il existe une correspondence biunivoque entre les espaces d'ordres abstraits et les groupes spéciaux réduits.*

Démonstration :
On considère les applications :

$$\{ \text{ Espaces d'ordres abstraits } \} \overset{\Psi}{\underset{\Phi}{\longleftrightarrow}} \{ \text{ Groupes spéciaux réduits} \}$$

$$(X, G) \overset{\Psi}{\longrightarrow} < G, -1, \underset{X}{\equiv} >$$

$$(X_G, G) \overset{\Phi}{\longleftarrow} < G, -1, \underset{G}{\equiv} >,$$

où $\Psi((X, G))$ est défini comme dans l'Exemple 2 et $\Phi(< G, -1, \underset{G}{\equiv} >) = (X_G, G)$, avec :
$X_G = \{\sigma \in \mathcal{X}(G) ; \sigma(-1) = -1 \wedge \forall a\, b\, c\, d \in G\, [(a, b) \underset{G}{\equiv} (c, d) \Rightarrow \sigma(a) + \sigma(b) = \sigma(c) + \sigma(d)]\}$

Il est entendu que dans cette définition la somme a lieu dans $\mathbb{Z}$. Pour démontrer que (X_G, G) est un espace d'ordres abstrait, on a besoin de deux lemmes :

Lemme 7 : $\overset{n}{\underset{G}{\equiv}} \Longrightarrow \overset{n}{\underset{X_G}{\equiv}}$. *Ce qui revient à dire :*

$$< a_1, .., a_n > \underset{G}{\equiv} < b_1, .., b_n > \Rightarrow \forall \sigma \in X_G \quad [\sum_{i=1}^{n} \sigma(a_i) = \sum_{i=1}^{n} \sigma(b_i)].$$

Lemme 8 : *L'homomorphisme :*

$$\begin{array}{rl} \Theta \ : \ X_G \longrightarrow & Hom(W(G), \mathbb{Z}) \\ \sigma \longmapsto & \hat{\sigma} \ : W(G) \longrightarrow \mathbb{Z} \\ & \qquad \overline{f} \longmapsto sgn_\sigma f \end{array}$$

est une bijection ($\overline{f}$ désigne la classe de f modulo l'équivalence de Witt).

Démonstration :

Si $f =< a_1, .., a_n >$ on définit $sgn_\sigma f = \sum_{i=1}^{n} \sigma(a_i)$. Par le Lemme 7, $\hat{\sigma}$ est bien définie. Si $\varphi \in Hom(W(G), \mathbb{Z})$ on définit $\sigma_\varphi : G \longrightarrow \mathbb{Z}$ par $\sigma_\varphi(a) = \varphi(\overline{< a >})$. Comme $a^2 = 1$ et φ est un homomorphisme, on a $\sigma_\varphi(a) \in \{1, -1\}$. Par définition de X_G on peut conclure que $\sigma_\varphi \in X_G$. Il est facile de prouver que $\Theta(\sigma_\varphi) = \varphi$.

Proposition 9 : *Soit $< G, -1, \underset{G}{\equiv} >$ un groupe spécial réduit. Alors :*

$$(a, b) \underset{G}{\equiv} (c, d) \iff \forall \sigma \in X_G, \ \sigma(a) + \sigma(b) = \sigma(c) + \sigma(d)$$

Démonstration :

"$\Rightarrow$" C'est la définition de X_G.

"$\Leftarrow$" Considérez $f =< a, b > \oplus < -c, -d >$; alors $\sigma(f) = 0$, $\forall \sigma \in X_G$. Donc, par le Lemme 8, $\varphi(\overline{f}) = 0, \forall \varphi \in Hom(W(X), \mathbb{Z})$. Le Principe Local-Global de Pfister ([M_3], Thm.4.2, pg.76) implique que $\overline{f} \in W(G)_{Tor}$ (la partie de torsion de $W(G)$). On considère deux cas :

1) $X_G \neq \emptyset$. Alors $Hom(W(G), \mathbb{Z}) \neq \emptyset$ (Lemme 8), et [M_3], Cor.4.20 prouve que $Nil(W(G)) = W(G)_{Tor}$. Comme $< G, -1, \underset{G}{\equiv} >$ est réduit, $W(G)$ est réduit, i.e. $Nil(W(G)) = \{0\}$ ([M_3], Thm.4.27). On a donc $\overline{f} = 0$ dans $W(G)$, ce qui entraîne :

$$(\diamond) \qquad \overline{< a, b >} = \overline{< c, d >} \Leftrightarrow < a, b > \underset{G}{\equiv} < c, d > .$$

2) $X_G = \emptyset$. Par [M_3], Cor.4.20 on a $Nil(W(G)) = I(W(G)) = \{0\}$, c'est-à-dire, toutes les formes de dimension paire sont Witt-équivalentes. Alors, l'équivalence $(\diamond)$ est vraie aussi dans ce cas.

Remarque. La preuve qu'on vient de faire, montre que les applications Ψ et Φ font correspondre l'espace d'ordres vide avec le groupe spécial $G = \{1\}$ (muni de la seule structure spéciale possible). Remarquez aussi que si $(\emptyset, G)$ est un espace d'ordres, alors $G = \{1\}$, et que si $< G, -1, \underset{G}{\equiv} > \models \mathcal{SG}_{red}$ et $1 = -1$, alors $G = \{1\}$ (voir l'exemple 5.2).

Maintenant, pour montrer que (X_G, G) est un espace d'ordres il faut vérifier les axiomes suivants (voir Définition 1 dans [L]) :

$\mathcal{O}_1$) X_G est un fermé de $\mathcal{X}(G)$ (par définition).

$\mathcal{O}_2$) $\sigma(-1) = -1$, $\forall \sigma \in X_G$ (par définition).

$\mathcal{O}_3$) $\bigcap_{\sigma \in X} Ker\sigma = \{-1\}$.
Supposons $\sigma(a) = 1$, $\forall \sigma \in X_G$; alors $\sigma(a) + \sigma(a) = \sigma(1) + \sigma(1)$, $\forall \sigma \in X_G$. Donc, par la Proposition 9, $(a, a) \underset{G}{\equiv} (1, 1)$, et comme $< G, -1, \equiv_G >$ est réduit, $a = 1$.

$\mathcal{O}_4$) $X_G \models \mathcal{O}_4$ ssi $\underset{X_G}{\equiv} \Longleftrightarrow \underset{X_G}{\overset{*}{\equiv}}$ (voir Proposition 6 dans [L]).

En faisant une récurrence sur la dimension, on a la Proposition 9 pour toutes les relations $\underset{G}{\overset{n}{\equiv}}$, $n \geq 3$, c'est-à-dire, $\underset{X_G}{\overset{*}{\equiv}} \Leftrightarrow \underset{X_G}{\equiv}$.

Définition 10 (Morphisme d'espace d'ordres) : *Soient (X_1, G_1) et (X_2, G_2) deux espaces d'ordres abstraits. Un morphisme d'espaces d'ordres est un homomorphisme continu de groupes $\varphi : \mathcal{X}(G_1) \longrightarrow \mathcal{X}(G_2)$, tel que $\varphi(X_1) \subseteq X_2$.*

Définition 11 (Morphisme de groupes spéciaux) : *Un morphisme entre deux groupes spéciaux $< G_1, -1, \underset{G_1}{\equiv} >$ et $< G_2, -1, \underset{G_2}{\equiv} >$ est un homorphisme de groupes $\psi : G_1 \longrightarrow G_2$, tel que :*

1) $(a, b) \underset{G_1}{\equiv} (c, d) \implies (\psi(a), \psi(b)) \underset{G_2}{\equiv} (\psi(c), \psi(d))$,

2) $\psi(-1_{G_1}) = -1_{G_2}$.

Théorème 12 : *La correspondance Ψ du Théorème 6 est un foncteur contravariant entre la catégorie des espaces d'ordres (avec morphismes donnés par la Définition 10) et la catégorie des groupes spéciaux réduits (avec les homomorphismes de groupes spéciaux considéré dans la Définition 11). La correspondence Φ est un foncteur dans la direction inverse. En plus, la paire (Ψ, Φ) établit un isomorphisme de catégories entre celle des espaces d'ordres et celle opposée aux groupes spéciaux; c'est-à-dire, les composés $\Psi \circ \Phi$ et $\Phi \circ \Psi$ sont les foncteurs identité des catégories respectives.*

Démonstration :

(A) Construction du foncteur Ψ.

On construit pour chaque morphisme d'espace d'ordres $\varphi : (X_1, G_1) \longrightarrow (X_2, G_2)$ un morphisme $\Psi(\varphi) = \varphi^* : < G_2, -1, \underset{X_2}{\equiv} > \longrightarrow < G_1, -1, \underset{X_1}{\equiv} >$ de groupes spéciaux. Pour ce faire on peut utiliser deux techniques différentes, toutes les deux aboutissant, bien entendu, au même résultat. On fera la construction en utilisant le Théorème de représentation, en donnant en fin de preuve une construction alternative qui utilise un cas simple de la dualité de Pontrjagin.

Théorème 13 (Théorème de Représentation) : *Soient (X, G) un espace d'ordres abstrait, $f \in \mathcal{C}(X, \{\pm 1\})$ et*

$$\begin{aligned} \mathcal{E} : \; G &\longrightarrow \mathcal{C}(X, \{\pm 1\}) \\ a &\longmapsto ev_a : X \longrightarrow \{\pm 1\} \\ &\qquad\qquad \sigma \longmapsto \sigma(a) \end{aligned}$$

Alors, $f \in Im\mathcal{E}$ ssi pour tout fan de 4 éléments $V \subseteq X$, on a $\prod_{\sigma \in V} f(\sigma) = 1$.

(Pour une preuve voir Marshall, $[M_2]$, th. 7.2.)

Etant donnés (X_1, G_1) et (X_2, G_2), deux espaces d'ordres abstraits et φ un morphisme entre ces espaces, on considère $\mathcal{E}_i : G_i \longrightarrow \mathcal{C}(X_i, \{\pm 1\})$, $i \in \{1, 2\}$, comme dans le théorème ci-dessus.

Lemme 14 : *Pour tout $a \in G_2$, $\exists!\, b \in G_1$, tel que $ev_b = ev_a \circ \varphi$, c'est-à-dire $\forall \sigma \in X_1$ on a :*

$$\sigma(b) = \varphi(\sigma)(a)$$

Démonstration :

Existence : Il suffit de démontrer que $ev_a \circ \varphi \in Im\mathcal{E}_1$. Nous utilisons le théorème de représentation pour G_1.

Prenons un fan $V = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \subseteq X_1$. Alors $\sigma_4 = \sigma_1\sigma_2\sigma_3$ et :

$$\varphi[V] = \{\varphi(\sigma_1), \varphi(\sigma_2), \varphi(\sigma_3), \varphi(\sigma_4)\}$$

Donc $\varphi[V]$ est encore un fan.

cas 1) $|\varphi[V]| = 1$. Alors
$$\prod_{\sigma \in V} ev_a \circ \varphi(\sigma) = (\varphi(\sigma_1)(a))^4 = 1.$$

cas 2) $|\varphi[V]| = 2$. Alors
$$\prod_{\sigma \in V} ev_a \circ \varphi(\sigma) = (\varphi(\sigma_1)(a))^2.(\varphi(\sigma_3)(a))^2 = 1.$$

cas 3) $|\varphi[V]| = 4$. Alors $\varphi(\sigma_4) = \varphi(\sigma_1)\varphi(\sigma_2)\varphi(\sigma_3)$, et
$$\prod_{\sigma \in V} ev_a \circ \varphi(\sigma) = (\varphi(\sigma_1)(a).\varphi(\sigma_2)(a).\varphi(\sigma_3)(a))\varphi(\sigma_4)(a) = 1.$$

Unicité : Supposons que $\exists b' \in G_1$; $ev_b = ev_{b'}$. Alors,

$$\forall \sigma \in X_1,\ \sigma(b) = \sigma(b') \implies bb' \in \bigcap_{\sigma \in X_1} Ker\sigma = \{1\} \implies b = b'.$$

En revenant à la démonstration du théorème 12(A) , d'après le lemme précédent on peut définir :

$$\begin{aligned} \varphi^* : \; G_2 &\longrightarrow G_1 \\ a &\longmapsto \varphi^*(a) \; ; \quad ev_{\varphi^*(a)} = ev_a \circ \varphi \end{aligned}$$

φ^* est un morphisme des groupes spéciaux. En effet, pour $a, b, c, d \in G_2$ on a

$$(\varphi^*(a), \varphi^*(b)) \underset{X_1}{\equiv} (\varphi^*(c), \varphi^*(d)) \text{ ssi } \forall \sigma \in X_1,\ \varphi(\sigma)(a) + \varphi(\sigma)(b) = \varphi(\sigma)(c) + \varphi(\sigma)(d).$$

Mais $\forall \sigma \in X_1$, $\varphi(\sigma) \in X_2$. Donc, si $(a, b) \underset{X_2}{\equiv} (c, d)$, le résultat suit.

Il est clair que Ψ est un foncteur contravariant :

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f} & X_2 & & G_1 & \xleftarrow{f^*} & G_2 \\ g \circ f & \searrow & \Big\downarrow g & \overset{\Psi}{\Longrightarrow} & (g \circ f)^* & \nwarrow & \Big\uparrow g^* \\ & & X_3 & & & & G_3 \end{array}$$

(B) Construction du foncteur Φ.

A chaque homomorphisme de groupes spéciaux $\psi : < G_2, -1, \underset{G_2}{\equiv} > \longrightarrow < G_1, -1, \underset{G_1}{\equiv} >$, on assigne l'application $\Phi(\psi) = \psi^* : (X_{G_1}, G_1) \longrightarrow (X_{G_2}, G_2)$ définie par $\psi^*(\sigma) = \sigma \circ \psi$, pour $\sigma \in \mathcal{X}(G_1)$. On vérifie aisément que ψ^* est un morphisme d'espaces d'ordres dans le sens de la Définition 10 et que Φ est, en fait, un foncteur contravariant.

(C) Les foncteurs Ψ et Φ donnent un isomorphisme de catégories.

Les points **(1)-(4)**, ci-dessous montrent que les composés $\Psi \circ \Phi$ et $\Phi \circ \Psi$ sont les fonctions identités sur les objets et sur les morphismes des catégories respectives (cf. Mac Lane, $[M_c]$, pg.14).

(1) $\Phi \circ \Psi(X, G) = (X, G)$ pour tout espace d'ordres (X, G).

Preuve : Désignons par $G(X)$ le groupe spécial $< G, -1, \underset{X}{\equiv} >$. D'aprés la définition de Ψ et Φ (Théorème 6) on doit montrer que $X = X_{G_X}$.

Or, comme la relation $\underset{G_X}{\equiv}$ est définie par :

$$< a, b > \underset{G_X}{\equiv} < c, d > \Longleftrightarrow \forall \sigma \in X(\sigma(a) + \sigma(b) = \sigma(c) + \sigma(d)),$$

il en résulte de la définition de X_G que $X \subseteq X_{G_X}$.

Pour l'inclusion inverse on utilise le Lemme 4.1 de Marshall $[M_2]$, pg.511. Étant donnés un espace d'ordres (X, G) et $\sigma \in \mathcal{X}(G)$, $\sigma \neq 1$, si σ satisfait la condition :

$$(\star) \qquad x \in Ker(\sigma) \Rightarrow D_X(< 1, x >) \subseteq Ker(\sigma),$$

alors $\sigma \in X$.

Soit $\sigma \in X_{G_X}$. Comme $\sigma(-1) = -1$, on a $\sigma \neq 1$. Pour montrer que σ vérifie la condition $(\star)$, soient $x \in Ker(\sigma)$ et $y \in D_X(< 1, x >)$, i.e. $< y, yx > \underset{X}{\equiv} < 1, x >$. Alors $\sigma \in X_{G_X}$ entraîne $\sigma(y)(1 + \sigma(x)) = 1 + \sigma(x)$, et comme $1 + \sigma(x) \neq 0$, il s'ensuit que $\sigma(y) = 1$. On conclut que $\sigma \in X$.

(2) $\Phi \circ \Psi(\varphi) = \varphi$ pour tout morphisme $\varphi : (X_1, G_1) \longrightarrow (X_2, G_2)$ d'espace d'ordres.

Preuve : En décodant les définitions de Ψ et Φ sur les morphismes, on doit prouver l'égalité des fonctions φ' et φ, toutes les deux définies sur $(X_1, G_1) = (X_{G(X_1)}, G_1)$, à valeurs dans $(X_2, G_2) = (X_{G(X_2)}, G_2)$, où $\varphi' = \Phi(\varphi^*)$ (et $\varphi^* = \Psi(\varphi)$).

Or, φ' est définie par

$$(\dagger) \qquad \varphi'(\sigma) = \sigma \circ \varphi^* \qquad \text{pour } \sigma \in \mathcal{X}(G_1),$$

et $\varphi^* :< G_2, -1, \underset{X_2}{\equiv} > \longrightarrow < G_1, -1, \underset{X_1}{\equiv} >$ est définie par les équations fonctionnelles

$$(\dagger\dagger) \qquad ev_{\varphi^*(a)} = ev_a \circ \varphi \qquad \text{pour } a \in G_2$$

On a donc d'un côté, pour $a \in G_2$ et $\sigma \in \mathcal{X}(G_1)$:

$$ev_a \circ \varphi = ev_a(\varphi(\sigma)) = \varphi(\sigma)(a) \ ,$$

et de l'autre :

$$ev_{\varphi^*(a)}(\sigma) = \sigma(\varphi^*(a)) = (\sigma \circ \varphi^*)(a) = \varphi'(\sigma)(a).$$

L'égalité cherché, $\varphi' = \varphi$, suit alors de $(\dagger\dagger)$ et $(\dagger)$.

(3) $(\Psi \circ \Phi)(< G, -1, \underset{G}{\equiv} >) =< G, -1, \underset{G}{\equiv} >$.

Preuve : D'après les définitions de Ψ et Φ sur les objets, ceci révient à pouver l'identité des relations $\underset{G}{\equiv}$ et $\underset{X_G}{\equiv}$, ce qui est le contenu de la Proposition 9.

(4) $(\Psi \circ \Phi)(\psi) = \psi$ pour tout homomorphisme $\psi : < G_2, -1, \underset{G_2}{\equiv} > \longrightarrow < G_1, -1, \underset{G_1}{\equiv} >$ de groupes spéciaux.

Preuve : En décodant les définitions de Ψ et Φ sur les morphismes, ceci revient à prouver l'égalité des fonctions ψ' et ψ, toutes les deux définies sur $< G_2, -1, \underset{G_2}{\equiv} >$ à valeurs dans $< G_1, -1, \underset{G_1}{\equiv} >=< G_1, -1, \underset{X_{G_1}}{\equiv} >$, où $\psi' = \Psi(\psi^*)$ (et $\psi^* = \Phi(\psi)$).

Or, ψ' est définie par les équations fonctionnelles

$$(\dagger\dagger') \qquad ev_{\psi'(a)} = ev_a \circ \psi^* \qquad \text{pour } a \in G_2$$

et ψ^* est définie par

$$(\dagger') \qquad (\dagger') \quad \psi^*(\sigma) = \sigma \circ \psi \qquad \text{pour } \sigma \in \mathcal{X}(G_1)$$

Le même calcul qu'au (2) montre :

$$\begin{cases} ev_{\psi'(a)}(\sigma) = \sigma(\psi'(a)) \ , \\ ev_a \circ \psi^*(\sigma) = ev_a(\psi^*(\sigma)) = ev_a(\sigma \circ \psi) = \sigma(\psi)(a)) \end{cases}$$

Pour $a \in G_2$ fixe on a donc $\sigma(\psi')(a) = \sigma(\psi)(a)$ pour tout $\sigma \in \mathcal{X}(G_1)$, d'où $\psi'(a) = \psi(a)$. Il en résulte l'égalite cherchée, $\psi' = \psi$.

Observation 15 : On a démontré en particulier, l'identité $\theta^{**} = \theta$ aussi bien pour les morphismes de groupes spéciaux comme pour ceux des espaces d'ordres.

Remarque 16 : *Construction alternative du foncteur Ψ.*

Etant donné un groupe topologique commutatif Γ (noté multiplicativement), on désigne par $\mathcal{X}_c(\Gamma)$ l'ensemble des homomorphismes *continus* de Γ à valeurs dans $\{+1, -1\}$, celui-ci muni de la topologie discrète.

L'application :

$$\begin{aligned} ev \ : G &\longrightarrow \mathcal{X}_c(\mathcal{X}(G)) \\ g &\longmapsto ev_g \ : \mathcal{X}(G) \longrightarrow \{+1, -1\} \\ &\qquad\qquad\quad \sigma \longmapsto \sigma(g) \end{aligned}$$

identifie G avec un sous-groupe de $\mathcal{X}_c(\mathcal{X}(G))$. Observons que ev_g est une application continue, car $ev_g^{-1}[\{1\}] = \{\sigma \in \mathcal{X}(G) \ ; \ \sigma(g) = 1\}$ est un ouvert-fermé de base, par définition de la topologie de $\mathcal{X}(G)$.

Le résultat dont nous avons besoin est:

Théorème 17 : *Soit G un groupe d'exposant 2. Alors l'application ev est un isomorphisme de groupes entre G et $\mathcal{X}_c(\mathcal{X}(G))$.*

Note : Ceci est un cas particulier du Théorème de dualité de Pontrjagin *pour les groupes compacts*, cas où la preuve est assez simple ; voir Pontrjagin ([*P*]; Ch.6, Section 36 et 37). Remarquez que comme G est un groupe d'exposant 2, tout homomorphisme de G à valeurs dans le groupe multiplicatif de complexes de modulo 1 ne prend valeurs que dans $\{+1, -1\}$.

En revenant à la construction de Ψ, un morphisme d'espaces d'ordres $\varphi : (X_1, G_1) \longrightarrow (X_2, G_2)$ est, en particulier, un homomorphisme continu de $\mathcal{X}(G_1)$ dans $\mathcal{X}(G_2)$. Soit $\overline{\varphi}^* : \mathcal{X}_c(\mathcal{X}(G_2)) \longrightarrow \mathcal{X}_c(\mathcal{X}(G_1))$ l'application définie par :

$$\overline{\varphi}^*(\gamma) = \gamma \circ \varphi \qquad \text{pour } \gamma \in \mathcal{X}_c(\mathcal{X}(G_2)).$$

Manifestement, $\gamma \circ \varphi \in \mathcal{X}_c(\mathcal{X}(G_1))$. Par le Théorème 17, on a que les applications $ev_i \ : \ G_i \longrightarrow \mathcal{X}_c(\mathcal{X}(G_1))$ $(i \in \{1, 2\})$ sont des isomorphismes. L'homomorphisme cherché est donc :

$$\Psi(\varphi) = ev_2^{-1} \circ \overline{\varphi}^* \circ ev_1.$$

Résultats dans le dual

Dans ce qui suit on regardera dans la catégorie des groupes spéciaux réduits les duales de certaines constructions introduites par Marshall dans la catégorie des espaces d'ordres abstraits (voir section 'Structures Associées à un Espace d'Ordres' dans [*L*]) et lorsqu'on se restreindra pour $< G, -1, \underset{G}{\equiv} >$ à des groupes spéciaux réduits on utilisera l'abréviation **g.s.r.**

Définition 18 (Sous-groupe spécial) : *Soit* $< G, -1, \underset{G}{\equiv} >$ *un groupe spécial et* $H \subseteq G$, *tel que* $-1 \in H$. *On dit que* H *est un sous-groupe spécial de* G *si* $< H, -1, \underset{G_{|H}}{\equiv} >$ *est un groupe spécial.*

Remarque 19 : Notez que la sous-structure $< H, -1, \underset{G_{|H}}{\equiv} >$ vérifie automatiquement les axiomes $\mathcal{SG}_0 - \mathcal{SG}_5$ car ceux-ci sont des énoncés universels du langage $\mathcal{L} = \{\cdot, \equiv, 1, -1\}$ des groupes spéciaux. De même, si $< G, -1, \equiv_G >$ est réduit, $< H, -1, \underset{G_{|H}}{\equiv} >$ l'est aussi. Seul l'axiome $\mathcal{SG}_6$ ne passe pas forcément aux sous-structures (il est un énoncé $\forall\exists$).

Proposition 20 : $< H, -1, \underset{G_{|H}}{\equiv} >$ *est un sous-groupe spécial de* $< G, -1, \equiv_G >$, *ssi :*
$\forall a\ b\ c\ d \in H\ [\underset{G}{D} < a, b > \cap \underset{G}{D} < c, d > \cap H \neq \emptyset \Longrightarrow \underset{G}{D} < a, -d > \cap \underset{G}{D} < c, -b > \cap H \neq \emptyset]$.

Considérons les axiomes suivants:

$\mathcal{SG}_7$) $\forall a\ \forall a'\ \forall x\ \forall t\ \forall t'\ \forall y\ [\ (a, a') \underset{G}{\equiv} (x, t)\ \wedge\ (t, t') \underset{G}{\equiv} (1, y)\ ]$
$\Longrightarrow \exists\ a''\ s\ s'\ [\ (a, a'') \underset{G}{\equiv} (y, s)\ \wedge\ (s, s') \underset{G}{\equiv} (1, x)\ ]$
C'est-à- dire $\displaystyle\bigcup_{t \in D_G < 1, y >} D_G < x, t > = \bigcup_{y \in D_G < 1, x >} D_G < y, s >$

Cet énoncé a été introduit par Marshall dans $[M_1]$, pg 160.

$\mathcal{SG}_8$) (**Axiome de la 2-simplification**) $\forall\ a\ a_2\ a_3\ b_2\ b_3\ \forall\ f_1 ...\ f_n$ formes quadratiques de dimension 3 :
$< a, a_2, a_3 > \underset{G}{\equiv} f_1 \underset{G}{\equiv} ... \underset{G}{\equiv} f_n \underset{G}{\equiv} < a, b_2, b_3 > \Longrightarrow < a_2, a_3 > \underset{G}{\equiv} < b_2, b_3 >$

Proposition 21 (Caractérisation de $\mathcal{SG}_6$) : *Soit* G *un groupe multiplicatif d'exposant 2 et* $\underset{G}{\equiv}$ *une relation binaire sur* G^2 *vérifiant* $\mathcal{SG}_0 - \mathcal{SG}_5$. *Alors on a :*

$$\underset{G}{\equiv} \models \mathcal{SG}_6 \iff \underset{G}{\equiv} \models \mathcal{SG}_7 \wedge \mathcal{SG}_8.$$

Lemme 22 : *Supposons que* $\underset{G}{\equiv}$ *satisfait la condition suivante :*

(∗) $\qquad \forall f,\ g$ *formes sur* H, $f \underset{G_{|H}}{\equiv} g$ *ssi* $f \underset{G}{\equiv} g$.

Alors, $< H, -1, \underset{G_{|H}}{\equiv} >$ *est un sous-groupe spécial de* $< G, -1, \equiv_G >$.

Définition 23 (Sous-groupe spécial complet) : *Si la relation* $\underset{G_{|H}}{\equiv}$ *satisfait la condition* (∗), *on appelle* $< H, -1, \underset{G_{|H}}{\equiv} >$ *un sous-groupe spécial complet de* $< G, -1, \equiv_G >$.

Théorème 24 : *Soit* $< G', -1, \underset{G'}{\equiv} >$ *un sous-groupe spécial complet d'un g.s.r* $< G, -1, \underset{G}{\equiv} >$, *alors* $(X_{G'}, G')$ *est un espace quotient de* (X_G, G). *Réciproquement, si* (X', G') *un espace quotient d'un espace d'ordres* (X, G); *alors* $< G', -1, \underset{X'}{\equiv} >$ *est un sous-groupe spécial complet de* $< G, -1, \underset{X}{\equiv} >$.

Soit $< G, -1, \underset{G}{\equiv} >$ un groupe spécial et $\Delta \subseteq G$. On considère sur G/Δ la relation suivante :

$$< \overline{a}, \overline{b} > \underset{G}{\equiv} < \overline{c}, \overline{d} > \ \ ssi\ \exists\ a_1, .., a_n \in \Delta,\ < a, b > \otimes f \underset{G}{\equiv} < c, d > \otimes f,$$

où $f = < 1, a_1 > \otimes ... \otimes < 1, a_n >$ est une forme de Pfister.

Remarques 25 :

1) La relation $\underset{G/\Delta}{\equiv}$ ne définit pas nécessairement une structure de groupe spécial sur G/Δ; en effet, l'axiome $\mathcal{SG}_3$ n'est pas toujours vrai.

2) La façon naturelle de définir une relation d'isométrie sur G/Δ de manière à ce qu'elle soit préservée par l'application canonique $G \longrightarrow G/\Delta$, est :

$$< \overline{a}, \overline{b} > \overset{*}{\underset{G/\Delta}{\equiv}} < \overline{c}, \overline{d} > \text{ ssi } \exists\ a', b', c', d' \in G, \text{ où}$$
$$\overline{a} = \overline{a'},\ \overline{b} = \overline{b'},\ \overline{c} = \overline{c'},\ \overline{d} = \overline{d'} \text{ et } < a, b > \underset{G}{\equiv} < c, d >$$

Mais $\overset{*}{\underset{G/\Delta}{\equiv}}$ ne définit pas nécessairement une structure de groupe spécial parce qu'on ne peut garantir la transitivité dans $\mathcal{SG}_0$ ni la validité de $\mathcal{SG}_6$.

Des conditions pour que les relations $\underset{G/\Delta}{\equiv}$ et $\overset{*}{\underset{G/\Delta}{\equiv}}$ soient des relations spéciales sont données ci-dessous :

Proposition 26 : *Soit $< G, -1, \underset{G}{\equiv} >$ un groupe spécial. Les conditions suivantes sont équivalentes :*

i) $\underset{G/\Delta}{\equiv}$ *est une relation spéciale.*

ii) Δ *est un sous groupe saturé de* G, *c'est à dire* $\Delta = \bigcup \{D_G(f) \mid f \text{ Pfister sur } \Delta\}$.

iii) Les relations $\underset{G/\Delta}{\equiv}$ *et* $\overset{*}{\underset{G/\Delta}{\equiv}}$ *coïncident.*

iv) $\forall a \in G\ [a \in \Delta \Longrightarrow D_G(< 1, a >) \subseteq \Delta]$.

Définition 27 (Groupe quotient spécial) : *Si $< G, -1, \underset{G/\Delta}{\equiv} >$ est un groupe spécial, on l'appellera un groupe quotient spécial.*

Théorème 28 : *Si $< G, -1, \underset{G/\Delta}{\equiv} >$ est un groupe quotient spécial d'un g.s.r $< G, -1, \underset{G}{\equiv} >$, alors le dual $(X_{G/\Delta}, G/\Delta)$ est un sous-espace d'ordres du (X_G, G). Réciproquement, si $(Y, G/\Delta)$ est un sous-espace d'un espace d'ordres (X, G) alors, $< G, -1, \underset{X}{\equiv} >$ est un g.s.r et $< G/\Delta, -1, \underset{Y}{\equiv} >$ est un groupe quotient spécial de $< G, -1, \underset{X}{\equiv} >$.*

Proposition 29 (Produit direct des groupes spéciaux) : *Soit $\{< G_i, -1, \underset{G_i}{\equiv} >,\ i \in I\}$ une famille de g.s.r. Considérons $G = \prod_{i \in I} G_i$ le produit direct de groupes. Alors la relation suivante est une relation spéciale réduite sur G :*

$$< (a_i)_{i\in I}, (b_i)_{i\in I} > \underset{G}{\equiv} < (c_i)_{i\in I}, (d_i)_{i\in I} > \ \ ssi\ \forall i \in I, < a_i, b_i > \underset{G_i}{\equiv} < c_i, d_i >$$

Théorème 30 : *Le dual d'un produit direct de groupes spéciaux réduits est la somme directe des duaux de chaque groupe spécial et réciproquement.*

Définition 31 (Système projectif des groupes spéciaux) : *Un système projectif des groupes spéciaux est la donné de* $\{(I,\leq), < G_i, -1, \underset{G_i}{\equiv} >, \varphi_{ij}\}$, *où*

1) $(I,\leq)$ *est un ensemble filtrant.*

2) $\forall i \in I$, $< G_i, -1, \underset{G_i}{\equiv} >$ *est un groupe spécial.*

3) $\forall\, i \leq j$, $\varphi_{ij} : G_j \longrightarrow G_i$ *est un morphisme de groupes spéciaux.*

4) $\forall\, i \leq j \leq k$, $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$

Corollaire 32 : *Le dual d'un système projectif de g.s.r est un système inductif d'espaces d'ordres et vice-versa.*

Proposition 33 (Limite projective de groupes spéciaux finis) : *Soit un système projectif de groupes spéciaux finis* $\{< G_i, -1, \underset{G_i}{\equiv}>, \varphi_{ij}\}_{i,j\in I}$; *alors* $G' = \underset{\longleftarrow}{lim}\ G_i$ *(limite projective de groupes) muni de l'élément distingué* $-1 = (-1_{G_i})_{i\in I}$ *et de la relation donnée çi-dessous est un groupe spécial.*

$$< (a_i)_{i\in I}, (b_i)_{i\in I} > \underset{G'}{\equiv} < (c_i)_{i\in I}, (d_i)_{i\in I} > \text{ ssi } \forall i \in I, < a_i, b_i > \underset{G_i}{\equiv} < c_i, d_i >$$

Démonstration :

Soit $< G = \prod_{i\in I} G_i, -1, \underset{G}{\equiv} >$ le groupe spécial produit direct des groupes G_i. Alors, clairement $\underset{G'}{\equiv} = \underset{G_{|G'}}{\equiv}$. La démonstration consiste alors à montrer que $< G', -1, \underset{G'}{\equiv} >$ est un sous-groupe spécial complet de $< G, -1, \underset{G}{\equiv} >$. Pour cela on démontre que $< G', -1, \underset{G'}{\equiv} >$ satisfait la condition $(*)$ du lemme (22).

Remarque 34 : Si on définit un "**groupe profini spécial**" comme étant un groupe profini G munit d'une relation spéciale R_G fermée dans la toplologie produit de G^4, on peut démontrer alors que la limite projective de groupes profinis spéciaux existe.

Définition 35 (Système inductif de groupes spéciaux) : *Un système inductif de groupes spéciaux consiste de* $\{(I,\leq), < G_i, -1, \underset{G_i}{\equiv} >, \phi_{ij}\}$ *où :*

1) $(I,\leq)$ *est un ensemble filtrant.*

2) $\forall\, i \in I$, $< G_i, -1, \underset{G_i}{\equiv} >$ *est un groupe spécial.*

3) $\forall\, i \leq j$, $\phi_{ij} : G_i \rightarrow G_j$ *est un morphisme de groupe spécial.*

4) $\forall\, i \leq j \leq k$, $\phi_{ij} = \phi_{jk} \circ \phi_{ij}$.

Si en plus, l'axiome (5) ci-dessous est satisfait, on dit que ce système est un **système inductif complet**.

5) $\forall i \leq j$ et pour toutes les formes quadratiques f et g définies sur G_i, on a $f \underset{G_i}{\equiv} g$ ssi $\phi_{ij}(f) \underset{G_j}{\equiv} \phi_{ij}(g)$.

Corollaire 36 : *Le dual d'un système inductif de g.s.r est un système projectif d'espaces d'ordres et réciproquement. Si d'ailleurs ce système satisfait l'axiome (35.5) on peut montrer que:*
i) $\forall\, i \leq j$, ϕ_{ij} *est injective.*
ii) $\forall\, i \leq j$, $\phi^*_{ij}(X_j) = X_i$

Proposition 37 (Limite inductif de groupes spéciaux) : *Considérons un système inductif de groupes spéciaux* $\{< G_i, -1, \underset{G_i}{\equiv} >, \varphi_{ij}\}_{i,j \in I}$; *alors,* $G = \underrightarrow{lim}\, G_i$, *la limite inductive de groupes est un groupe spécial.*

Démonstration :

La démonstration est immédiate du fait que les axiomes des groupe spécial sont des énoncés de la forme $\forall\exists$ du langage $\mathcal{L} = \{\cdot, \equiv, 1, -1\}$ des groupes spéciaux.

Remarque 38 : On a montré que la limite d'un système inductif de groupes spéciaux réduits (pas nécessairement complet) est un groupe spécial réduit. Mais la limite des duaux de ce système n'est pas nécessairement un espace d'ordres. On sait d'après Marshall que l'axiome (35.5) est une condition nécessaire. En fait, on peut toujours considérer les espaces d'ordres duaux d'une limite inductive (pas nécessairement complet) de groupes spéciaux réduits, mais on ne peut pas assurer que cet espace système est la limite projective d'espaces duaux, car il est possible que cette limite ne soit pas un espace d'ordres. Cependant si notre système est un système complet on a le résultat suivant:

Théorème 39 : *Soit* $< G = \underrightarrow{lim}\, G_i, -1, \underset{G}{\equiv} >$ *le groupe spécial limite inductif d'un système inductif complet de g.s.r. Alors* (X_G, G) *est la limite projective du système projectif des espaces d'ordres duaux* $\{X_{G_i}, \varphi^*_{ij}\}_{i,j \in I}$, *c'est à dire* $X_G = \underleftarrow{lim}\, X_{G_i}$. *Réciproquement, si* X *est l'espace d'ordres limite projective d'un système projective des espaces d'ordres, alors son dual est le groupe spécial limite inductive du système dual.*

Remarque 40 : Supposons que pour tout $i \leq j$ dans I, $\varphi_{ij}(Gi)$ est un sous-groupe spécial complet de G_j (ceci équivaut à dire que le système inductif est complet). Soit $i \in I$, considérons le morphisme canonique $\varphi_i : G_i \longrightarrow G$; alors $\varphi_i(G_i)$ est un sous-groupe complet de G. De plus, pour tout i, $\varphi^*_i : \mathcal{X}(G) \longrightarrow \mathcal{X}(G_i)$ vérifie $\varphi^*_i(X_G) = X_{G_i}$. Donc X_{G_i} est un espace quotient de X_G.

Corollaire 41 : *Tout groupe spécial est la limite inductive d'un système inductif complet de sous-groupe spéciaux dénombrables. En particulier (cf. Marshall [M_4],Thm.4.7, pg. 612), tout espace d'ordres abstrait est la limite projective d'un système d'espaces d'ordres dénombrables.*

Démonstration :

Immédiate, à partir du Théorème de Löwenheim-Skolem.

Proposition 42 : *Soient* $\{< G_i, -1, \underset{G_i}{\equiv} >,\ i \in I\}$ *une famille de groupes spéciaux et* $\mathcal{U}$ *un ultrafiltre sur* I. *Alors l'ultraproduit* $G_{\mathcal{U}} = \prod G_i/\mathcal{U}$ *est groupe spécial avec la relation :*

$$< \overline{(a_i)_{i\in I}}, \overline{(b_i)_{i\in I}} > \underset{G_{\mathcal{U}}}{\equiv} < \overline{(c_i)_{i\in I}}, \overline{(d_i)_{i\in I}} > \ \text{ssi}\ \{i \in I, < a_i, b_i > \underset{G_i}{\equiv} < c_i, d_i >\} \in \mathcal{U}.$$

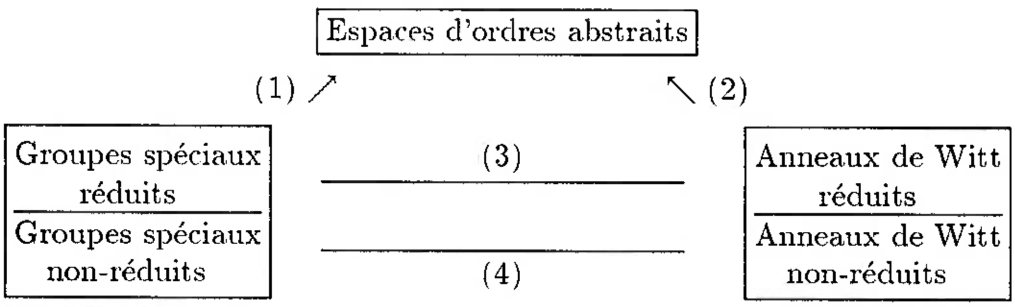
Théorème 43 : *Soient* $\{< G_i, -1, \underset{G_i}{\equiv} >, \ i \in I\}$ *une famille de g.s.r,* $\mathcal{U}$ *un ultrafiltre sur* I *et* $G_{\mathcal{U}}$ *le groupe spécial ultraproduit. Alors* $X_{G_{\mathcal{U}}}$ *est l'ultrasomme de* X_{G_i}*, c'est-à-dire*

$$X_{G_{\mathcal{U}}} = \bigcap_{J \in \mathcal{U}} X_J \ , \quad X_J = \sum_{i \in J} X_{G_i}$$

où $X_J = \sum_{i \in J} X_{G_i}$ *est la somme connexe des espaces d'ordres.*

Commentaires

(A) La catégorie des espaces d'ordres abstraits est équivalente à l'opposé de la catégorie des anneaux de Witt abstraits réduits et cette dernière catégorie est équivalente à la catégorie des groupes spéciaux réduits. On a donc, le shéma suivant :



- (1) c'est le Théorème 12, de cet article.
- (2) est prouvé dans le ch.IV de $[M_3]$.
- (3) c'est le Théorème 2.8 dans $[D]$.
- (4) c'est le Théorème 2.8 dans $[D]$.

(B) Il y a des différences considérables entre le groupes spéciaux réduits et le groupes spéciaux non nécessairement réduits. Par exemple, soient G un groupe spécial tel que $| G |= 2^n$ et $\Re(n)$ le nombre de relations spéciales sur G. On a :

i) Si G est réduit et
- $n = 1 \implies \Re(n) = 1$
- $n = 2 \implies \Re(n) = 1$
- $n = 3 \implies \Re(n) = 2$
- Par dualité, $\Re(n)$ est le nombre des espaces d'ordres contenu dans $\mathcal{X}(G)$. Sa valeur a été calculée par Bröcker dans $[B]$, §2.2, pg.459.

ii) Si G n'est pas nécessairement réduit, par équivalence de catégories on a que $\Re(n)$ est le nombre d'anneaux de Witt R, avec $|G_R| = 2^n$. Pour $n \leq 3$, Marshall $[M_3]$, Ch.V, pg.122, a calculé les valeurs de $\Re(n)$:
- $n = 1 \implies \Re(n) = 3$.
- $n = 2 \implies \Re(n) = 6$.
- $n = 3 \implies \Re(n) = 17$.
- En général, la valeur de $\Re(n)$ n'est pas connue.

Le tableau suivant donne les résultats connus :

n	Cas réduit	Cas général
1	$\Re$(n)=1	$\Re$(n)=3
2	$\Re$(n)=1	$\Re$(n)=6
3	$\Re$(n)=2	$\Re$(n)=17
n	Bröcker [B]	$\Re$(n)= ?

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THE DOWNWARD LOWENHEIN-SKOLEM THEOREM

Francisco Miraglia (*)

1. Sheaves and Presheaves over Topological Spaces.

If X is a topological space, $\Omega(X)$ denotes the collection of opens in X; $\Omega(X)$ comes equipped with the usual partial ordering by inclusion and the operations of finite joins and meets ($\cup, \cap$) as well as the infinitary operations of union ($\bigcup$) and meet ($\bigwedge$), given by

$$\bigwedge_{i\in I} u_i = \mathrm{int}(\bigcap_{i\in I} u_i),$$

the interior of the set theoretical intersection. If $S \subseteq X$, $\Omega(S)$ denotes the induced topology on S, $\Omega(S) = \{ v \cap S : v \in \Omega(X)\}$.

We can consider $\Omega(X)$ as a category whose objects is $\Omega(X)$ itself and whose morphisms are given by

$$\mathrm{Mor}(u, v) = \begin{cases} \{u \hookrightarrow v\} & u \subseteq v \\ \emptyset & \text{otherwise} \end{cases}$$

Let L be a 1st order language with equality and Mod(L) be the category of L-structures and L-morphisms.

Definition 1 : *A* **presheaf of L-structures over X** *is a contravariant functor $Q : \Omega(X) \longrightarrow Mod(L)$, that is, for each $u \in \Omega(X)$ we have a L-structure $Q(u)$ and if $u \subseteq v$ in $\Omega(X)$, we have a L-morphism $q_{vu} : Q(v) \longrightarrow Q(u)$ such that the following conditions are verified for all $u \subseteq v \subseteq w$ in $\Omega(X)$:*

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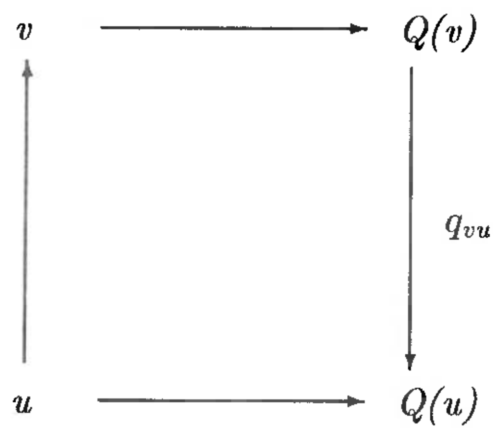

fig. 1

[P1] : $q_{uu} = Id_{Q(u)}$;

[P2] : $q_{wu} = q_{vu} \circ q_{wv}$;

[P3] : $Q(\emptyset) = \{*\}$, *the one element L-structure;*

[ext] : Let $\phi(v_1, \ldots, v_n)$ *be an atomic formula in L,* $s_1, \ldots, s_n \in Q(u)$ *and* $\{u_i\} \subseteq \Omega(u)$ *be an open covering of u. Then,*

$$Q(u) \models \phi[s_1, \ldots, s_n] \quad \textit{iff} \quad \forall\, i \in I,\ Q(u_i) \models \phi[q_{uu_i}s_1, \ldots, q_{uu_i}s_n].$$

Notice that in particular we have, for s, t $\in$ Q(u),

$$s = t \quad \text{iff} \quad \exists \text{ a covering } \{u_i\} \subseteq \Omega(u) \text{ such that } q_{uu_i}\, s = q_{uu_i}\, t\ ,\ \forall\, i \in I.$$

Notation : a) For each u $\in \Omega$(X), Q(u) is called the **structure of sections of Q over u** and an element of Q(u) is refered to as a **section over u**; Q(X) is the structure of **global** sections of Q. The maps q_{uv} are called the **restriction maps** of Q.

b) In general we abuse notation and write, for s $\in$ Q(u) and v $\in \Omega$(u), $s_{|v}$ in place of q_{uv}.

We now introduce an important simplification in notation, due to Dana Scott. Let Q be a presheaf over X. Define

$$|Q| = \coprod_{u\in\Omega(X)} Q(u) = \bigcup_{u\in\Omega(X)} Q(u) \times \{u\},$$

the disjoint union of all structures of sections of Q over X. An element of $|Q|$ will be referred to as a **section of Q.**

For $t = (s, u) \in |Q|$, put Et = **extension of t** = u. We may identify each Q(u) with its copy inside $|Q|$ by the map $s \mapsto (s, Es)$. We shall thus consider $Q(u) \subseteq |Q|$.

For each $s \in |Q|$ and $u \in \Omega(X)$ define a restriction $(s, u) \mapsto s_{|u}$ by the formula

$$s_{|u} = q_{Es\,Es\cap u}\; s$$

Notice that $s_{|Es\cap u} = s_{|u}$, for all $s \in |Q|$ and $u \in \Omega(X)$.

In this notation we may rewrite properties [P1] and [P2] as

i) $s_{|Es} = s$ ii) $s_{|u|v} = s_{|u\cap v}$.

For $\vec{s} \in |Q|^n$ and $u \in \Omega(X)$, define $E\vec{s} = \cap_{i=1}^{n} Es_i$ and $\vec{s}_{|u} = (s_{1|u}, \ldots, s_{n|u})$.

The condition [ext] (extensionality) can be rendered as

[ext] : Let $\phi(v_1, \ldots, v_n)$ be an atomic formula in L, u an open set in X and $\vec{s} \in |Q|^n$ be such that $Es_i = Es_j$, $1 \le i, j \le n$. Let $\{u_i\} \subseteq \Omega(u)$ be an open covering of u. Then

$$Q(u) \models \phi[\vec{s}] \quad \text{iff} \quad \forall\, i \in I,\ Q(u_i) \models \phi[\vec{s}_{|u}].$$

When dealing with sheaves or presheaves one of the basic ideas is that of **germ of a section at a point of X**. Let ν_x be the filter of open neighbourhoods of $x \in X$. If Q is a presheaf over X, let

$|Q|(x) = \coprod_{u\in\nu_x} Q(u) = \{s \in |Q| : Es \in \nu_x\}$.

If $s, t \in |Q|(x)$ put

$$s \approx_x t \quad \text{iff} \quad \exists\, v \in \nu_x \cap \Omega(Es \cap Et) \text{ such that } s_{|v} = t_{|v}.$$

This defines an equivalence relation on $|Q|(x)$. The class of a section s with respect to this equivalence relation is called **the germ of s at x** and indicated by s_x. The set of these equivalence classes, $|Q|(x)/\approx_x$, is the **stalk of Q at x**, to be indicated by Q_x. Observe that for any finite set $a_1, \ldots, a_n \in Q_x$ we can select $u \in \nu_x$ and $s_1, \ldots, s_n \in Q(u)$ such that $s_{ix} = a_i$, $1 \leq i \leq n$.

There is a natural way to make Q_x in to a L-structure :

i) If c is a constant in L, define its interpretation in Q_x as the germ of the interpretation of c in any $Q(u)$, $u \in \nu_x$. Note that since restriction is a L-morphism, we have $c^{Q(u)}_{|v} = c^{Q(v)}$ for all $v \in \Omega(u) \cap \nu_x$. Thus, the above assignement is independent of the $u \in \nu_x$.

ii) If ω is a n-ary operation symbol in L and $a_1, \ldots, a_n \in Q_x$, choose $u \in \nu_x$ and $s_1, \ldots, s_n \in Q(u)$ such that $s_{ix} = a_i$, $1 \leq i \leq n$. If we define

$$\omega^{Q_x}(a_1, \ldots, a_n) = (\omega^{Q(u)}(\vec{s}))_x,$$

this is independent of representatives and gives an interpretation of ω in Q_x.

iii) If R is a n-ary relation symbol in L, put

$$Q_x \models R[a_1, \ldots, a_n] \quad \text{iff} \quad \exists\, u \in \nu_x \text{ and } s_1, \ldots, s_n \in Q(u) \text{ such that } s_{ix} = a_i \text{ and } Q(u) \models R[s_1, \ldots, s_n].$$

With these definitions for any atomic formula $\phi(v_1, \ldots, v_n)$ and any $a_1, \ldots, a_n \in Q_x$ we have

$$Q_x \models \phi[a_1, \ldots, a_n] \quad \text{iff} \quad \exists\, u \in \nu_x \text{ and } s_1, \ldots, s_n \in Q(u) \text{ such that } s_{ix} = a_i \text{ and } Q(u) \models \phi[s_1, \ldots, s_n].$$

We have natural maps $Q(u) \longrightarrow Q_x$, $s \mapsto s_x$, that become L-morphisms with the definitions given above. In fact, the stalk of Q at x is precisely the inductive limit of the system of L-structures indexed by the elements of the filter ν_x.

It is natural to define equality of s, t **in Q** as the set of x where they have the same germ :

$$[s = t] = \{x \in Es \cap Et : s_x = t_x\}.$$

Simple calculations will show that [s = t] is an open set in X and in fact we have

$$[s = t] = \bigcup \{u \in \Omega(Es \cap Et) : s_{|u} = t_{|u}\}.$$

Similarly, for an atomic formula $\phi(v_1, \ldots, v_n)$ and $s_1, \ldots, s_n \in |Q|$, we may define the value of ϕ relative to Q as

$$[\phi(s_1, \ldots, s_n)] = \{x \in E\vec{s} : Q_x \models \phi[s_{1x}, \ldots, s_{nx}].$$

Again, this is an open set in X and we have

$$[\phi(s_1, \ldots, s_n)] = \bigcup \{u \in \Omega(E\vec{s} : Q(u) \models \phi[s_{1|u}, \ldots, s_{n|u}].$$

It's also easily verified that the extensionality condition ([ext]) can be stated as

$$E\vec{s} = [\phi(s_1, \ldots, s_n)] \quad \text{iff} \quad Q(E\vec{s}) \models \phi[s_1, \ldots, s_n].$$

For the equality relation this comes to

$$Es = Et = [s = t] \quad \text{iff} \quad s = t.$$

Perhaps the very basic example of the sheaf of continuous real valued functions on a topological space can help the reader to grasp the ideas succintly presented above.

We now present the notion of morphism. Let P and Q be presheaves over X.

Definition 2 : *A morphism* $P \xrightarrow{\eta} Q$ *is a natural transformation of contravariant functors that is, for each* $u \in \Omega(X)$, *a L-morphism* $\eta_u : P(u) \longrightarrow Q(u)$ *such that for* $v \in \Omega(u)$ *we have* $q_{uv} \circ \eta_u = \eta_v \circ p_{uv}$, *where* p_{uv} *is the appropriate restriction map of P.*

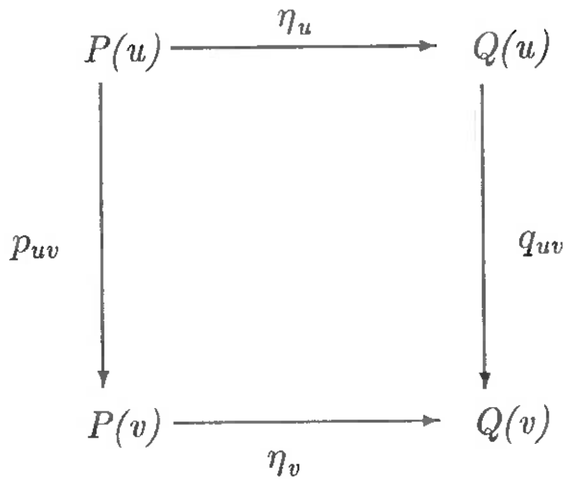

fig. 2

The commutativity of the diagrams in fig. 2 can be expressed as

$$(\eta_u s)_{|v} = \eta_v \, s_{|v},$$

for all $u \in \Omega(X)$, $v \in \Omega(u)$ *and* $s \in Q(u)$. *In short, a morphism is a natural transformation that 'commutes' with restriction.*

If $P \xrightarrow{\eta} Q$ is a morphism define $|P| \xrightarrow{\eta} |Q|$ by

$$\eta s = \eta_{Es} \, s.$$

It can be shown that the map $\eta : |P| \longrightarrow |Q|$ has the following properties, for all s, t $\in |P|$ and u $\in \Omega(X)$:

[mor 1] : $E\eta s = Es$

[mor 2] : $[s = t] \subseteq [\eta s = \eta t]$.

[rest] : $\eta(s_{|u}) = (\eta s)_{|u}$.

Conversely, a map $|P| \xrightarrow{\eta} |Q|$ satisfying the above properties induces a unique morphism of presheaves as defined above. In fact, it suffices that η satisfies [mor 1] and [mor 2] as [rest] is a consequence of these two properties.

The notion of sheaf arrises when we require that all compatible family of sections of a presheaf P can be 'glued' to a section of P. The formal definitions are given below.

Definition 3 : *Let Q be a presheaf over X and $S \subseteq |Q|$ a set of sections of Q. We say that S is* **compatible** *iff for all s, $t \in S$ we have*

$$s_{|Et \cap Es} = s_{|Et} = t_{|Es \cap Et} = t_{|Es}.$$

A presheaf is said to be **complete or a sheaf** *iff for all compatible $S \subseteq |Q|$ there is a (unique) $t \in |Q|$ such that*

i) $Et = \bigcup_{s \in S} Es$

ii) For all $s \in S$, $t_{|Es} = s$.

This unique $t \in |Q|$ will sometimes be indicated by $\sigma(S)$.

A morphism of sheaves is simply a morphism of the underlying presheaves.

If $\eta : P \longrightarrow Q$ is a morphism and $S \subseteq |P|$, let $\eta(S) = \{\eta s : s \in S\}$ be the image of S by η. One can verify that

i) If S is compatible in P then $\eta(S)$ is compatible in Q.

ii) If $\sigma(S)$ exists in P then $\sigma(\eta(S)) = \eta\sigma(S)$.

It was realized by Dana Scott that one could profitably treat the objects described above in a context similar to boolean valued models, except that the complete boolean algebra had to be replaced by a **complete Heyting algebra**. We shall adopt that point of view in what follows. The basic intuition however originates in sheaves (or presheaves) over topological spaces.

2. Complete Heyting Algebras

If Λ is a lattice, we shall use the symbols $\wedge$ and $\vee$ for the finitary operations of meet and join, respectively. The symbols $\bigwedge$ and $\bigvee$ will denote the corresponding infinitary operations. The least element of Λ and its largest element will be indicated (whenever they exist) by 0 and 1, respectively.

Recall that a lattice Λ is said to be **complete** if for all $T \subseteq \Lambda$ there is $\bigvee T$ and $\bigwedge T$ in Λ.

Definition 4 : *A complete lattice Ω is said to be a* **complete Heyting algebra** *(cHa) if for all $p \in \Omega$ and all $S \subseteq \Omega$ the following distributive law is verified :*

$[\wedge, \bigvee]$: $p \wedge \bigvee T = \bigvee_{q \in T} p \wedge q$.

In particular, all cHa's are distributive lattices with 0 and 1. There are many interesting examples of cHa's. An important **proper** subclass is made up of the lattices of opens of a topological spaces, with its usual operations of union and interior of the set theoretical intersection. Of course, all complete Boolean algebras (cBa) are complete Heyting algebras.

In any cHa we have an operation of implication, $p \to q$, given by

$$p \to q = \max \{r \in \Omega : p \wedge r \leq q\}.$$

Thus, $p \to q$ is the largest element of Ω that satisfies modus ponens relative to the hypothesis p and conclusion q that is :

$$r \wedge p \leq q \quad \text{iff} \quad r \leq p \to q.$$

We also have an operation of $\leftrightarrow$ which can be given as

$$p \leftrightarrow q = (p \to q) \wedge (q \to p) = \max \{r \in \Omega : r \wedge p = r \wedge q\}.$$

In particular, we may define a **negation** ($\neg$) on Ω by

$$\neg p = p \to 0 = \max \{r \in \Omega : r \wedge p = 0\}.$$

In the algebra of opens of a topological space X, the operations of implication, negation and double negation can be described as follows :

i) $u \to v = \text{int}\,(X - u \cup v)$

ii) $\neg\, u = \text{int}\,(X - u)$

iii) $\neg\neg\, u = \text{int}\, \overline{u}$, where int is the operation of taking interior and $\overline{*}$ that of closure.

Note that, in general, $p \leq \neg\neg\, p$, but they may be distinct. An element of a cHa Ω is said to be regular precisely when $\neg\neg p = p$.

Frequently a cHa is refered to as a **frame or locale**. We will however stick with the older terminology.

Definition 5 : *Let Λ be a complete lattice.*

a) A subset $B \subseteq \Lambda$ is asid to be a **basis** *for Λ if for all $p \in \Lambda$ there is $S \subseteq B$ such that $p = \bigvee S$. Define the* **weight** *of Λ as*

$$w(\Lambda) = min\ \{\ cardinal(B) : B \text{ is a basis for } \Lambda\}.$$

b) An element $q \in \Lambda$ is said to be **compact** *iff for all $S \subseteq \Lambda$ if $q \leq \bigvee S$ then there is a finite subset $C \subseteq S$ such that $q \leq \bigvee C$.*

In all that follows, Ω will denote a complete Heyting algebra.

3. Sheaves and presheaves over Ω

We begin with

Definition 6 : *Let Ω be a cHa. An* **Ω-set A** *consists of a set $|A|$ (the domain of A), together with a map $[\cdot = \cdot] : A \times A \longrightarrow \Omega$ satisfying, for all $x,\ y,\ z \in |A|$,*

$[=1] : [x = y] = [y = x].$

$[=2] : [x = y] \wedge [y = z] \leq [x = z].$

The map $[\cdot = \cdot]$ is called the equality relation on A. When there is need to register the Ω-set to which an equality relation refers to, we shall use the notation $[\cdot = \cdot]_A$.

For $x \in |A|$, **the extent of x** *is defined as $Ex = [x = x]$. It's clear that $[x = y] \leq Ex \wedge Ey$.*

We refer to an element in $|A|$ as a **section of A**. *For $p \in \Omega$, $A(p) = \{x \in |A| : Ex = p\}$, is the set of* **sections of A over p**. *An element of $A(1)$ is called a* **global section** *of A.*

An Ω-set is said to be **extensional** *iff it satisfies*

[ext] : For all $x, y \in |A|$, $Ex = Ey = [x = y]$ implies $x = y$.

In what follows, unless explicit mention to the contrary, all Ω-sets will be extensional.

Clearly, if A is an Ω-set and $|B| \subseteq |A|$, then $|B|$ is the domain of a Ω-set obtained by restricting $[\cdot = \cdot]$ to $|B| \times |B|$. Furthermore, if A is extensional, the same will be true of B. The symbol $A \subseteq B$ will always indicate the Ω-set structure induced by A on $|B|$.

Definition 7 : *Let A, B be Ω-sets. A* **morphism** $A \xrightarrow{f} B$ *consists of a map $|A| \xrightarrow{f} |B|$ such that for all $s, t \in |A|$*

[mor 1] : $E_B f s = E_A s$

[mor 2] : $[s = t]_A \leq [fs = ft]_B$.

Whenever clear from context, all references to the names of the Ω-sets in conditions [mor 1] and [mor 2] will be omitted.

By the remarks in section 1, all presheaves P over a topological space X give rise to a Ω(X)-set, also denoted by P, with $|P| = \coprod_{u \in \Omega(X)} P(u)$ and for $s \in P(u)$, $t \in P(v)$,

$$[s = t] = \bigcup \{w \in \Omega(u \cap v) : s_{|w} = t_{|w}\}.$$

Furthermore, the morphisms there considered are precisely the same when they are treated as Ω(X)-sets.

An important notion when dealing with presheaves is that of a compatible family of sections. With this in hand one can then define various forms of completeness and the concept of sheaf. We also introduce the notion of **finitely complete** Ω-set, a convenient category of objects to work with.

Definition 8 : *Let A be an Ω-set, $S \subseteq |A|$ a set of sections of A and p an element of Ω.*

a) S is said to **compatible over p** *iff*

$$\forall\ s,\ t \in S, \quad p \wedge [s = t] = p \wedge Es \wedge Et.$$

We say S is **compatible** *if it is compatible over 1.*

b) A is said to be **finitely complete (fc)** *if for all $p \in \Omega$ and all* **finite** *$S \subseteq |A|$, if S is compatible over p then there is $t \in |A|$ such that*

i) $Et = p \wedge \bigvee \{Es : s \in S\}$ *and* *ii)* $\forall\ s \in S,\ p \wedge Es = p \wedge [t = s]$.

c) A is said to be **complete or a sheaf over Ω** *if for all $p \in \Omega$ and all $S \subseteq |A|$, if S is compatible over p then there is $t \in |A|$ satisfying conditions (i) and (ii) of item (b).*

It is easily verified that the element t satisfying conditions i) and ii) in items (b) and (c) is unique. It shall be indicated by $\sigma(S)$.

Lemma 1 : *Let A be a finitely complete Ω-set. Then there is a map $\cdot_{|}\cdot : |A| \times \Omega \longrightarrow |A|$, called* **restriction** *such that for all $x,\ y \in |A|$ and $p,\ q \in \Omega$ we have*

[rest 1] : $x_{|Ex} = x$

[rest 2] : $x_{|p|q} = x_{|p \wedge q}$.

[rest 3] : $[x_{|p} = y_{|q}] = p \wedge q \wedge [x = y]$.

[rest 4] : $x_{|[x=y]} = y_{|[x=y]}$.

[rest 5] : There is a unique $*$ in $|A|$ such that $E = 0$ and $s_{|0} = *$, for all $s \in |A|$.*

We now introduce the notion of presheaf over Ω.

Definition 9 : *An Ω-set A is a* **presheaf over** Ω *if there is a map* $*_{|*} : |A| \times \Omega \longrightarrow |A|$, *called* **restriction**, *satisfying properties [rest 1] - [rest 5] in the statement of Lemma 1. A morphism of presheaves is simply a morphism of their underlying Ω-sets.*

Thus, all finitely complete Ω-sets and all sheaves over Ω are presheaves over Ω.

Let A be a presheaf over Ω. We register the following observations :

1. It's clear that presheaves, finitely complete Ω-sets and sheaves over Ω, together with their morphisms are categories, denoted by pSh(Ω), fc Ω-sets and Sh(Ω), respectively.

2. A subset $S \subseteq |A|$ is compatible over $p \in \Omega$ iff

$S_{|p} = \{s_{|p} : s \in S\}$ is compatible (over 1) iff

$\forall$ s, t $\in$ S, $s_{|Et \wedge p} = t_{|Es \wedge p}$.

In particular, S is compatible iff $\forall$ s, t $\in$ S, $s_{|Et} = t_{|Es}$.

2. If $A \xrightarrow{f} B$ is a morphism of presheaves then

a) $\forall$ s $\in$ |A| and p $\in \Omega$, $f(s_{|p}) = (fs)_{|p}$.

b) If $S \subseteq |A|$ is compatible in A, the same is true of $f(S) = \{fs : s \in S\}$. Further, if $\sigma(S)$ exists in A then $\sigma(f(S))$ exists in B and $f(\sigma(S)) = \sigma(f(S))$.

c) f is a **monic** in pSh(Ω), fc Ω-sets or Sh(Ω) iff f is an injective set map from |A| to |B|.

d) f is an **epic** in the above categories iff

$\forall$ t $\in$ |B| $Et = \bigvee_{s \in |A|} [t = fs]$ iff

$\exists$ $S \subseteq |A|$ such that $Et = \bigvee_{s \in S} Es$ and $t_{|Es} = fs$.

The definition below describes a basic notion.

Definition 10 : *Let A be an Ω-set and $S \subseteq |A|$ be a set of sections in A. We say that S is* **dense** *in A iff for all $t \in |A|$ we have*

$$Et = \bigvee_{s \in S} [t = s].$$

If A is a presheaf, then this is equivalent to :

$$\forall\, t \in |A|\ \exists\ \alpha \subseteq \Omega \text{ and } \{s^p : p \in \alpha\} \subseteq S \text{ such that}$$
$$\bigvee \alpha = Et \text{ and } t_{|p} = s^p_{|p},\ \forall\ p \in \alpha.$$

Define the **density** *of A as*

$$d(A) = min\,\{cardinal(S) : S \text{ is dense in } A\}.$$

Note that we can always find $S \subseteq |A|$ with cardinal(S) = d(A).

A is said to be **separable** *if $d(A) \leq \aleph_0$.*

Let A be a presheaf over Ω. For $\vec{s}, \vec{\in} |A|^n$ and $p \in \Omega$, define

*) $E\,\vec{s} = \bigwedge_{i=1}^{n} Es_i$;

**) $[\vec{s} = \vec{]} = \bigwedge_{i=1}^{n} [s_i = t_i]$; thus, $E\vec{s} = [\vec{s} = \vec{s}]$.

***) $\vec{s}_{|p} = (s_{1|p}, \ldots, s_{n|p})$.

We describe the notion of product in the categories we are considering. Let $A_1, \ldots, A_n$ be a finite collection of Ω-sets. Define an Ω-set $\prod_{i=1}^{n} A_i$ by the following rules :

i) $|\prod_{i=1}^{n} A_i| = \{(s_1, \ldots, s_n) \in \prod_{i=1}^{n} |A_i| : Es_i = Es_j,\ 1 \leq i, j \leq n\}$.

ii) For $\vec{s}$, $\vec{}$in $|\prod_{i=1}^{n}|\ A_i$, $[\vec{s} = \vec{]} = \bigwedge_{i=1}^{n} [s_i = t_i]$ (as in (**) above).

iii) If each A_i is a presheaf, put $\vec{s}_{|p} = (s_{1|p}, \ldots, s_{n|p})$ (as in (***) above).

With these definitions $\prod_{i=1}^{n} A_i$ is an extensional Ω-set which is a presheaf, finitely complete or a sheaf iff the same is true of each component. Moreover, we have natural projections $\pi_i : \prod A_i \longrightarrow A_i$ given by $s_1, \ldots, s_n \mapsto s_i$,

$1 \leq i \leq n$, which are morphisms of Ω-sets. This construction gives the product in the categories we are considering and, recalling that Ω is complete, it's clear that in fact these categories have **all** products.

To be able to deal with subobjects as well as to define the value of formulas with respect to a presheaf, it's convenient to introduce the notion of **characteristic function**.

Definition 11 : *A* **characteristic function** *on an Ω-set A is a map $|A|^n \xrightarrow{h} \Omega$, $n \geq 0$ a positive integer, such that for all $\vec{x}$, $\vec{y} \in |A|^n$,*

[ch 1] : $h(\vec{x}) \leq E\vec{x}$.

[ch 2] : $h(\vec{x}) \wedge [\vec{x} = \vec{y}] \leq h(\vec{y})$.

For each $n \geq 0$, $Ch(n, A, \Omega)$ denotes the set of characteristic functions defined on $|A|^n$. Whenever clear from context, the reference to Ω will be omitted from the notation and we write $Ch(n, A)$ for $Ch(n, A, \Omega)$.

It is simple to verify that [ch 2] is equivalent to

$$\text{[ch 2']} : \quad h(\vec{x}) \cap [\vec{x} = \vec{y}] = h(\vec{y}) \cap [\vec{x} = \vec{y}],$$

for all $h \in \mathrm{Ch}(n, A)$ and all $\vec{x}$, $\vec{y} \in |A|^n$.

Notice that if $n = 0$, a characteristic function $h : |A|^0 \longrightarrow \Omega$ corresponds to a map from $\{*\}$ to Ω that is, **an element of** Ω.

As maps from $|A|^n$ to Ω, characteristic functions inherit a natural partial order :

$$h \leq k \quad \text{iff} \quad \forall\, \vec{x} \in |A|^n,\ h(\vec{x}) \leq k(\vec{x}).$$

In fact, it's easily seen that if h_i, $i \in I$, is a family of elements of $\mathrm{Ch}(n, A)$, A an Ω-set, then

$$\textstyle\bigwedge_{i\in I} h_i : |A|^n \longrightarrow \Omega, \text{ defined by } \quad [\bigwedge_{i\in I} h_i](\vec{x}) = \bigwedge_{i\in I} h_i(\vec{x}),$$

as well as

$$\bigvee_{i \in I} h_i : |A|^n \longrightarrow \Omega(X), \quad \text{defined by} \quad [\bigvee_{i \in I} h_i](\vec{x}) = \bigvee_{i \in I} h_i(\vec{x}),$$

are characteristic functions, respectively the inf and the sup of the family h_i in the poset Ch(n, A).

Straightforward computations will show that for $n \geq 0$ and $A \in \Omega$-sets, **Ch(n, A) is a complete Heyting algebra** with the operations described above, that is, for all k, h_i, $i \in I$, in Ch(n, A)

$$k \wedge \bigvee h_i = \bigvee_{i \in I} (k \wedge h_i).$$

Notice that in the cHa Ch(n, A), we have 0 as the characteristic function identically equal to $0 \in \Omega$ and 1 as the characteristic function $\vec{x} \mapsto E\vec{x}$, the largest of all characteristic functions on $|A|^n$.

With the operations of implication and negation (or pseudocomplementation) one cannot directly translate from the operations on Ω to Ch(n, A), for this will not preserve the quality of being a characteristic function. Nevertheless, it's readily checked that if $h, k \in$ Ch(n, A) then the implication $h \rightarrow k$ and the negation $\neg\, h$ in the cHa Ch(n, A) are given, respectively, by

$$[h \rightarrow k](\vec{x}) = E\vec{x} \wedge (h(\vec{x}) \rightarrow k(\vec{x})) \quad \text{and}$$
$$[\neg\, h](\vec{x}) = E\vec{x} \wedge \neg\, h(\vec{x}),$$

where the symbols $\rightarrow$ and $\neg$ in the righthand side of the above formulas correspond to the implication and negation in Ω.

For a sheaf A over Ω and for each $n \geq 0$, the cHa Ch(n, A) is isomorphic to the cHa of subsheaves of the power A^n. For presheaves there is a similar isomorphism, except that we cannot take all sub presheaves but only those that have certain closure properties; we omit the details.

If A is a presheaf over Ω, one should keep in mind the distinction between $|A|^n$ and $|A^n|$. Note that, in general, $\vec{x}_{|E\vec{x}}$ **is not** equal to $\vec{x}$.

As a matter of fact,

$$\vec{x}_{|E\vec{x}} \text{ is always in } |A^n| \quad \text{and} \quad \vec{x}_{|E\vec{x}} = \vec{x} \quad \text{iff} \quad \vec{x} \in |A^n|.$$

The vector notation introduced above is a very convenient way to deal with products and characteristic functions. The basic properties of characteristic maps and its relation to the concept of density are described in the next Lemma. It also includes the fact that morphisms are uniquely determined in dense sets as well as a result on extension of morphisms (item (e)).

Lemma 2 : *With notation as above, let A be a presheaf over Ω, h, $k \in Ch(n, A)$ and $D \subseteq |A|$ a dense set of sections in A. Then*

a) For all $\vec{x} \in |A|^n$ and $\vec{p} \in \Omega^n$,

$$h(x_{1|p_1}, \ldots, x_{n|p_n}) = h(\vec{x}) \wedge \bigwedge_{i=1}^{n} p_i.$$

In particular, $h(\vec{x}_{|E\vec{x}}) = h(\vec{x}_{|h(\vec{x})}) = h(\vec{x})$.

b) Let $S \cup \{x\} \subseteq |A|$ and $p \in \Omega$ be such that $p = \bigvee_{s\in S} [x = s]$. Then, for all $\vec{a} \in |A|^{n-1}$,

$$p \wedge h(x, \vec{a}) = \bigvee_{s\in S} (h(s, \vec{a}) \wedge [s = x]).$$

In particular, if p_i, $i \in I$, is such that $p = \bigvee_{i\in I} p_i$, then

$$h(x_{|p}, \vec{a}) = \bigvee_{i\in I} h(x_{|p_i}, \vec{a}).$$

c) Suppose $T \subseteq |A^n|$ is dense in A^n. If h and k coincide in T then $h = k$. In particular, if h and k coincide in D^n, then they are equal.

d) Let $h_0 : D^n \longrightarrow \Omega$ be a map such that for all $\vec{x}$, $\vec{y} \in D^n$, $h_0(\vec{x}) \leq E\vec{x}$ and $h_0(\vec{x}) \wedge [\vec{x} = \vec{y}] \leq h_0(\vec{y})$. Then there is a unique characteristic function $h \in Ch(n, A)$ such that $h_{|D^n} = h_0$.

(e) Let B be an Ω-set. If f, $g : A \longrightarrow B$ are morphisms such that $f_{|\Gamma} = g_{|\Gamma}$, then $f = g$.

Further, suppose B is a sheaf over Ω and $f : \Gamma \longrightarrow B$ is a map such that for all s, $s' \in \Gamma$

1. $Efs = Es$ *2. $[s = s']_A \leq [fs = fs']_B$.*

Then there is a unique Ω-set morphism $g : A \longrightarrow B$ extending f.

4. L-structures in Presheaves over Ω

Let L be a 1^{st} order language with equality and Ω a cHa.

Definition 12 : *An interpretation of L in a Ω-set A consists of the following data :*

1. *For each n-ary operation ω in L, a morphism $\omega^A : A^n \longrightarrow A$;*

2. *For each constant c in L, a global section $c^A \in A(1)$;*

3. *For each n-ary relation symbol R, distinct from =, a characteristic map $[R(\ldots)]_A : |A|^n \longrightarrow \Omega$. The equality relation shall be interpreted by the characteristic map $[\cdot = \cdot]$.*

Whenever clear from context, all references to the name of the underlying Ω-set shall be omitted. We refer to A as a L-structure in pSh(Ω), fc Ω-sets or Sh(Ω) if A belongs to any of these categories.

It can be shown that presheaves of L-structures, as defined in section 1, are instances of the general definition given above.

Let $p \in \Omega$ and suppose that A is an L-structure in pSh(Ω) such that A(p), the set of sections of A over p, is not empty. A(p) inherits a natural L-structure from A as follows :

1. If ω is a n-ary operation symbol in L, since it is a morphism we know that for all $s_1, \ldots, s_n \in$ A(p), $E\omega(s_1, \ldots, s_n) = p$; thus, $\omega(s_1, \ldots, s_n) \in$ A(p), giving the interpretation of the n-ary operation ω in A(p). Observe that the interpretation of ω in A(p) is simply the restriction of the morphism ω^A to $(A(p))^n \subseteq |A^n|$.

2. If c is a constant symbol in L, then $c^A_{|p} \in$ A(p) is the interpretation of c in A(p).

3. If R is a n-ary relation symbol in L and $s_1, \ldots, s_n \in$ A(p), define

$$A(p) \models R[s_1, \ldots, s_n] \quad \text{iff} \quad [R(s_1, \ldots, s_n] = p.$$

Notice that in particular, condition [ext] guarantees that the interpretation induced by $[\cdot = \cdot]$ in A(p) is the identity.

4. Note that if $p \leq q$ in Ω, then the restriction map $A(q) \longrightarrow A(p)$ is a L-morphism. This is quite clear for operations (they are morphisms!) and constants. For relations, just recall that Lemma 1.(a) yields $[R(s_{1|p}, \ldots, s_{n|p})] = p \wedge [R(s_1, \ldots, s_n)]$, $\forall\, s_1, \ldots, s_n \in A(q)$.

By the usual induction on complexity, one can show that for each term $\tau(v_1, \ldots, v_n)$ in L we can associate a morphism $\tau^A : A^n \longrightarrow A$, its interpretation in A. Just as above, for each $p \in \Omega$ we have that the interpretation of τ in A(p) is simply the restriction of the morphism τ^A to $(A(p))^n \subseteq |A^n|$.

Definition 13 : *Let A be a L-structure in pSh(Ω). By induction on complexity, we define the value of a formula $\phi(v_1, \ldots, v_n)$ in L with respect to A as a characteristic function*

$$[\phi(\cdot, \ldots, \cdot)]_A : |A|^n \longrightarrow \Omega,$$

its interpretation in A, as follows :

For $\vec{a} \in |A|^n$,

[atom] : If $\tau_j(\vec{v})$, $1 \leq j \leq m$ are terms in L and $R \in rel(m)$ then

$$[\tau_1 = \tau_2(\vec{a})]_A = [\tau_1^A(\vec{a}) = \tau_2^A(\vec{a})]$$

and

$$[R(\tau_1(\vec{v}), \ldots, \tau_m(\vec{v}))(\vec{a})]_A = [R^A(\tau_1^A(\vec{a}), \ldots, \tau_m^A(\vec{a}))].$$

[conn] : If $\diamond$ is a binary connective in L then

$$[\phi \diamond \psi(\vec{a})]_A = E\vec{a} \wedge ([\phi(\vec{a})]_A \diamond [\psi(\vec{a})]_A)$$

and

$$[\neg\phi(\vec{a})]_A = E\vec{a} \wedge \neg [\phi(\vec{a})]_A,$$

where the $\diamond$ and $\neg$ in the righthand side of the above equations are operations in Ω presented in section 2.

[∃] : $[\exists x \phi(x; \vec{v})(\vec{a})]_A \equiv_{def} [\exists x \phi(\vec{a})]_A = \bigvee_{t \in |A|} [\phi(t; \vec{a})]_A$.

[∀] : $[\forall x \phi(x; \vec{v})(\vec{a})]_A \equiv_{def} [\forall x \phi(\vec{a})]_A = E\vec{a} \wedge \bigwedge_{t \in |A|} Et \rightarrow [\phi(t; \vec{a})]_A$.

As always, the name of the underlying Ω-set will be dropped from the notation when clear from context.

The extent to which existential assertions have witnesses in our context is described in

Theorem 1 : *(The maximum principle) Let* $\exists\, v\phi(v; (_1, \ldots, (_n v))$, $n \geq 0$, *be a formula in L and A an L-structure in Sh(Ω). For all* $\vec{a} \in |A|^n$ *there is* $b \in |A|$ *such that* $[\phi(b; \vec{a})] \leq [\exists v\phi(v; \vec{a})] \leq \neg\neg\, [\phi(b; \vec{a})]$.

We now define the different notions of L-morphisms.

Definition 14 : *Let A, B be L-structures in pSh(Ω) and* $A \xrightarrow{f} B$ *a morphism of presheaves. If* $\vec{s} \in |A|^n$, *write* $f\vec{s}$ *for* $(fs_1, \ldots, fs_n)$. *We say that f is*

a) A **L-morphism** *iff for all* $\vec{s} \in |A|^n$,

i) For all n-ary operations ω *in L,* $\omega(f(\vec{s}_{|E\vec{s}})) = f(\omega(\vec{s}_{|E\vec{s}}))$;

ii) For each constant c in L, $fc^A = c^B$;

iii) For all n-ary operations R in L, $[R(\vec{s})] \leq [R(f\vec{s})]$.

b) A **L-monic** *iff it's a L-morphism such that for all n-ary relations R in L and all* $\vec{s} \in |A|^n$, $[R(\vec{s})] = [R(f\vec{s})]$.

c) An **elementary monomorphism** *iff for all formulas* $\phi(v_1, \ldots, v_n)$ *in L and all* $\vec{s} \in |A|^n$, $[\phi(\vec{s})] = [\phi(f\vec{s})]$.

Whenever $A \subseteq B$ and the natural L-morphism is an elementary monomorphism we say (as usual) that A is an elementary substructure of B and write $A \prec B$.

The following simple result is important.

Theorem 2 : *Let A be a L-structure in pSh(Ω). Then there is an L-structure δA in fc Ω-sets and a L-structure cA in Sh(Ω), together with* **elementary monomorphisms** $A \xrightarrow{\delta} \delta A$, $A \xrightarrow{c} cA$ *and* $\delta A \xrightarrow{\gamma} cA$ *such that*

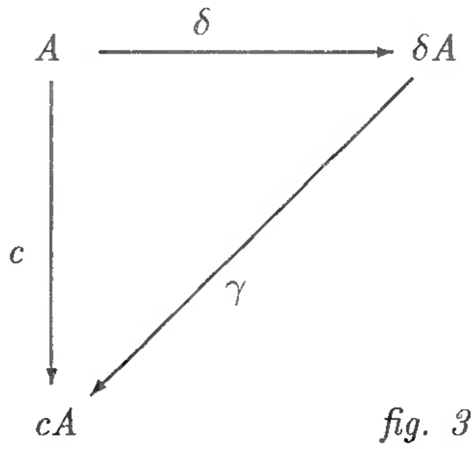

fig. 3

1. *The image of A by δ and c are dense in δA and cA, respectively.*
2. $\delta \circ \gamma = c$*; consequently the image of δA by γ is also dense in cA.*

The L-structures δA and cA are called the **finite completion** and the **completion or sheafification** of A. We shall always consider $A \subseteq \delta A \subseteq cA$ and so we have $A \prec \delta A \prec cA$.

Let A_n, $n \geq 0$, be a sequence of L-structures in pSh(Ω) such that $A_n \subseteq A_{n+1}$. A moment of thought is enough to convince oneself that $|A| = \bigcup_{n \geq 0} |A_n|$ is the domain of a L-structure A in pSh(Ω) such that $A_n \subseteq A$. Furthermore, if each A_n is finitely complete, the same will be true of A. We write $\bigcup_{n \geq 0} A_n = A$ (the union in pSh(Ω) or fc Ω-sets of the A_n).

If each A_n is a sheaf it will no longer be true, in general, that A will be a sheaf. However, the completion of A, cA, will be a sheaf and in fact, the 'least' sheaf containing all the A_n. We still write $cA = \bigcup_{n \geq 0} A_n$, with the understanding that, in the category of sheaves, the construction of taking the union of the domains of the A_n is followed by that of completion.

With these preliminaries we state

Proposition 1 *(Tarski's union of chains) : Let A_n be a sequence of L-structures in pSh(Ω) (fc Ω-sets, Sh(Ω)) such that $A_n \prec A_{n+1}$, $n \geq 0$. Then, for each $n \geq 0$, we have $A_n \prec \bigcup_{n\geq 0} A_n$.*

It will be important to discuss substructures generated by a set of sections as well as the density of these.

If A is a L-structure in pSh(Ω) and S $\subseteq$ |A| is a set of sections, S* denotes the L-substructure of A generated by S.

Since it's easily verified that the intersection of the domains of any collection of L-substructures of A is the domain of a L-substructure of A, it's clear that the term 'generated by' makes sense. The question is to describe S* in a explicit form.

Let Ct be the set of constants in L and op(m) the set of m-ary operation symbols in L.

Given $S \subseteq |A|$, define by induction on $n \geq 0$,

$S_0 = \{s_{|p} : s \in S \text{ or } s = c^A, \text{ where } c \in Ct \text{ and } p \in \Omega\}$.

$S_{n+1} = S_n \cup \{\omega(\vec{s}_{|E\vec{s}}) : \omega \in op(m) \text{ and } \vec{s} \in S_n^m, m \geq 1\}$.

Lemma 3 : *Let A be a L-structure in pSh(Ω) and $S \subseteq |A|$. Then*

a) $S^* = \bigcup_{n\geq 0} S_n$.

b) Let $D \subseteq |A|$ be such that

i) $\{c^A : c \in Ct\} \subseteq D$;

ii) $\forall\, n \geq 0, \forall\, (y, \vec{s}) \in D^{n+1}$, *we have* $y_{|E\vec{s}} \in D$;

iii) $\forall\, \vec{s} \in D^m, \forall\, \omega \in op(m)$, *we have* $\omega(\vec{s}_{|E\vec{s}}) \in D$;

iv) D is dense in S.

Then D is dense in S.*

c) Let $K \subseteq |A|$ be dense in S with cardinal(K) = $d(S)$. Define, by induction on $n \geq 0$, two sequences K_n and D_n as follows :

1. $K_0 = K$ *and* $D_0 = K \cup \{c^A : c \in Ct\}$.
2. $K_{n+1} = \{y_{|E\vec{s}} : (y, \vec{s}) \in D_n^{k+1},\ k \geq 0\}$

   $D_{n+1} = K_{n+1} \cup \{\omega(\vec{s}_{|E\vec{s}}) : \vec{s} \in K_{n+1}^m$ *and* $\omega \in op(m),\ m \geq 0\}$.

Then $D = \bigcup D_n$ satisfies the conditions in item (b) and has cardinality not greater than max $\{d(S),$ cardinal$(L)\}$.

d) density of S^* = $d(S^*) \leq$ max $\{d(S),$ cardinal$(L)\}$. Therefore, $d(\delta S^*) = d(cS^*) \leq$ max $\{d(S),$ cardinal$(L)\}$. Moreover, we have cardinal$(S^*) \leq$ max $\{card(S \times \Omega),$ cardinal$(L)\}$.

For compact elements of Ω we can state

Lemma 4 : *Let A be a L-structure in $pSh(\Omega)$ and p a compact element in Ω. Let $A(p)$ be the structure of sections of A over p and $w(\Omega)$ the weight of Ω as in Def. 5. Then*

a) cardinal $(A(p)) \leq$ cardinal {finite subsets of $d(A) \times w(\Omega)$}.

b) $\delta A(p) = cA(p)$, that is the finite completion and the completion of A have the same set of sections over p.

5. The Downward Lowenheim-Skolem Theorem

In order to prove a sheaf theoretic version of the downward Lowenheim-Skolem theorem, we shall have to impose certain conditions on the cHa Ω, described in the definition that follows. To simplify exposition we shall assume that the language L is countable.

Definition 15 : *A cHa Ω is said to have*

1. **countably determined meets (cdm)** *iff for all $S \subseteq \Omega$ there is a countable $T \subseteq S$ such that $\bigwedge T = \bigwedge S$.*

2. **countably determined joins (cdj)** *iff for all $S \subseteq \Omega$ there is a countable $T \subseteq S$ such that $\bigvee T = \bigvee S$.*

3. **countable caracter (cc)** *iff it has cdm and cdj.*

4. **the countable chain condition (ccc)** *iff for all $S \subseteq \Omega$ - $\{0\}$, if $s \wedge s' = 0$, for $s \not\subseteq s'$ in S, then S must be at most countable.*

If X is a topological space, recall that X is **separable** if it has a countable dense subset. It is said to be **Lindeloff** if all open covers of X have a countable subcover. If P is a property of topological spaces, X is said to be **hereditarily P** if all subspaces of X have P.

Lemma 5 : *For a cHa Ω we have*

a) If Ω has cdm or cdj then Ω is ccc.

b) If B is a complete Boolean algebra (cBa), are equivalent :

i) B has cdm; ii) B has cdj iii) B is ccc.

c) Let X be a topological space and $\Omega(X)$ the cHa of opens in X. Then $\Omega(X)$ has the ccc iff X has the ccc. Further, are equivalent

i) $\Omega(X)$ has the cdm (cdj);

ii) X is hereditarily separable (resp., hereditarily Lindeloff).

Remarks : 1. Every second countable space is, of course, hereditarily separable and hereditarily Lindeloff. The converse is false even for regular Hausdorff spaces.

2. The properties cdm, cdj and cc are **not** faithfully reflected by Stone duality. As an example consider the cBa 2^{ω} (parts of the natural numbers) which is clearly ccc and so has cc. Its Stone space, $\beta\omega$, the Stone-Cech compatification of the natural numbers is neither hereditarily separable nor hereditarily Lindeloff.

Theorem 3 : *Let Ω be a cHa with cc and L a countable first order language with equality. If A is a L-structure in $pSh(\Omega)$ and $S \subseteq |A|$ is a countable set of sections in A, then there is a separable elementary substructure $B \prec A$ such that $S \subseteq |B|$.*

Corollary 1 : *Let Ω and L be as in Theorem 3. If A is a L-structure in $Sh(\Omega)$ (fc Ω-sets) then for all separable (resp., countable) $S \subseteq |A|$, there is a separable subsheaf (resp., finitely complete) B such that $S \subseteq |B|$ and $B \prec A$.*

Theorem 4 : *Suppose that Ω has countable weight (Def. 5) and that L is countable. Let $\{p_n : n \geq 0\}$ be a sequence of compact elements in Ω. Let A be a L-structure in $pSh(\Omega)$ and $S \subseteq |A|$ be a countable set of sections in A. Then there is an elementary substructure $B \prec A$ such that*

1. *B is separable and $S \subseteq |B|$;*
2. *For each $n \geq 0$, $B(p_n)$ is a countable classical elementary substructure of $A(p_n)$.*

Theorem 5 : *Let Ω, L and p_n, $n \geq 0$, be as in Theorem 4. Let A be a L-structure in $Sh(\Omega)$ and $S \subseteq |A|$ a separable set of sections in A. Then there is a separable L-structure B in $Sh(\Omega)$ such that*

1. *$S \subseteq B$ and $B \prec A$*
2. *For all $n \geq 0$, $B(p_n) \prec A(p_n)$.*

For sheaves over topological spaces we can treat any sequence of compacts, whether open or not. This includes, of course, stalks at points of X.

Theorem 6 : *Let X be a second countable space and p_n, $n \geq 0$, a sequence of compacts (not necessarily open) in X. Let A be a sheaf of L-structures over X and $S \subseteq |A|$ a countable subset of the domain of A. If L is a countable language, there is a separable elementary subsheaf $B \subseteq A$ such that $S \subseteq |B|$ and $B(p_n) \prec A(p_n)$, $n \geq 0$.*

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Back-and-forth for systems of antichains

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Introduction

Throughout this paper "root-system" stands for a partially ordered set such that the set of successors of any element is totally ordered. In other words, root-systems are the order-duals of trees.

In [2] Conrad, Harvey and Holland proved that any abelian lattice-ordered group can be represented as a lattice-subgroup of the Hahn power over a root-system of copies of the real line with its totally ordered group structure. Moreover, there are particular but interesting cases of lattice-ordered groups where the root-system can be determinated, see [1]. In a forthcoming paper, see [5], it is proved that the elementary theory of a Hahn product of divisible totally ordered abelian groups is determined by the elementary theory of the lattice of antichains of its root-system. In this paper we prove that this last theory is determined by the theory of its underlying root-system. So, we can conclude that the theory of such a group is determined by the theory of its root-system.

In fact, the proof of transfer for elementary equivalence of root-systems to their respective lattices of antichains, gives also the transfer for elementary embedding. Concerning model-theoretic properties, we end up with some (trivial) remarks about unstability and the independence property for the theories of systems of antichains.

In another development, in 1960 Dilworth (see [3]) proved that the system of antichains over a finite poset has a structure of distributive lattice; moreover, he proved that any finite distributive lattice admits such a representation. In general, for infinite ordered sets the corresponding systems of antichains do not admit a distributive lattice structure but only a sup-semilattice one (see [7]). However, for very special ordered sets, e.g. root-systems (or their order-duals, trees) an analog of the Dilworth result holds. In this paper we present these systems of antichains as a natural generalization of the notion of an atomic Boolean algebra.

Let $S = \langle S, \leq \rangle$ be a root-system, we shall denote by $A(S) = \langle\, A(S), \preceq \rangle$ the set of all antichains of S endowed with the binary relation given by:

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$$a \preceq b \text{ if and only if } \forall x \in a \exists y \in b (x \leq y).$$

Definition 1 *For a root-system S, we shall say that $A = \langle A, \preceq \rangle$ is a* presystem of antichains over S *if $A \subseteq A(S)$, the order relation is the restriction of that of $A(S)$, and $S \subseteq A$ in the sense that all the antichains formed by only one element belong to A.*

Definition 2 *For a presystem of antichains A and $a, b \in A$, we say that b is a* quasicomplement *of a if $a \cup b$ is a maximal antichain.*

(Observe that, in general, an antichain can have many quasicomplements).

Definition 3 *We say that two antichains a and b are* orthogonal *if and only if $a \cap b = \emptyset$ and $a \cup b$ is an antichain. The set of all antichains orthogonal to a will be denoted by $a^{\perp}$.*

Let S be a root-system, A a presystem of antichains over S and $\{a_1, \ldots, a_n\} \subseteq A$. Since each antichain is a subset of S, call $S(\{a_1, \ldots, a_n\})$ the union of those antichains, and endow that set with the restriction of the order of S. Since any chain in $S(\{a_1, \ldots, a_n\})$ has at most n elements the following definition is not void:

Definition 4 *Call $\mathcal{M}\{a_1, \ldots, a_n\}$ to the set of all minimal points of $S(\{a_1, \ldots, a_n\})$.*

Remark 1 $\mathcal{M}\{a_1, \ldots, a_n\}$ is an antichain.

Remark 2 $a_i \succeq \mathcal{M}\{a_1, \ldots, a_n\}$ $(1 \leq i \leq n)$, where $a \succeq b$ iff $\forall x \in a \exists y \in b (x \geq y)$.

Definition 5 *Let a be an antichain and x a point of the root-system, we define $a(x)$ as the unique point $y \in a$ such that $x \leq y$ if that point exists and $\emptyset$ if not (obviously there is at most one such point y).*

Definition 6 *Let $\varphi(v_1, \ldots, v_n)$ be a quantifier-free formula. Define $[\![\varphi[a_1, \ldots, a_n]]\!]$ by induction:*

$[\![v_i \preceq v_j[a_1, \ldots, a_n]]\!] = \{x \in \mathcal{M}\{a_1, \ldots, a_n\} \,/\, a_i(x) \neq \emptyset \;\&\; a_j(x) \neq \emptyset \;\&\; a_i(x) \leq a_j(x)\}$;

$[\![\varphi \wedge \psi[a_1, \ldots, a_n]]\!] = [\![\varphi[a_1, \ldots, a_n]]\!] \cap [\![\psi[a_1, \ldots, a_n]]\!]$;

$[\![\neg\varphi[a_1, \ldots, a_n]]\!] = \mathcal{M}\{a_1, \ldots, a_n\} \setminus [\![\varphi[a_1, \ldots, a_n]]\!]$.

In particular, we call $\pi(v_i[a_1, \ldots, a_n])$ (the projection of a_i on $\mathcal{M}\{a_1, \ldots, a_n\}$) the set $[\![v_i \preceq v_i[a_1, \ldots, a_n]]\!] = \{x \in \mathcal{M}\{a_1, \ldots, a_n\} \,/\, a_i \neq \emptyset\}$.

Remark 3 $[\![\varphi[a_1, \ldots, a_n]]\!]$ is always a subantichain of $\mathcal{M}\{a_1, \ldots, a_n\}$.

Remark 4 $a_i \preceq a_j$ if and only if $[\![v_i \preceq v_j[a_1, \ldots, a_n]]\!] = \pi(v_i[a_1, \ldots, a_n])$ (Where $\{a_1, \ldots, a_n\}$ is any set containing a_i and a_j.)

Definition 7 *Define $\mathcal{MP}\{a_1,\ldots,a_n\}$ as the partition (where some of its elements may be empty) induced on $\mathcal{M}\{a_1,\ldots,a_n\}$ by:*

Given $C \subseteq \mathcal{M}\{a_1,\ldots,a_n\}$, we say that $C \in \mathcal{MP}\{a_1,\ldots,a_n\}$ if and only if there exists a quantifier-free formula $\psi(v_1,\ldots,v_n)$ such that $C = [\![\psi[a_1,\ldots,a_n]]\!]$ and for all quantifier-free formulas $\varphi(v_1,\ldots,v_n)$, $[\![\psi[a_1,\ldots,a_n]]\!] \subseteq [\![\varphi[a_1,\ldots,a_n]]\!]$ or $[\![\psi[a_1,\ldots,a_n]]\!] \subseteq [\![\neg\varphi[a_1,\ldots,a_n]]\!]$ holds.

Remark 5 It is clear that every $C \in \mathcal{MP}\{a_1,\ldots,a_n\}$ is defined by a condition of the form:

$$[\![(\bigwedge_{i,j=1}^{n} \varphi^{\epsilon(i,j)}(v_i,v_j)[a_1,\ldots,a_n]]\!]$$

where $\epsilon : \{1,\ldots,n\} \times \{1,\ldots,n\} \longrightarrow 2$, $\varphi^0(v_i,v_j)$ is $\neg(v_i \preceq v_j)$ and $\varphi^1(v_i,v_j)$ is $(v_i \preceq v_j)$.

Let $\varphi^\epsilon(v_1,\ldots,v_n)$ be $\bigwedge_{i,j=1}^{n} \varphi^{\epsilon(i,j)}(v_i,v_j)$, and $\mathcal{F}_n = \{\varphi^\epsilon \ / \ \epsilon \in 2^{\{1,\ldots,n\}\times\{1,\ldots,n\}}\}$. Clearly, $\mathcal{F}_n$ has cardinality 2^{n^2}, and hence $\sharp(\mathcal{MP}\{v_1,\ldots,v_n\}) \leq 2^{n^2}$.

Remark 6 Given antichains $a_1,\ldots,a_{n+1}$ and $\psi \in \mathcal{F}_n$, let $\{\psi_1,\ldots,\psi_{2^{2n+1}}\}$ be the subset of $\mathcal{F}_{n+1}$ consistent with ψ; then we have $[\![\psi_k[a_1,\ldots,a_{n+1}]]\!] \preceq [\![\psi[a_1,\ldots,a_n]]\!]$ $(1 \leq k \leq 2^{2n+1})$ and

$$[\![\psi[a_1,\ldots,a_n]]\!] \succeq \bigcup_{k=1}^{2^{2n+1}} [\![\psi_k[a_1,\ldots,a_{n+1}]]\!].$$

So, we can say that $\mathcal{MP}\{v_1,\ldots,v_{n+1}\}$ "refines" $\mathcal{MP}\{v_1,\ldots,v_n\}$.

Definition 8 *For a presystem of antichains A, define a binary function Rest(,) given by:*

$$Rest(a,b) = \{x \in a \ / \ \exists y \in b \ (x \leq y \text{ or } y \leq x)\}$$

that is, the set of all elements of a comparable with b. We shall also denote Rest(a,b) by $a \upharpoonright b$ (it shall be thought of as a "restriction function").

Remark 7 Given $a_1,\ldots,a_n$ and $(\varphi_k(v_1,\ldots,v_n))$ $(1 \leq k \leq 2^{n^2})$ the formulas of $\mathcal{F}_n$, if we call $a_{i(k)}$ the set $a_i \upharpoonright [\![\varphi_k[a_1,\ldots,a_n]]\!]$, we have that, for each k the set $\{a_{1(k)},\ldots,a_{n(k)}\}$ of antichains (where some of the $a_{i(k)}$ may be empty) is totally ordered. Analogously, since $Rest(\ ,\ )$ is a function, we write $v_{i(k)} = Rest(v_i,[\![\varphi_k[v_1,\ldots,v_n]]\!])$ $(1 \leq i \leq n)$ where the v_i's are variables. Observe that, independently of any concrete model, there are i and k such that $a_{i(k)}$ is always empty. For example, consider the formula $\varphi_1(v_1,v_2)$ given by $\neg(v_1 \preceq v_2)$ & $\neg(v_2 \preceq v_1)$ & $\neg(v_1 \preceq v_1)$ & $v_2 \preceq v_2$. In this case $a_{1(1)} = \emptyset$ for any a_1 but $a_{2(1)}$ depends on the model and the particular element.

Definition 9 *Let $\varphi_k \in \mathcal{F}_n$. Consider the set*

$$\{k \,/\, (1 \leq k \leq n) \text{ and } PSA \not\vdash v_{i(k)} \neq \emptyset\}$$

where PSA is the theory of presystems of antichains. Seen as a subsequence of $1,\dots,n$ write it as $i_1,\dots,i_p$. Define now the formula $\varphi_k^*(v_1,\dots,v_p)$ of the language $\langle\leq\rangle$ as the total order implied by φ_k on the antichains $(v_{i_1},\dots,v_{i_p})$.

In the sequel, we refer the reader to [4, Ch. XI] for details. Now, recall that, for the language $\langle\leq\rangle$, we can define inductively the back-and-forth equivalence classes $(\alpha_{n,m,i})_{n,m\in\omega,(1\leq i\leq\delta_{n,m})}$ (where $\delta_{n,m}$ is their (finite) number), for any $\langle\leq\rangle$-structure (in particular for a root-system) in the following way:

For $m = 0$: $(x_1,\dots,x_n) \equiv_{n0} (y_1,\dots,y_n)$ iff the two n-tuples are isomorphic for $\leq$;

For $m+1$, the class of $(x_1,\dots,x_n)$, denoted by $\mathcal{C}_{n,m+1}(x_1,\dots,x_n)$, is defined by the $\langle\leq\rangle$-formula:

$$\bigwedge_{i\in I_1}(\exists x_{n+1}\mathcal{C}_{n+1,m}(x_1,\dots,x_{n+1}) = \alpha_{n+1,m,i}) \& \bigwedge_{i\in I_2}(\neg\exists x_{n+1}\mathcal{C}_{n+1,m}(x_1,\dots,x_{n+1}) = \beta_{n+1,m,i}).$$

Where $(\alpha_{n+1,m,i})_{i\in I_1}$ and $(\beta_{n+1,m,i})_{i\in I_2}$ are all $(n+1)$-ary classes of depth m separated by the fact that there exists or not an x_{n+1} such that $(x_1,\dots,x_{n+1})$ satisfies it.

Let us compute $\delta_{n,m}$. For $m = 0$, the isomorphism classes are given by the complete formulas of the form

$$\bigwedge_{i,j=1}^{n} \pm(x_i \leq x_j).$$

Hence $\delta_{n,0} \leq 2^{n^2}$. Observe that the above definition of a class of depth $m+1$ as a function of the classes of depth m gives $\delta_{n,m} \leq 2^{\delta_{n+1,m-1}}$. So we have that $\delta_{n,m}$ is bounded by an hyperexponential function of height $m+1$ and last exponent $(n+m)^2$.

Using the back-and-forth relation on root-systems, we shall define a new back-and-forth relation on presystems of antichains.

Definition 10 *Let A be a presystem of antichains over S, $a_1,\dots,a_n \in A$ and $m > 0$. Call $\mathcal{A}_m(a_1,\dots,a_n)$ the set formed by all the sets of 2^{m-1}-tuples of pairwise orthogonal antichains such that their union is orthogonal with $\mathcal{M}\{a_1,\dots,a_n\}$. Say that $(c_1,\dots,c_{2^{m-1}}) \in \mathcal{A}_m(a_1,\dots,a_n)$* is equivalent *to $(d_1,\dots,d_{2^{m-1}}) \in \mathcal{A}_m(a_1,\dots,a_n)$ if and only if, for each i $(1 \leq i \leq 2^{m-1})$ and for each unary class $\alpha_{1,m-1,j}$ of depth $m-1$ (of the root-system) the sets $c_i \cap \alpha_{1,m-1,j}$ and $d_i \cap \alpha_{1,m-1,j}$ have the same cardinality (identifying numbers greater or equal than $2^{2(n+1)(m-1)+(m-1)^2}$). Call $\tilde{\mathcal{A}}_m(a_1,\dots,a_n)$ the set $\mathcal{A}_m(a_1,\dots,a_n)$ quotiented by this equivalence relation. So, each element of $\tilde{\mathcal{A}}_m(a_1,\dots,a_n)$ is coded by a 2^{m-1}-tuple of $\delta_{1,m-1}$-tuples of integers between 0 and $2^{2(n+1)(m-1)+(m-1)^2}$.*

Definition 11 *Let A and B be presystems of antichains over S and T (root-systems) respectively. For all $n, m \in \mathbb{N}$, $(a_1, \ldots, a_n) \in A^n$ and $(b_1, \ldots, b_n) \in B^n$, define $(a_1, \ldots, a_n) \rightleftharpoons_{nm} (b_1, \ldots, b_n)$ in the following way:*

i) If $\mathcal{F}_n = \{\varphi_1, \ldots, \varphi_{2^{n^2}}\}$, then for each $k \in \{1, \ldots, 2^{n^2}\}$ and $j \in \{1, \ldots, \delta_{q,m}\}$ (where q is the arity of the formula φ_k^*), the sets

$$\{(x_1, \ldots, x_q) \,/\, \text{for every } l,\ (1 \leq l \leq q),\ x_l \in a_{i_l} \text{ and there is } x \in [\![\varphi_k[a_1, \ldots, a_n]]\!] \\ \textit{such that } x \leq x_l \textit{ and } S \models \varphi_k^*(x_1, \ldots, x_q) \,\&\, \mathcal{C}_{q,m}(x_1, \ldots, x_q) = \alpha_{q,m,j}\}$$

and

$$\{(y_1, \ldots, y_q) \,/\, \text{for every } l,\ (1 \leq l \leq q),\ y_l \in b_{i_l} \text{ and there is } y \in [\![\varphi_k[b_1, \ldots, b_n]]\!] \\ \textit{such that } y \leq y_l \textit{ and } S \models \varphi_k^*(y_1, \ldots, y_q) \,\&\, \mathcal{C}_{q,m}(y_1, \ldots, y_q) = \alpha_{q,m,j}\}$$

have the same number of elements (identifying cardinalities greater or equal than 2^{2nm+m^2}), where $i_1, \ldots, i_q$ is the subsequence of $1, \ldots, n$ implied by Definition 9.

And, if $m > 0$
ii) There is an isomorphism between the sets $\dot{A}_m(a_1, \ldots, a_n)$ and $\dot{A}_m(b_1, \ldots, b_n)$ in the sense that $F \in \dot{A}_m(a_1, \ldots, a_n)$ corresponds to $H \in \dot{A}_m(b_1, \ldots, b_n)$ if and only if F and H are coded by the same tuple of integers (see Definition 10).

Remark 8 Point i) of the foregoing Definition states that, for k and j fixed, the number (up to 2^{2nm+m^2}) of q-tuples of the respective root-systems satisfying φ_k^* and $\alpha_{q,m,j}$, and constituted by points above $[\![\varphi_k[a_1, \ldots, a_n]]\!]$ and $[\![\varphi_k[b_1, \ldots, b_n]]\!]$, is the same for both systems.

Since the proof of the following Proposition is somewhat involved we shall intersperse some comments (in italics) to give the intuitive meaning of what we are doing.

Proposition 1 *Suppose $S \equiv T$, $A = A(S)$ and $B = A(T)$. Then the relation $(\rightleftharpoons_{nm})_{n,m \in \omega}$ is a back-and-forth relation between A and B.*

Proof: The n-ary classes of depth 0 for root-systems, are exactly the $\leq$-isomorphism classes and $2^{2n0+0^2} = 1$. So, suppose we have $(a_1, \ldots, a_n) \rightleftharpoons_{n0} (b_1, \ldots, b_n)$ and, for example, $a_1 \preceq a_2$. Then, for each $\varphi_i \in \mathcal{F}_n$, we have either $a_{1(i)} = \emptyset$ or $\emptyset \neq a_{1(i)} \preceq a_{2(i)}$. So, we have $[\![a_1 \preceq a_2[a_1, \ldots, a_n]]\!] \neq \emptyset$, $[\![((a_2 \preceq a_1) \wedge \neg(a_1 \preceq a_2))[a_1, \ldots, a_n]]\!] = \emptyset$ and $[\![((a_1 \preceq a_1) \wedge \neg(a_2 \preceq a_2))[a_1, \ldots, a_n]]\!] = \emptyset$.

Since we have, in particular $(a_1, a_2) \rightleftharpoons_{20} (b_1, b_2)$ and this implies that the corresponding elements of $\mathcal{MP}\{b_1, \ldots, b_n\}$ must be, respectively, non empty, empty and empty, we have that $b_1 \preceq b_2$. By repeating this argument for each pair of antichains, we conclude that $(a_1, \ldots, a_n) \cong_{\preceq} (b_1, \ldots, b_n)$.

Observe that, for this first step, we have only written the order $\preceq$ on the systems of antichains in terms of the order $\leq$ on the root-systems (see the beginning of this article).

Now, let be $(a_1, \ldots, a_n) \rightleftharpoons_{nm} (b_1, \ldots, b_n)$ and $a_{n+1} \in A$. Let $\mathcal{F}_n = \{\varphi_1, \ldots, \varphi_{2^{n^2}}\}$. Since

$\mathcal{MP}\{v_1,\ldots,v_{n+1}\}$ refines $\mathcal{MP}\{v_1,\ldots,v_n\}$, for each $i=1,\ldots,2^{n^2}$, there exists formulas $(\varphi_{i_j}(v_1,\ldots,v_{n+1}))$ $(1\le j\le 2^{2n+1})$ "refining" φ_i.

That is, all the formulas of $\mathcal{F}_{n+1}$ consistent with φ_i.

For each $i = 1,\ldots,2^{n^2}$ and $j = 1,\ldots,2^{2n+1}$ call $(a_{n+1})_{i_j}$ the antichain $\{x\in a_{n+1} \,/\, \exists y\in [\![\varphi_{i_j}[a_1,\ldots,a_{n+1}]]\!](y\le x)\}$.

$(a_{n+1})_{i_j}$ is the subantichain of a_{n+1} consisting of all of its elements comparable with $[\![\varphi_{i_j}[a_1,\ldots,a_{n+1}]]\!]$.

For each $i=1,\ldots,2^{n^2}$ call $(a_{n+1})_i$ the (disjoint) union of all $(a_{n+1})_{i_j}$ $(1\le j\le 2^{2n+1})$ and $(a_{n+1})'_i$ the antichain $[\![\varphi_i[a_1,\ldots,a_n]]\!]\setminus(a_{n+1})_i$.

$(a_{n+1})_i$ is the subantichain of a_{n+1} consisting of all of its elements comparable with $[\![\varphi_i[a_1,\ldots,a_n]]\!]$ and $(a_{n+1})'_i$ its complement relative to the same set.

Call a' the antichain $a_{n+1}\setminus\bigcup_{i=1}^{2^{n^2}}(a_{n+1})_i$.

a' is the set of all elements of a_{n+1} which are not comparable with any element of $\mathcal{M}\{a_1,\ldots,a_n\}$, and we have

$$a_{n+1}=a'\cup\bigcup_{i=1}^{2^{n^2}}(a_{n+1})_i\,,$$

where the union is disjoint.

If a' is not empty we have two possible cases:
If $m=1$, since a' is orthogonal with $\mathcal{M}\{a_1,\ldots,a_n\}$, we have that $a'\in\mathcal{A}_m(a_1,\ldots,a_n)$ (see Definition 10). Let F be the class of a' in $\tilde{\mathcal{A}}_m(a_1,\ldots,a_n)$. By property 11 (ii), there exists $H\in\tilde{\mathcal{A}}_m(b_1,\ldots,b_n)$ coded by the same tuple of integers that codes F, so take a representative $b'\in H$.

$\mathcal{A}_1(a_1,\ldots,a_n)$ is the set of all antichains orthogonal to $\mathcal{M}\{a_1,\ldots,a_n\}$.

If $m>1$ then consider all $(c_1,\ldots,c_{2^{m-1}})\in\mathcal{A}_m(a_1,\ldots,a_n)$ such that $a'=\bigcup_{i=1}^{2^{m-2}}c_i$. If F is the equivalence class (in $\tilde{\mathcal{A}}_m(a_1,\ldots,a_n)$) of such a $(c_1,\ldots,c_{2^{m-1}})$ then, by property 11 (ii), there exists an element $H\in\tilde{\mathcal{A}}_m(b_1,\ldots,b_n)$ coded by the same $(2^{m-1}\times\delta_{1,m-1})$-tuple of integers which codes F (see Definition 10). Choose a representative $(d_1,\ldots,d_{2^{m-1}})$ of the equivalence class H and set $b'=\bigcup_{i=1}^{2^{m-2}}d_i$.

There exist many non-equivalent 2^{m-1}-tuples in $\mathcal{A}_m(a_1,\ldots,a_n)$ satisfying $a'=\bigcup_{i=1}^{2^{m-2}}c_i$ (it suffices to take a permutation). However any of them gives univocally $\#(a'\cap\alpha_{1,m-1,j})$ for each $j\le\delta_{1,m-1}$ (identifying all numbers greater or equal than $2^{2(n+1)(m-1)+(m-1)^2}$).

Consider the $i=1,\ldots,2^{n^2}$ and $j=1,\ldots,2^{2n+1}$ such that $(a_{n+1})_{i_j}$ is not empty. In that case we have that, if the arity of $\varphi^*_{i_j}$ is q, then it applies to q-tuples $(x_1,\ldots,x_q)$ $(x_l\in a_{k_l})$

where $k_q = n+1$ and the arity of φ_i^* is $q-1$ (see Definition 11 (i)).
Since we have $(a_1,\ldots,a_n) \rightleftharpoons_{nm} (b_1,\ldots,b_n)$ then, by 11 (i), for each $(q-1)$-ary class of depth m, $\alpha_{n,m,r}$, the number of $(q-1)$-tuples of that class $(x_1,\ldots,x_{q-1})$ $(x_l \in a_{k_l} \in A)$ and $(y_1,\ldots,y_{q-1})$ $(y_l \in b_{k_l} \in B)$ is the same (identifying numbers greater or equal than 2^{2nm+m^2}).

Since $S \equiv T$, then the back-and-forth (for root-systems) holds, implying that if one of these $(q-1)$-tuples is extended to a q-tuple, then the other can be extended in a way that both q-tuples will be of the same class (for root-systems) of arity q and depth $m-1$.

For example, given $x_q \in S$ there exists $y_q \in T$ such that $\mathcal{C}_{q,m-1}(x_1,\ldots,x_{q-1},x_q) = \mathcal{C}_{q,m-1}(y_1,\ldots,y_{q-1},y_q)$.

Consider, for each class $\alpha_{q,m-1,r}$, the elements $x_q \in (a_{n+1})_{i_j}$ such that $\mathcal{C}_{q,m-1}(x_1,\ldots,x_{q-1},x_q) = \alpha_{q,m-1,r}$. Since each element of the partition $\mathcal{MP}(v_1,\ldots,v_n)$ is refined by 2^{n^2} elements of the partition $\mathcal{MP}(v_1,\ldots,v_{n+1})$, we can find (up to cardinality $2^{2nm+m^2}/2^{n^2} = 2^{2(n+1)(m-1)+(m-1)^2}$) the same number of q-tuples $(y_1,\ldots,y_q) \in \alpha_{q,m-1,r}$ that we have of q-tuples $(x_1,\ldots,x_q) \in \alpha_{q,m-1,r}$.
Knowing that $(a_{n+1})_{i_j} = \{x \,/\, x = x_q$ for some of those q-tuples $\}$, it can be expressed (in S) that their elements are pairwise orthogonal and so the same must hold in T. Define now $(b_{n+1})_{i_j}$ as the (finite) antichain $\{y \,/\, y = y_q$ for some of those q-tuples $\}$.

This set can be assumed finite because our back-and-forth relation identifies (at this stage) $2^{2(n+1)(m-1)+(m-1)^2}$ with infinity.

Define now $b = \bigcup (b_{n+1})_{i_j} (1 \le i \le 2^{n^2}, 1 \le j \le 2^{2n+1})$. By the same considerations about orthogonality in S, we have that b is an antichain and it is orthogonal to b'. So, define b_{n+1} as $b \cup b'$.

The construction implies that the number of q-tuples in each element of the partition $\mathcal{MP}\{b_1,\ldots,b_{n+1}\}$ is the same (identifying the cardinalities greater or equal than $2^{2(n+1)(m-1)+(m-1)^2}$) to the corresponding number in $\mathcal{MP}\{a_1,\ldots,a_{n+1}\}$. For the property 11 (ii), it is easy to verify that c is a quasicomplement of $\mathcal{M}(a_1,\ldots,a_{n+1})$ if and only if it is a quasicomplement of $\mathcal{M}(a_1,\ldots,a_n,a')$ (recall that a' was defined as the "part" of a_{n+1} incomparable with $\mathcal{M}(a_1,\ldots,a_n)$). So, we have that $\mathcal{A}_{m-1}(a_1,\ldots,a_{n+1})$ consists exactly of the 2^{m-2}-tuples of the form $(c_{2^{m-2}+1},\ldots,c_{2^{m-1}})$ where $(c_1,\ldots,c_{2^{m-1}}) \in \mathcal{A}_m(a_1,\ldots,a_n)$ for some pairwise orthogonal antichains $c_1,\ldots,c_{2^{m-2}} \in A$. By the choice of b' it results that the isomorphism between $\hat{\mathcal{A}}_{m-1}(a_1,\ldots,a_{n+1})$ and $\hat{\mathcal{A}}_{m-1}(b_1,\ldots,b_{n+1})$ is verified. □

Corollary 2 *Let S and S' be two root-systems, if $S \equiv S'$ then $A(S) \equiv A(S')$.*

Remark 9 Observe that, in the proof of Proposition 1 we have not fully used the fact that A and B were $A(S)$ and $A(T)$ respectively. We have used that they are presystems of antichains closed under disjoint unions and that on them the family of relations

$(\rightleftharpoons_{n,m})_{(n,m)\in\omega^2}$ gives a back-and-forth with $A(S)$ and $A(T)$, respectively. Since, given such a family of relations, we have that two models M and N are elementary equivalent if and only if, for each $m \in \mathbb{N}$ and $x \in M\,(y \in N)$ there exists $y \in N\,(x \in M)$ such that $x \rightleftharpoons_{1,m} y$, we have that $A \equiv A(S)$ if and only that last property holds between them. In order to make this point explicit, let us restate Definition 11 for $n = 1$.

We have that $\mathcal{F}_1 = \{x_1 \preceq x_1, x_1 \not\preceq x_1\}$ and the set $[\![(a_1 \not\preceq a_1)[a_1]]\!]$ is empty for any a_1. Also $[\![(a_1 \preceq a_1)[a_1]]\!] = a_1 = \mathcal{M}(a_1)$. So we have, for $a \in A$ and $b \in B$, $a \rightleftharpoons_{1,m} b$ if and only if:

i) For any $i \in \{1, \dots, \delta_{1,m}\}$ $\#(a \cap \alpha_{1,m,i}) = \#(b \cap \alpha_{1,m,i})$ (identifying the numbers greater or equal than $2^{m(m+2)}$).
And, if $m > 0$

ii) There is an isomorphism between the sets $\tilde{A}_m(a)$ and $\tilde{A}_m(b)$.

Since the isomorphism of (ii) is given by the unary classes (of the root-system) of depth $m-1$ (see Definition 10) and the existence of antichains composed by a given number of elements of certain class can be expressed for the root-system in terms of orthogonality, we can give the following

Definition 12 *A presystem of antichains over a root-system S is a* system of antichains over *S if it is closed under disjoint union and admits all the antichains compatible with the theory of S (in the sense of Remark 9).*

So we can state:

Corollary 3 *Let A and B be systems of antichains over S and T respectively. Then $S \equiv T$ implies $A \equiv B$.*

Remark 10 Observe that the type of an element in a presystem of antichains is given by the number of elements of each unary class (of the root-system). So, if A is a system of antichains over S, we have that for each $a \in A$ its type is the same when considered as an element of $A(S)$. Analogously, if S is an elementary submodel of T, since the back-and-forth classes of the elements of S are the same when considered as elements of T, and the type of the elements of $A(S)$ are given by those classes, then the types are the same when considered as elements of $A(T)$. Hence we can conclude:

Theorem 4 *Let S be a root-system and A a system of antichains over S, then $A \preceq A(S)$ (A is an elementary submodel of $A(S)$).*

Theorem 5 *Let S and T be root-systems such that S is an elementary submodel of T and A and B systems of antichains over S and T respectively. If the embedding of S in T can be extended to an embedding of A into B, then this embedding is also elementary. In particular $A(S) \preceq A(T)$.*

Definition 13 *Let A be a presystem of antichains over S; define the following binary operations:*

$$a \vee b = \begin{cases} \{max\{a(x), b(x)\} \,/\, \{x\} \preceq a \;\; or \;\; \{x\} \preceq b\} & if\, a \neq \emptyset \neq b \\ a & if\, b = \emptyset \\ b & if\, a = \emptyset \end{cases}$$

and

$$a \wedge b = \begin{cases} \{min\{a(x), b(x)\} \,/\, \{x\} \preceq a \;\; \& \;\; \{x\} \preceq b\} & if\, a \neq \emptyset \neq b \\ \emptyset & if\, b = \emptyset \;\; or \;\; a = \emptyset \end{cases}$$

Proposition 6 *The operations $\vee$, $\wedge$ and the empty antichain $\emptyset$ give to any presystem of antichains a structure of distributive lattice with a least element.*

Proof: Let be $c = \{max\{a(x), b(x)\} \,/\, \{x\} \preceq a$ or $\{x\} \preceq b\}$. First we shall prove $a \preceq c$. Let $x \in a$, then $x = a(x) \leq max\{a(x), b(x)\} \in c$ implying $a \preceq c$ (in an analogous way we prove $b \preceq c$). Now suppose $a, b \preceq d$. Then, by definition of the order on A, we have that for each point $x \in a \cup b$ there exists a point $y \in d$ such that $x \leq y$. If $x = max\{a(x), b(x)\}$ we are done. So suppose that x is strictly smaller that $max\{a(x), b(x)\}$ and, for example, $x \in a$. Then there exists $z \in b$ such that $x < z = b(x)$. Let $y' \in d$ such that $z \leq y'$. Since the points form a root-system, we have that y and y' are comparable implying –because d is an antichain– that they coincide, so we have that for every point in c there exists a point in d which is greater or equal. Then we have that c is the lattice-theoretic join of a and b. For the meet operation the proof is similar.

For the distributivity, we have
$(a \vee b) \wedge c = \{max\{a(x), b(x)\} \,/\, \{x\} \preceq a$ or $\{x\} \preceq b\} \wedge c$ and
$(a \wedge c) \vee (b \wedge c) =$
$= \{max\{min\{a(x), c(x)\}, min\{b(x), c(x)\}\} \,/\, (\{x\} \preceq a \;\&\; \{x\} \preceq c$ or $\{x\} \preceq b \;\&\; \{x\} \preceq c)$.

Since the meet is defined in terms of the minimum of a totally ordered set and logical conjunction, the join is defined in terms of the maximum of a totally ordered set and logical disjunction, and since distributivity for those pairs holds, we have then distributivity for our lattice. □

Remark 11 Observe that, since any presystem of antichains admits a unique distributive lattice structure compatible with $\preceq$, we have that the operations $\vee$ and $\wedge$ are definable in the language $\langle \preceq, \emptyset \rangle$.

Proposition 7 *Given a presystem of antichains A over a root-system S, the antichains of the form $\{x\}$ $(x \in S)$ are exactly the join-irreducible antichains.*

Proof: Let be $c = a \vee b = \{max\{a(x), b(x)\} \,/\, \{x\} \preceq a$ or $\{x\} \preceq b\}$ and suppose $a \neq c \neq b$. Then a and b are not comparable, which implies that there exist elements

$x_0 \in a$ and $y_0 \in b$ such that x_0 (resp., y_0) is incomparable with all $y \in b$ (resp., $x \in a$). Hence, $b(x_0) = \emptyset$, and $max\{a(x_0), b(x_0)\} = a(x_0) = x_0 \in a \vee b = c$. Similarly, $y_0 \in c$, and c has at least two points.
For the converse, if $\{x, y\} \in c \, (x \neq y)$, we have $c = \{x\} \vee (c \setminus \{x\})$ (where $(c \setminus \{x\}) \neq \emptyset$. □

Remark 12 Observe that in a system of antichains the relation "$x \in a$" (where x is a point of the root-system) can be expressed in the language of systems of antichains:

$$Point(x) \& x \preceq a \& \forall y (Point(y) \& x \neq y \to \neg(x \preceq y \preceq a))$$

where $Point(x)$ states that x is join-irreducible.

Corollary 8 *Given a presystem of antichains* $A = \langle \mathrm{A}, \preceq \rangle$ *over a root-system* $S = \langle \mathrm{S}, \leq \rangle$, *there exists a definable subset* $\mathrm{S(A)}$ *of* A *such that* $\langle \mathrm{S(A)}, \preceq \rangle \cong \langle \mathrm{S}, \leq \rangle$.

For a presystem of antichains A over a root-system S we shall identify S and $S(A)$.

Theorem 9 i) *Let A and B be systems of antichains over S and T, respectively. Then $S \equiv T$ if and only if $A \equiv B$.*

ii) *Let S and T be root-systems, then $A(S) \preceq A(T)$ if and only if $S \preceq T$.*

iii) *Let B be a presystem of antichains over S. Then $B \equiv A(S)$ if and only if $B \preceq A(S)$ if and only if B is a system of antichains over S.*

Proof: i) One direction is Corollary 3, the other is a consequence of the definability of S over A and that of T over B.

ii) One direction is Theorem 5, the other is also consequence of the definability.

iii) One direction is Theorem 4. For the other, if $B \equiv A(S)$, since we have $S \cong S(B)$ and all sentences stating the existence of the antichains whose construction was indicated as in Remark 9 are true for $A(S)$, then they must be true for B, and applying Theorem 4, we conclude that $B \preceq A(S)$.□

Remark 13 Having proved that elementary equivalence and elementary embedding are transfered from root-systems to systems of antichains, it is natural to look at the stability and independence property. In [6] Parigot proved that a theory of trees is stable (and superstable) if and only if its models are of bounded height and that no theory of trees has the independence property. Since a tree is the order-dual of a root-system, it is obvious that the same holds for theories of root-systems. However, even for a system of antichains over a root-system of bounded height, if it is infinite, there are antichains of any finite cardinal, implying that the theory of that system of antichains has the strict order property and then, that it is unstable (see [9]). The unstability results also from the independence property which holds in any theory of antichains over a root-system of unbounded width. Consider the (system of antichains) formula $\varphi(x,y) = x \in y$; then for any $n \in \omega$ and any such a theory of antichains T, the following sentence holds:

$$\exists x_0,\ldots,x_{n-1}, \exists (y_\alpha)_{\alpha\in\mathcal{P}(n)} (\bigwedge_{\alpha\in\mathcal{P}(n)} (\bigwedge_{i\in\alpha} \varphi(x_i, y_\alpha) \& \bigwedge_{i\notin\alpha} \neg\varphi(x_i, y_\alpha))).$$

Remark 14 In [6] is referred the result of [8] about arborescent structures, where it is proved the interpretability of those structures in the theory of trees. Since the theory of root-systems (and hence that of trees) is interpretable in the theory of systems of antichains, it is a natural problem to know whether a system of antichains is an arborescent structure. The answer turns out to be *no*, because arborescent structures are characterized by the absence of a partially ordered set of four elements such that $a < b$; $c < b$ and $c < d$. It suffices to consider a root-system having three incomparable elements x, y, z and take the antichains $a = \{x\}$, $b = \{x, y\}$, $c = \{y\}$ and $d = \{y, z\}$.

Remark 15 We have that the notion of a system of antichains (over a root-system) is a natural generalization of that of an atomic Boolean algebra. In this case the root-system is the set of atoms with the trivial order and, for a given set S of atoms, $A(S) = \mathcal{P}(S)$ (the power-set of S); in particular, $\mathcal{P}(S)$ is isomorphic to the complete atomic distributive lattice with set of atoms S. Recall that a lattice is called *laterally complete* if any subset of pairwise orthogonal elements admits a lowest upper bound. It is easy to verify that, for any root-system S, $A(S)$ is laterally complete.

In order to illustrate this way of thinking the systems of antichains, the list below shows how some of the properties of systems of antichains proved in this paper generalize well-known properties of atomic Boolean algebras.

Let B be an atomic Boolean algebra with set of atoms T and A a system of antichains over a root-system S.

$\mathcal{B}_1$: $At(B) = T$;

$\mathcal{B}_2$: If B is complete, then $B \cong \mathcal{P}(T)$;

$\mathcal{B}_3$: $B \preceq \mathcal{P}(T)$;

$\mathcal{B}_4$: B contains all finite and cofinite subsets of T.

$\mathcal{A}_1$: $S(A) \cong S$;

$\mathcal{A}_2$: If A is laterally complete, then $A \cong A(S)$;

$\mathcal{A}_3$: $A \preceq A(S)$;

$\mathcal{A}_4$: For each positive integer m, and $\delta_{1,m}$-tuple $(r_1, \ldots, r_{\delta_{1,m}})$ of positive integers less or equal than $2^{m(m+2)}$, A contains an antichain formed of r_j points of class $\alpha_{1,m,j}$ (or possibly more if $r_j = 2^{m(m+2)}$) if and only if the existence of such an antichain is consistent with the theory of S.

In fact the points $\mathcal{B}_2$ and $\mathcal{A}_2$ can be strengthened:

$\mathcal{B}_{2'}$: If B is a complete atomic distributive lattice with set of atoms T, then $B \cong \mathcal{P}(T)$.

$\mathcal{A}_{2'}$: If A is a laterally complete presystem of antichains over S, then $A \cong A(S)$.

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Elementary equivalence of Hahn-powers of divisible totally ordered groups on a root system

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Introduction:

Definitions:

A *root-system* $(I,<)$ is an ordered set, such that for all $i\in I$ the set $\{j \ / \ i\leq j\}$ is totally ordered.

If I is a root-system and $(G_i)_{i\in I}$ a family of totally ordered groups, the *Hahn product* of the family on the root-system I: $\Lambda(I,(G_i)_{i\in I})$ is the subset of the product ΠG_i given by the elements g such that any nonvoid part of the support of g (supp(g)) has a maximal element. It is easy to verify that this subset is, in fact, a subgroup. If we define g to be positive when for each i maximal in supp(g) g(i) is positive then $\Lambda(I,(G_i)_{i\in I})$ is lattice-ordered.

If for all i and j, $G_i=G_j=G$ then $\Lambda(I,(G_i)_{i\in I})=\Lambda(I,G)$ is called the *Hahn power* of G on I. Such Hahn powers have been considered by Conrad, Harvey and Holland [C.H.H.] who proved that any abelian lattice ordered group is isomorphic to an l-subgroup of a Hahn power $\Lambda(I,\mathbb{R})$.

The *maximal support* of g: ms(g) is the antichain of maximal elements of supp(g).

If I is a root-system and A(I) the set of antichains of $(I,\leq)$, A(I) is partially ordered by: $a \ll b$ iff $(\forall t\in a \ \exists t'\in b \ t\leq t')$ as defined in [R.1].

($\ll$ can be defined by the same way on $\mathcal{P}(I)$.)

Define $\mathcal{P}'(I)$ to be $\{P\in\mathcal{P}(I) \ \forall Q\subseteq P \ \forall t\in Q \ \exists t' \ t\leq t' \ t' \text{ maximal in } Q\}$.

A root system I is a *dense branching* root system iff for all $j<i$ there exists k such that $k<i$ and $j\perp k$.

The Feferman-Vaught [F.V.] theorem on first order theory of generalized products can be applied to Hahn products : if $(I,\leq)$ and $((J,\leq)$ satisfy $(\mathcal{P}'(I),\subseteq,\cap,\ll)\equiv(\mathcal{P}'(J),\subseteq,\cap,\ll)$ and all the G_i and G_j are divisible then $\Lambda(I,(G_i)_{i\in I})\equiv\Lambda(J,(G_j)_{j\in J}$. Here we are proving the same result using only the hypothesis of elementary equivalence of the partially ordered sets of antichains and a partial converse.

Theorem 1: Let I and J be dense branching root systems, and for each $i\in I$, $(j\in J)$, G_i, (G_j) is a totally ordered group, we can interpret $(I,\leq)$, $(A(I),\ll)$, and each G_i in $\Lambda(I,(G_i)_{i\in I})$ and:
if $\Lambda(I,(G_i)_{i\in I})\equiv\Lambda(J,G_j)_{j\in J})$ then $(A(I),\ll)\equiv(A(J),\ll)$.

Theorem 2: If $(I,\leq)$ and $(J,\leq)$ are root systems, for all $i\in I$, $j\in J$, the G_i and G_j, are all divisible, and $(I,\leq)\equiv(J,\leq)$ then $\Lambda(I,(G_i)_{i\in I})\equiv\Lambda(J,(G)_{j\in J})$.

For this we use a first order transfer result between root systems and there systems of antichains, proved by D.Gluschankof in a forthcoming paper [G.]:

Theorem : $(A(I),\ll)\equiv(A(J),\ll)$ iff $(I,\leq)\equiv(J,\leq)$.

0) Some remarks and definitions

-about root systems and branching properties:

We have defined dense branching root systems. The following will give a better understanding of this notion.

Definition: In a root system I, i is a *branching point* if $(\exists j,k<i)(\forall x)((j\leq x<i \ \&\ k\leq y<i)\Rightarrow x\perp y)$ (i.e. $[j,i[\cap[k,i[=\varnothing$)

Remark: If I has a dense subset of branching points then: $(\forall u,v;\ u<v)(\ \exists a\in]u,v],\ a$ is branching). I is discrete and each point is a branching point or I has dense branching, with a dense subset of branching points. A root system I without branching point can be dense branching.

-about antichains:

Let $(I,\leq)$ be a root-system and $(A(I),\ll)$ the associated set of antichains: $(A(I),\ll)$ is a lattice ordered set. A(I) has a smallest element, the empty antichain:$\varnothing$. $(I,\leq)$ is definable in $(A(I),\ll)$ as the set of sup irreductible elements.
The infimum $\wedge$ and supremum $\vee$ for the relation $\ll$, and the relativised orthogonal $a^{\perp b}=\{t\in a,\ t\perp b\}$ are definable in $A(I,\ll)$.
If $b\subseteq a$ then $a\backslash b=a^{\perp b}$.
we have $a^{\perp b\vee c}=\{t\in a,\ t\perp(b\vee c)\}=\{t\in a,\ t\wedge(b\vee c)=0\}=a^{\perp b}\wedge a^{\perp c}$, $a^{\perp b\wedge c}=a^{\perp b}\vee a^{\perp c}$.

Definition: for each n-tuple $(a_1,\ldots,a_n)\in A(I)^n$ let $T(a_1,\ldots,a_n)$ be the substructure of A(I) generated by $a_1,\ldots,a_n$, in the language $\{\wedge,\vee,\perp,\cap\}$. $T(a_1,\ldots,a_n)$ is finite. We can also define T(A) for an infinite substructure A of A(I) to be the substructure generated by A in the language $\{\wedge,\vee,\perp,\cap\}$.

Definition:for $a,b\in T(A)$ and $a\ll b$, define $b(a)=\{t\in b,\ \exists t'\in a,\ t'\leq t\}$

Definition: Let $mp(a_1,\ldots,a_n)$ "*min-partition*" be the family of minimal antichains for $\ll$ in $T(a_1,\ldots,a_n)$;
Remark: in the infinite case T(A) such a min-partition does not always exist.

Remark: if $\alpha\in mp(a_1,\ldots a_n)$, and $x\in T(a_1,\ldots,a_n)$, for each α, $\{x(\alpha),x\in T(a_1,\ldots,a_n)\}$ is totally ordered .

Now we give some definitions following [G.]:

Definition: Given $a_1,\ldots,a_n$ in A(I), consider $a_1\cup\ldots\cup a_n\subseteq I$ and define $M(a_1,\ldots,a_n)=\{i\in I,\ i$ minimal in $a_1\cup\ldots\cup a_n\}$.

Remark: $M(a_1,\dots,a_n)$ is an antichain;

Definition: For each quantifier free formula $\phi(x_1..,x_n)$ in the language $\leq$, and any n-tuples of antichains $a_1,\dots,a_n$, define $[\phi(a_1,\dots,a_n)]$, by induction:

$[a_i\leq a_j]=\{t\in M(a_1,\dots,a_n),\ a_i(t)\neq\emptyset,\ a_j(t)\neq\emptyset,\ a_i(t)\leq a_j(t)\}$

$[\phi(a_1,\dots,a_n)\wedge\psi(a_1,\dots,a_n)]=[\phi(a_1,\dots,a_n)]\cap[\psi(a_1,\dots,a_n)]$

$[\neg\phi(a_1,\dots,a_n)]=M(a_1,\dots,a_n)\setminus[\phi(a_1,\dots,a_n)]$, (in this last case, $M(a_1,\dots,a_n)\subseteq[\phi(a_1,\dots,a_n)]$ and if $a\leq b$ $a\setminus b=a^{\perp b}$).

Remarks: $[\phi(a_1,\dots,a_n)]$ is always a subantichain of $M(a_1,\dots,a_n)$

ϕ can always be writen in a conjunctive form $\phi=\wedge(\phi_{i,j}\ (a_i,a_j))$ where $\phi_{i,j}(a_i,a_j)=a_i\leq a_j$ or $\phi_{i,j}(a_i,a_j)=\neg(a_i\leq a_j)$.

ϕ is complete if for each (i,j) $a_i\leq a_j$ or $\neg(a_i\leq a_j)$ is in the conjunction.

Definition: $MP(a_1,\dots,a_n)=\{[\phi(a_1,\dots,a_n)]$ with ϕ complete $\}$.

Remarks: $MP(a_1,\dots,a_n)$is the set of minimal $[\phi(a_1,\dots,a_n)]$.

$[a_i\leq a_i]$ $=\{t\in M(a_1,\dots,a_n),\ a_i(t)\neq 0\}$ is the projection of a_i on $M(a_1,\dots,a_n)$.

If $a_i\leq a_i$ is one of the conjoncts of ϕ then for each $t\in[\phi]$ $a_i(t)\neq\emptyset$.

If ϕ is false ,$[\phi]$ is the empty antichain.

If ϕ is true , then for each i, $a_i([\phi])=a_i$, (for simplification of the notations we shall use $a_i[\phi]$ for $a_i([\phi])$.

$a_i[\phi]=\{t\in a_i,\ \exists t'\in M(a_1,\dots,a_n)$ such that $a_1(t'),\dots,a_n(t')$ satisfy ϕ and $a_i(t')=t.\}$ i.e.:

$a[\phi]=\{t\in a$, there is a branch going through t and satisfying ϕ }.

$a[\neg\phi]=\{t\in a$, there is a branch going through t and satisfying $\neg\phi$ }.

$a-a[\neg\phi]=\{t\in a$, each branch going through t satisfies ϕ }.

$a-a[\phi]=\{t$, each branch going through t satisfies $\neg\phi$ }.

Lemma: $T(a_1,\dots,a_n)=\{\vee_{k=1,\dots m}\ a_{i_k}[\phi_{i_k}(a_1,\dots,a_n)],\ \phi_{i_k}$ quant.free}.

Proof:1)Let $T'=\{\vee_{k=1,\dots m}\ a_{i_k}[\phi_{i_k}(a_1,\dots,a_n)],\ \phi_{i_k}$ quant.free}:

for each a_i, $a_i\in T'$ and:

if a and $b\in T'$, $a\vee b\in T'$,

$a\wedge b\in T'$: $a_i[\phi]\wedge a_i[\psi]=a_i[\phi\wedge\psi]$,

$a_i[\phi])\wedge a_j[\psi]=a_i[\phi\wedge\psi\wedge(a_i\leq a_j)]\vee a_j[\phi\wedge\psi\wedge(a_j\leq a_i)]$,

$a_i[\phi]\cap a_j[\psi]=a[(a\leq b)\wedge(b\leq a)\wedge\phi\wedge\psi]$

$a^{\perp b}\in T'$: $a^{\perp(b\vee c)}=a^{\perp b}\wedge a^{\perp c}$ and $(a\vee b)^{\perp c}=a^{\perp b}\vee a^{\perp c}$

$a^{\perp b}=\{t\in a$, on each branch going through t, $t\perp b\}$

$=a-a[(a\leq b)\vee(b\leq a)]$

$a[\phi]^{\perp b[\psi]}=a[\phi]\cap(a-a[(\neg(\phi\wedge((a\leq b)\vee(b\leq a)))])$

$(a\vee b)\cap c=((a[\neg(a\leq b)])\cap c)\vee((b[\neg(b\leq a)])\cap c)\vee((a[a=b])\cap c)$ and

$a[\phi]\cap b[\psi]=a[(a=b)\wedge\phi\wedge\psi]$

hence $T(a_1,\dots,a_n)\subseteq T'$.

2) for each ϕ and each a_i, $a_i(\phi)\in T(a_1,\dots,a_n)$,

$a_i([a_i\leq a_j])=a_i\cap(a_i\wedge a_j)$ and $a_i([a_j\leq a_i])=a_i\cap([a_i\vee a_j])$,

$a_i([a_j\leq a_k])=a_i([a_i\leq a_j([a_j\leq a_k])])\vee a_i([a_i\leq a_k([a_j\leq a_k])])$

$a_i[\phi\wedge\psi]=a_i[\phi]\wedge a_i[\psi]$

$a_i([\neg\phi])=a_i\setminus a_i([\phi])$,hence $T'\subseteq T(a_1,\dots,a_n)$.□

Corollary: $mp(a_1,\dots,a_n)=MP(a_1,\dots,a_n)$.

-about Hahn products:

Definition: If M is a subset of $\wedge(I,(G_i)_{i\in I})$ we shall consider ms(M) the set of maximal supports of elements in M and if M is finite mp(M) his min-partition.

Definition: $T(G)=T(ms(G))$

Definition: For each $g=(g_i)$ and each $a\subseteq ms(g)$ define $g|a$ to be (h_i) with $h_j=g_j$ if $\{j\}\ll a$ and 0 if not.

Remark: If $a_1,\dots,a_n$ is a partition of ms(g) then $g=g|a_1+\dots+g|a_n$.

2) Definable maximal support and interpretability

Definition: If G is an l-group define the relation

$R(g,h)$ iff $h^{\perp}\subseteq g^{\perp} \wedge (\forall x,y;\ (h=x+y \wedge x\perp y))(\exists x',y';\ g=x'+y' \wedge x'\in x^{\perp\perp} \wedge y'\in y^{\perp\perp}))$

(If $h=x+y$ and $x\perp y$ we shall denote this later by $h=s\oplus y$).

Lemma: If $G=\Lambda(I,G_i)$ where I is dense branching: $ms(g)\ll ms(h)$ iff $R(g,h)$.

Proof: a) If $ms(g)\ll ms(h)$, then $h^{\perp}\subseteq g^{\perp}$,

suppose $h=x\oplus y$, define $a=ms(x)$, $b=ms(y)$. We have $a\cap b=\emptyset$ and $ms(g)=ms(g)\wedge ms(h)=ms(g)\wedge(a\vee b)=(ms(g)\wedge a)\vee(ms(g)\wedge b)=(ms(g)\wedge a)\cup(ms(g)\wedge b)$, put $a'=ms(g)\wedge a$ and $b'=ms(g)\wedge b$ then $g=(g|a')\oplus(g|b')$, $(g|b')\in y^{\perp\perp}$, $(g|a')\in x^{\perp\perp}$.

b) Suppose $R(g,h)$ is satisfied and consider $u\in ms(g)$. There is a $v\in ms(h)$ such that $u\leq v$ or $v\leq u$. There is no $v\in ms(h)$ such that $v<u$: if there where such a $v\in ms(h)$, $v<u$ consider the two following cases: first case v is the only point satisfying $v\in ms(h)$ and $v<u$, then I being dense branching , there is a w such that $w<u$ and w cannot be compared to any point of ms(h), hence there is a k such that $supp(k)=w$, $k\in h^{\perp}$ and $k\notin g^{\perp}$; second case, there is v', $v'\neq v$, $v'\in ms(h)$ and $v'<u$, consider a partition of ms(h): $ms(h)=a\cup b$ such that $v\in a$ and $v'\in b$, $h=(h|a)+(h|b)$; for each g' and g" such that $g=g'+g''$, we cannot have $g'\in(h|a)^{\perp\perp}$ and $g''\in(h|b)^{\perp\perp}$, because $u\in ms(g)$ and therefore $u\in ms(g')$ or $u\in ms(g'')$, say $u\in ms(g')$ for example, but $v<u$ and $v'<u$ hence $g'\notin(h|b)^{\perp}$ and $g'\notin(h|a)^{\perp\perp}$.□

As a corollary we obtain:

Theorem 1: Let I and J be dense branching root systems and for each $i\in I$, $(j\in J)$, G_i, (G_j) is a totally ordered group, we can interpret $(I,\leq)$, $(A(I),\ll)$, and each G_i in $\Lambda(I,(G_i)_{i\in I})$ and:

if $\Lambda(I,(G_i)_{i\in I})\equiv\Lambda(J,G_j)_{j\in J})$ then $(A(I),\ll)\equiv(A(J),\ll)$.

Proof: Under the hypothesis $ms(G)=A(I)$. Let $S(g,g')$ iff $R(g,g')$ and $R(g',g)$. S is definable , $A(I)=G/S$, $ms(g)\ll ms(g')$ iff $R(g,g')$.

From this we can interpret $(I,\leq)$ in G, because $(I,\leq)$ is definable in A(I). G_i is the quotient of two definable convex subgroups:

G_l={g; ms(g)«i}/{g; ms(g)«i and ms(g)≠i}.□

We can also interpret (I,≤) directly:

Definition: Let G an l-group, g° denotes the set:

$\{x\in g^{\perp\perp}, \exists y,z(\ y,z\in g^{\perp\perp},\ y^{\perp\perp}\neq g^{\perp\perp}\ \&\ z^{\perp\perp}\neq g^{\perp\perp}\ \&\ x=y+z)\ \}$

(x∈g° iff it is the sum of two elements wich are not weak units).

Facts:

If g∉g° the convex subgroup generated by g° is the lex kernel of $g^{\perp\perp}$ as defined in [C.].

If $x\in g^{\perp\perp}$and g is not a weak unit in $g^{\perp\perp}$ then x∈g°: (take y=x and z=0).

If $x\in g^{\perp\perp}$ and $x^{\perp\perp}=g^{\perp\perp}$,then x∈ g° iff|ms(x)|≥2.

Lemma: Let G=Λ(I,G_i), $x\in g^{\perp\perp}$ and ms(x) having at least two elements then x∈g°.In particular x∈x°.

Proof: for a∈A(I) define x|a=(x'_i) where $x'_i=x_i$ if {i}«a and $x'_i=0$ if not. Then if ms(x) has at least two elements it is a disjoint union : ms(x)=a∪b, x=x|a+x|b where neither x|a nor x|b are weak units in $x^{\perp\perp}$ and $g^{\perp\perp}$.□

Lemma: If G=Λ(I,G_i), g° is an l-ideal.

Proof: let x∈g°, y∈g°, let a=[ms(x)«ms(y) and ms(x)≠ms(y)], b=[ms(x)=ms(y)], c=[ms(y)«ms(x) and ms(x)≠ms(y)],

-if neither x nor y are weak units, then x+y ∈g°;

-if x is a weak unit, if two of these a,b,c, are nonvoid, the maximal support of x and then that of x+y is the union of two nonvoid orthogonal parts, then x+y ∈g° .

-if x is a weak unit and only a is nonvoid then y is also a weak unit and ms(x+y)=ms(y) has at least two elements, and x+y ∈g°.

-if x is a weak unit and only c is nonvoid, ms(x+y)=ms(x) has at least two elements,and x+y∈g°...

-if x is a weak unit and only b is nonvoid then ms(x)=ms(y), ms(x) has at least two elements and x is a weak unit and ms(x+y)«ms(x), then either ms(x+y) has at least two elements, or x+y is not a weak unit: x+y∈g°.

-if 0≤z<y and y∈g°, if y is not a weak unit z is not a weak unit, and if y is a weak unit, ms(y) has at least two elements. Then either ms(z) has at least two elements or z is not a weak unit.□

Definitions: Let g~g' iff $g^{\perp\perp}=g'^{\perp\perp}$ and R(G)={g, g∉g°, g∈G},

Lemma: If $G=\Lambda(I,(G_i)_{i\in I})$, where I is a dense branching root system, then $R(G)/\sim\simeq I$.

Proof: If $i\in I$ and $g\in G$ and $supp(g)=\{i\}$, then $g\in g^{\perp\perp}$ but $g\notin g^\circ$. I being dense branching, if $supp(h)=\{j\}\neq i$, $h^{\perp\perp}\neq g^{\perp\perp}$: we have an embedding of I in $R(G)/\sim$. We prove now that this embedding is onto. For each g such that $g^{\perp\perp}\neq g^\circ$ and $g\notin g^\circ$, ms(g) has only one element: ms(g)=i, therefore the embedding is onto.□

We are now looking at an l-group G' such that $G'\equiv\Lambda(I,(G_i)_{i\in I})=G$ where I is a dense branching root system. We come back to the foreward define equivalence relation S: S(g,h) iff (R(g,h)&R(h,g); We know that S is first order definable.

Definition: Define A(G') to be the quotient set G'/S and dms(g) the equivalence class of g. Define dms(g)«dms(h) iff R(g,h).

Remark: If G' is G then A(G)=A(I) and dms(g)=ms(g).

Lemma: $(A(G'),\ll)\equiv(A(I),\ll)$, the relation « is an order on A(G') and for this order A(G') is a lattice .

Proof: by interpretability.□

Definition: Let I(G') be the set of sup irreducible elements of A(G').

Lemma: $I(G')\equiv I$ and $A(G')\subseteq A(I(G'))$.

Proof: We can define $i\in dms(g)$ iff $i\ll dms(g)$ and $\forall k\ k\ll dms(g)\Rightarrow(k\ll i$ or $k\perp i$), and then for all $i,j\in dms(g)$ $i\neq j$ implies $i\perp j$.

We now restict oursef to the **divisible case**:

Embedding lemma: Let G be a divisible Hahn product $G=\Lambda(I,(G_i)_{i\in I})$ where I is a dense branching root system and $G'\equiv G$, then G' can be embedded in the Hahn product $\Lambda(I(G'),(G'_i)_{i\in I(G')})$.

Proof: for each $i\in dms(G')$, G'_i is defined $G'_i=\{g;dms(g)\leq i\}/\{g;dms(g)\leq\neq i\}$. Consider $C_i=\{g\in G';\ dms(g)=i\}$ and B_i a maximal $\mathbb{Q}$-independant subset of B_i such that for all b and b' in D_i dms(b-b')=dms(b)=dms(b'). Let D_i be the $\mathbb{Q}$-vector space generated by B_i, we have $G_i=\{cl(g),\ g\in D_i\}$. Let D be the direct sum $D=\oplus(D_i,\ i\in I(G'))$ wich is contained in G'. Let $D'=\{g\in G';\ \forall i\in dms(g)\ g|i\in D_i\ \}$ and D" the $\mathbb{Q}$-vector space generated by D and D'. For each $\mathbb{Q}$-vector space Δ such that $D''\subseteq\Delta\subseteq G'$ and $g\in G'$, $g\notin\Delta$, g is immediate on Δ in the following sense: for each $\gamma\in\Delta$ there is a $\gamma'\in\Delta$ such that $dms(g-\gamma)=dms(\gamma')$ and for each $i\in dms(g-\gamma)$ $cl((g-\gamma)|i)=cl(\gamma'|i)$.

Now we can construct an embedding of G' in $\Lambda(I(G'),(G'_i)_{i\in I(G')})$. For $g\in D_i$ define $f(g)$ by $(f(g)_k=0$ if $k\neq i$ and $f(g)_i=cl(g)$. It is easy to extend the definition to D' and D". The property of G' to be immediate on D" unable us to extend the definition of f by induction: if f is defined on Δ, $g\in G'$, $g\notin\Delta$, if there is a $\gamma\in G'$ such that $dms(g-\gamma)\ll\neq dms(g)$ and $i\ll dms(g)$ and there is a $j\in dms(g-\gamma)$ $j\ll\neq i$, then define $(f(g))_i=(f(\gamma))_i$, and in the other cases $(f(g))_i=0$. We can verify that if another γ' satisfies the same properties for g and i then $(f(\gamma'))_i=(f(\gamma))_i$ because this corresponds to a formula whichis true in G. We have constructed a group embedding but this embedding conserve the maximal support and the order in each quotient, then it is an order embedding.□

3) r-projectability

Definition: An l-group $G\subseteq\Lambda(I,(G_i)_{i\in I})$ is *r-projectable* if it satisfies the following:

1) $(\forall x,y)(\exists z,w)(\ x=z\oplus w\ \wedge\ (\forall u,v)((\ w=u\oplus v)\Rightarrow(u\notin y^{\perp}\ \wedge\ v\notin y^{\perp})))$.

2) $(\forall x,y)(\exists z)(ms(z)=ms(x)\cap ms(y))$.

Examples: If for all i G_i is totally ordered the following l-groups are r-projectable:

The Hahn power $\Lambda(I,(G_i)_{i\in I})$;

The Hahn sum $\oplus(I,(G_i)_{i\in I})$ i.e. $\Lambda(I,(G_i)_{i\in I})\cap(\oplus G_i)$ the subgroup of elements of finite support;

The narrow Hahn product $N(I,(G_i)_{i\in I})=\{g\in\Lambda(I,G_i)_{i\in I});$ all antichains in supp(g) are finite} as defined in [R.2].

Remark: 1) The first condition of r-projectability could be considered in any l-group, the second one needs the notion of maximal support.

Lemma: If G $\subseteq\Lambda(I,(G_i)_{i\in I})$ satisfies that for all x, ms(x) is finite then G satisfies the condition 1) of r-projectability.

Proof: given x and y, y define a partition of ms(x), $ms(x)=a\oplus b$ a maximal orthogonal to ms(y); let U be the set (may be empty) of $u\in G$ such that $ms(u)\subseteq a$ and $u=x|ms(u)$, since ms(x) is finite there is an $v\in U$ such that ms(v) is maximal in {ms(u), $u\in U$}. For this v we have $x=v\oplus(x-v)$ $v\perp y$ and if $(x-v)=s\oplus t$ then $s\notin y^{\perp}$ and $t\notin y^{\perp}$ by the maximality property of ms(u).□

Lemma: (unicity of r-projections in a Hahn product) If $G=\Lambda(I,(G_i)_{i\in I})$, $x,y\in G$ $x^{\perp y}=(x'_i)$, $x_y=(x''_i)$ where $x'_i=x_i$, $x''_i=0$ if $i\perp ms(y)$, and $x'_i=0$, $x''_i=x_i$ if not.

Proof: the defined $x^{\perp y}$ and x_y are orthogonal and if $x_y=t\oplus t'$ then for each $i\in supp(t)$ (resp.t') i is not ortogonal to ms(y). On the other side let z' and z" satisfying the condition 1) of r-projectability, z''_i has to be 0 if $i\perp ms(y)$, and z'_i has to be 0 if $i\notin ms(y)^{\perp}$.□

Remark: Let $G\subseteq G'\subseteq\Lambda(I,(G_i)_{i\in I})$ satisfying the condition 1) of r-projectability, and $x,y\in G$ we can have different r-projections when considered in G or G'. Consider for exemple $G'=(\mathbb{R}\blacktriangleright\mathbb{R})x(\mathbb{R}\blacktriangleright\mathbb{R})x\mathbb{R}$ where x is the product and ▶ is the lexicographical product. Let G be the l-subgroup generated by $x=((0,1),(1,0),1)$ and $y=((1,0),(0,1),0)$, in G' we have $x^{\perp y}=((0,0),(0,0),1)$, but we can prove that this element does not belong to G. An element of G is given by $\vee_I \wedge_{J_i} a_{ij}x+b_{ij}y$. Suppose that $\vee_I(\wedge_{J_i}(a_{ij}x+b_{ij}y)=((0,0),(0,0),1)$ then using the fact that the third projection is strictly positive: There is an i such that for all $j_i\in J_i$ $a_{ij}x_3+b_{ij}y_3>0$ i.e. $a_{ij}>0$, and using the fact that the second projection is negative: for all i' there is a $k\in J_i$such that $a_{i'k}x_2+b_{i'k}y_2\leq 0$, let i'=i then $a_{i'k}>0$ but $a_{i'k}(1,0)+b_{i'k}(0,1)\leq 0$ wich is impossible in the lexicographical product.

Definition: A subgroup G of $\Lambda(I,(G_i)_{i\in I})$ is *r-projectable in* $\Lambda(I,(G_i)_{i\in I})$ if for each $x,y\in G$ $x^{\perp y}\in G$ and $x_y\in G$ and there is a $z\in G$ such that $ms(z)=ms(x)\cap ms(y)$.

Lemma: A subgroup G of $\Lambda(I,G_i)$ is r-projectable in $\Lambda(I,G_i)_{i\in I})$ iff for each $x\in G$, and each $s\in T(G)$ such that $s\subseteq ms(x)$, $x|s$ belongs to G.

Proof: a) Suppose that for all $x\in G$ and $s\in T(G)$ with $s\subseteq ms(x)$, $x|s\in G$.
-G satisfies 1): given x and y, let $s=ms(x)^{ms(y)}$ then $s\subseteq ms(x)$ and $(x|s)\perp y$, let $x^{\perp y}=x|s$ and $x_y=x-x|s$, then each point of $ms(x_y)$ can be compared to a point of ms(y) , hence if $x_y=u+v$ with $u\perp v$, $ms(u)\subseteq ms(x_y)$ and $ms(v)\subseteq ms(x_y)$, thereby $u\notin y^{\perp}$ and $v\notin y^{\perp}$.
-G satisfies 2) with $z=x|ms(x)\cap ms(y)$
b) If G is r-projectable and $s\in T(G)$, $x\in G$ we shall prove by induction on the length of the formula defining s in T(G) that $x|s\in G$:
-if $s=ms(g)$ and $ms(g)\subseteq ms(x)$ then $x|s=x_g$;
-if $s=s_1\vee s_2$ and $s_1\perp s_2$, where $s_1=ms(g_1)$ and $s_2=ms(g_2)$, we have

$ms(x|s_1 \vee x|s_2)=s_1 \vee s_2=s$ and then $x|s=x|s_1 \vee x|s_2$;
and in the general case $s' \vee s''=s'^{\perp s''} \vee s''^{\perp s'} \vee s'[s'' \leq s'] \vee s''[s'<s'']$ with each term is orthogonal to the others;
-if $s=s_1 \cap s_2$ the answer is given by the condition 2),
-if $s=s'^{\perp s''}$ and $s \subseteq ms(x)$ then $s=(s' \cap ms(x))^{\perp s''}$, if $s''=ms(y)$ then $x|s=(x|s' \cap ms(x))^{\perp y}$.□

Lemma: If D is a divisible l-subgroup of $\Lambda(I,G_i)$, there exists a smallest divisible and r-projectable l-subgroup of G containing D: D_r, with the same cardinality than D.

Proof: D_r is obtained by composition of fonctions.□

Lemma: $\{\Sigma_{i=1,\dots,n}(x_i|s_i),\ x_i \in D,\ s_i \subseteq ms(x_i),\ s_i \in T(D)\}$
$= \{\Sigma_{i=1,\dots,n}(x_i|s_i),\ x_i \in D,\ s_i \subseteq ms(x_i),\ s_i \in T(D),\ s_i \perp s_j \text{ if } i \neq j\}$
(each finite sum of such elements is equal to an orthogonal one).

Proof: Let $x=x_1|s_1+\dots+x_n|s_n$, $s_i \in T(D)$, $x_i \in D$, take $T(s_1,\dots s_n)$ and $mp(s_1,\dots,s_n)$: for each $\alpha \in mp(s_1,\dots,s_n)$ let $I(\alpha)=\{i,\ 1 \leq i \leq n;\ \alpha \ll s_i\}$ then $x|ms(x)(\alpha)=\Sigma(x_i|ms(x_i)(\alpha),\ i \in I(\alpha)\}$, $x=\Sigma(x|ms(x)(\alpha),\ \alpha \in mp(s_1,\dots s_n))$ and this sum is orthogonal because if $\alpha \neq \alpha'$ then $\alpha \perp \alpha'$.

Lemma:(*) $D_r=\{\Sigma_{i=1,\dots,n}(x_i|s_i),\ x_i \in D,\ s_i \subseteq ms(x_i),\ s_i \in T(D),\ s_i \perp s_j \text{ if } i \neq j\}$.

Proof: Let $S=\{\Sigma_{i=1,\dots,n}(x_i|s_i),\ x_i \in D,\ s_i \subseteq ms(x_i),\ s_i \in T(D),\ s_i \perp s_j \text{ if } i \neq j\}$, we want to prove that $S=D_r$.
Obviousley if $x \in S$ then $-x \in S$. Let $x,y \in S$: $x=x_1|s_1+\dots+x_n|s_n$ and $y=y_1|t_1+\dots+y_k|t_k$, $s_i,t_j \in T(D)$ $x_i,y_i \in D$, $s_i \subseteq ms(x_i)$, $t_i \subseteq ms(y_i)$.
1) $z=x+y \in S$ by the previous lemma.
2) $u=x \vee y \in S$ and $v=x \wedge y \in S$:
take $T(s_1,\dots s_n,t_1,\dots t_k)$ and $mp(s_1,\dots,s_n,t_1,\dots,t_k)$: for each $\alpha \in mp(s_1,\dots,s_n,t_1,\dots,t_k)$ there is at most an s_i , and at most an t_j such that $\alpha \ll s_i$ and $\alpha \ll t_j$:
case 1): $\alpha \ll s_i$ and for all j, α does not satisfy $\alpha \ll t_j$:
$u|ms(u)(\alpha) = x_i|s_i=v|ms(v)(\alpha)$;
case 2): $\alpha \ll t_j$ and for all i α does not satisfy $\alpha \ll s_i$:
$u|ms(x)(\alpha) = y_j|t_j=v|ms(v)(\alpha)$;
case 3): $\alpha \ll s_i$ and $\alpha \ll t_j$ then t_i and t_j are comparable and
$u|ms(u(\alpha)=(x_i \vee y_j)|ms(x_i \vee y_j)(\alpha)$, $v=(x_i \wedge y_j)|ms(x_i \wedge y_j)(\alpha)$.
3) S is r-projectable: let $x \in S$, $x=x_1|s_1+\dots+x_n|s_n$, $s \in T(S)$ such that $s \subseteq ms(x)$, then $x|s=x_1|s_1 \cap s+\dots+x_n|s_n \cap s$. We shall prove that for each i

$x_1|s_1\cap s\in S$. Let $S_f\subseteq S$ be a finite set such that $s\in T(ms(S_f))$, consider $mp(S_f\cup\{x_1|s_1\})=m$, then $x_1|s_1=\Sigma(x_1|(s_1\cap s)(\alpha),\ \alpha\in m)$ belongs to S.

4) S is divisible since each x_i is divisible.□

Corollary: $T(D_r)=T(D)$.

Lemma: If D is an r-projectable l-subgroup of $\Lambda(I,G_i)$ then:

$ms(D)=T(ms(D))$, (wich is denoted by $T(D)$).

Proof: 1) $ms(D)\subseteq T(D)$; 2) if $s\in T(D)$, s is generated in T(D) by a set $\{ms(\gamma),\ \gamma\in\Gamma,\ \Gamma$ fini, $\Gamma\subseteq D\}$, for each $\alpha\in(\Gamma)$ there exists a $\gamma(\alpha)\in\Gamma$ such that $s(\alpha)\subseteq ms(\gamma(\alpha))$, from the fact that D is r-projectable it follows that $\gamma(\alpha)|s(\alpha)\in D$ and $s=ms(\Sigma(\ \gamma(\alpha)|s(\alpha),\ \alpha\in mp(\Gamma)))\in T(D)$.□

Lemma: If D is a subgroup of G and Δ the l-subgroup generated by D then $T(\Delta)=T(\{ms^+(g),\ ms^-(g),\ g\in D\})$.

Proof: Δ is the lattice generated by $\{g^+,\ g^-,\ g\in D\}$.□

4) Transfer results on elementary equivalence.

In this part we shall use back and forth arguments between r-projectable substructures. First we prove some lemmas.

Lemma:(**) If D is a divisible r-projectable l-subgroup of G and $y\in G$, $y\notin D$, Δ the divisible l-subgroup of G generated by D and y, then the divisible r-projectable l-subgroup generated by D and y is:

$D''=\{\Sigma_{i=1,..,n}(q_iy+d_i)|s_i,\ s_i\in T(\Delta),\ d_i\in D,\ q_i\in\mathbb{Q},\ s_i\subseteq ms(q_iy+d_i),\ s_i\perp s_j$ if $i\neq j\}$.

Proof: The divisible r-projectable l-subgroup generated by D and y:$\langle D,y\rangle_r$ is the divisible r-projectable l-subgroup generated by Δ and by lemma (*) this is: $\{\Sigma_{i=1,...,n}(z_i|s_i),\ s_i\subseteq ms(z_i),\ s_i\in T(\Delta),\ s_i\perp s_j$ if $i\neq j,\ z_i\in\Delta\}$.

1) $D''\subseteq\langle D,y\rangle_r$ because each q_iy+d_i is in Δ.

2) If $x\in D''$ and $z\in D''$ then $u=x\vee z\in D''$ and $v=x\wedge z\in D''$: Let $x=(q_1y+d_1)|s_1+\ldots+(q_ny+d_n)|s_n$ and $z=(q'_1y+d'_1)|t_1+\ldots+(q'_ky+d'_k)|t_k$, $s_i,t_j\in T(\Delta)$ $d_i,d'_i\in D$, $q_i,q'_i\in\mathbb{Q}$, $s_i\subseteq ms(q_iy+d_i)$, $t_i\subseteq ms(q'_iy+d'_i)$.

Take $T(s_1,\ldots s_n,t_1,\ldots t_k)$ and $mp(s_1,\ldots,s_n,t_1,\ldots,t_k)$: for each $\alpha\in mp(s_1,\ldots,s_n,t_1,\ldots,t_k)$ there is at most an s_i, and at most a t_j such that $\alpha\ll s_i$ and $\alpha\ll t_j$:

case 1): $\alpha\ll s_i$ and for all j, α does not satisfy $\alpha\ll t_j$:

$u|ms(u)(\alpha)\ =\ (q_iy+d_i)|s_i=v|ms(v)(\alpha)$;

case 2): $\alpha \ll t_j$ and for all i α does not satisfy $\alpha \ll s_i$:

$u|ms(u)(\alpha) = (q'_j y+d'_j)|t_j = v|ms(v)(\alpha)$;

case 3): $\alpha \ll s_i$ and $\alpha \ll t_j$ then s_i and t_j are comparable and

$$u|ms(u(\alpha)) = (q_i y+d_i)|ms(q_i y+d_i)(ms(((q_i-q'_j)y+d_i-d'_j)\vee 0)) + (q'_j y+d'_j)|ms(q'_j y+d'_j)(ms(((q_j'-q_i)y+d'_j-d_i)\vee 0))$$

$$v|ms(v(\alpha)) = (q_i y+d_i)|ms(q_i y+d_i)(ms(((q_i-q'_j)y+d_i-d'_j)\wedge 0)) + (q'_j y+d'_j)|ms(q'_j y+d'_j)(ms(((q_j'-q_i)y+d'_j-d_i)\wedge 0))$$

$ms(((q_i-q'_j)y+d_i-d'_j)\vee 0) \in ms(\Delta)$, $ms(((q_j'-q_i)y+d'_j-d_i)\vee 0) \in ms(\Delta)$.

3) D" is r-projectable: let $x \in D"$, $x=(q_1y+d_1)|s_1+\ldots+(q_ny+d_n)|s_n$, $s \in T(D")=T(\Delta)$, such that $s \subseteq ms(x)$, then $x|s=(q_1y+d_1)|s_1 \cap s+\ldots+(q_ny+d_n)|s_n \cap s$, for each i, $(q_iy+d_i)|s_i \cap s \in D"$. Let $D_f \subseteq D"$ be a finite set such that $s \in T(ms(D_f))$, consider $mp(D_f \cup \{(q_iy+d_i)|s_i\})=m$, then $(q_iy+d_i)|s_i=\Sigma((q_iy+d_i)|(s_i \cap s)(\alpha), \alpha \in m)$ belongs to D".□

Lemma (toward Theorem 2): If $(I,\leq)$ and $(J,\leq)$ are dense branching root systems and for all $i \in I$, $j \in J$, G_i and G_j are all divisible, and $(A(I),\ll) \equiv (A(J),\ll)$ then $\Lambda(I,(G_i)_{i\in I}) \equiv \Lambda(J,(G)_{j\in J})$.

Proof: Let $G_1=\Lambda(I,(G_i)_{i\in I})$ and $G_2=\Lambda(J,(G_j)_{j\in J})$ and $A(I,\ll) \equiv A(J,\ll)$.
Let G'_1 and G'_2, ω_1-saturated such that $G'_1 \equiv G_1$ and $G'_2 \equiv G_2$.
By the embedding lemma $G'_1 \subseteq H'_1 = \Lambda(I(G'_1),(G'_1)_i)$, (resp. $G'_2 \subseteq H'_2 = \Lambda(I(G'_2),(G'_2)_i)$. Recall that $A(G'_1)$ (resp. $A(G'_2)$) is the quotient of G'_1 (resp. G'_2) by the definable relation R(g,h)&R(h,g) and that $A(G'_1,\ll) \equiv A(G'_2,\ll) \equiv A(I,\ll)$.
$(A(G'_1),\ll)$ and $(A(G'_2),\ll)$ are ω_1 saturated. We have a family F of partial isomorphisms between countable substructures of $(A(G'_1),\ll,\wedge,\vee,\perp,\cap)$ and $(A(G'_2),\ll,\wedge,\vee,\perp,\cap)$ with the back and forth property. Using this we shall construct a family of partial isomorphisms F' between l-subgroups of G'_1 (resp. G'_2), countable, divisible and r-projectable in $H'_1 = \Lambda(I(G'_1),(G'_1)_i)$ (resp.in $H'_2 = \Lambda(I(G'_2),(G'_2)_i)$), with the back and forth property.

If $D_1 \subset G'_1$, (resp. $D_2 \subset G'_2$) are divisible, countable, and r-projectable in H'_1 (resp. in H'_2) and f' a partial isomorphism from D_1 to D_2, say that $f' \in F'$ iff there exist an $f \in F$ such that for each $g \in D_1$ and $g' \in D_2$ if $g'=f'(g)$, $ms(g')=f(ms(g))$.

F' is not empty because of the trivial isomorphism from {0} to {0}. We want to prove that F' has the back and forth property. Let $y \in G_1$, $y \notin D_1$, and $f' \in F'$ from D_1 to D_2 and $f \in F$ given by f', from $ms(D_1)$ to $ms(D_2)$. Let Δ be the divisible subgroup of G'_1 generated by D_1 and y. By the back and forth argument on F, f can be extended to $T(\Delta)$ wich coincides with $T(D"_1)$, where

D''_1 is the divisible r-projectable l-subgroup of G'_1 generated by D_1 and y. by lemma(**) we have $D_1''=$
$\{\Sigma_{i=1,..,n}(q_iy+d_i)|s_i,\ \ s_i\in T(\Delta),\ \ d_i\in D_1,\ \ q_i\in\mathbb{Q},\ \ s_i\leq ms(q_iy+d_i),\ \ s_i\perp s_j\ \text{ if }\ i\neq j\}$
and $ms(D''_1)=T(D''_1)=T(\Delta)=T(\{ms^+(q_iy+d_i),\ ms^-(q_iy+d_i),\ d_i\in D_1,\ q_i\in\mathbb{Q}\})$.

We want to define $f'\in F'$ with $dom(f)=D''_1$, f' has to be a group isomorphism and to verify for each $x\in D''_1$: $ms(f'(x)=f(ms(x))$ and $f'(x)\leq 0$ iff $x\leq 0$.

If $x\in D''_1$, $x=\Sigma_{i=1,..,n}((q_iy+d_i)|s_i)$, $f'(x)=\Sigma_{i=1,..,n}f'((q_iy+d_i)|s_i)$, $f'(q_iy+d_i)=q_if'(y)+f'(d_i)$, s_i and $ms(q_iy+d_i)$ are in $dom(f)$ and $f(s_i)\leq f(ms(q_iy+d_i))=ms(q_if'(y)+f'(d_i))$ hence $f'((q_iy+d_i)|s_i)=((q_if'(y)+f'(d_i)|f(s_i))$. Therefore f' has to satisfy:
$\Sigma_{i=1,..,n}(q_if'(y)+f'(d_i))|f(s_i)\leq 0$ iff $\Sigma_{i=1,..,n}(q_iy+d_i)|s_i\leq 0$ and
$ms^+(q_if'(y)+f'(d_i))=f(ms^+(q_iy+d_i))$ and $ms^-(q_if'(y)+f'(d_i))=f(ms^-(q_iy+d_i))$
i.e. for each $d\in D_1$ and $s\in T(\Delta)$, $(f'(y)+f'(d))|f(s)\leq 0$ iff $(y+d)|s\leq o$, and $ms^+(f'(y)+f'(d))=f(ms^+(y+d))$ and the same for ms^-.
For each $s\in T(\Delta)$ let $g'_s\in G'_2$ satisfy $f(s)=ms(g'_s)$, and for each $d\in D_1$, g'_d and $h'_d\in G'_2$ satisfy $f(ms^+(y+d))=ms(g'_d)$, and $f(ms^-(y+d))=ms(h'_d)$.

For each d,g,g', $((g+d)|ms(g'))\leq 0$ can be expressed as a formula $\delta(g,d,g')$, $((g+d)|ms(g'))\geq 0$ by $\delta'(g,d,g')$, $ms^+(g+d)=ms(g')$ by $\beta(g,d,g')$ and $ms^-(g+d)=ms(g')$ by $\gamma(g,d,g')$

Consider the following set Φ of formulas:
$\Phi=\{\delta(t,f'(d),g'_s)$ for d,s such that $G\models((y+d)|s)\leq o$, $\delta'(t,f'(d),g'_s)$ for d,s s.t. $G\models((y+d)|s)\geq o$, $\beta(t,f'(d),g'_d)$, $\gamma(t,f'(d),h'_d)$ for all $d\in D_1$ and $s\in T(\Delta)\}$

We want to prove that each finite subset $\Phi°$ of Φ can be satisfied.Let $D°$ and $S°$ be the finite sets of dand s (respectively) occuring in $\Phi°$.
Let $T°=T(\{f'(d),g'_d,h'_d,g'_s\ ,d\in D°\cup\{0\},\ \ s\in S°\})$ and $dat(D°)$ the formula expressing the conjonction of all true atomic or negation of atomic formulas in $T°$, For each $f'(d),g'_d,h'_d,g'_s$, $d\neq 0$, wich appears in $\Phi°$ define variables x_d,u_d,v_d,w_s and let Ψ be the quantified formula:
$\forall x_d,\dots,\forall u_d,\dots,\forall v_d,\dots,\forall w_s,\dots,\exists t\ (\ dat(D°)\ \&\ \Lambda(\phi,\ \phi\in\Phi°))$.
We shall prove that this formula is true in the Hahn product G_2 and then in G'_2.
$\delta(t,x_d,w_s)$ (resp. $\delta'(t,x_d,w_s)$) says that $(t+x_d|w_s)\leq 0$, (resp.≥ 0)
$\beta(t,x_d,w_s)$ (resp. γ) says that $ms^+(t+x_d)=ms(u_d)$, (resp. $ms^-(t+x_d)=ms(v_d)$).
Write $T°=T(\{x_d,u_d,v_d,w_s,t^+,t^-\})$ and $\alpha\ \in mp(T°)$ then $\{s(\alpha),\ \alpha\in T°\}$ is totally ordered:
If $ms^+(t)(\alpha)\neq ms(x_d)(\alpha)$ (respectively $ms^-(t)(\alpha)\neq s(\alpha)$) can be deduced from Ψ for each $d\in D°$ then define $t|\alpha$ to be any positive (resp. negative) element of the Hahn product on the given support.
If not, there is a minimal $ms(u_d)=ms(w_s)$(resp.$ms(v_d)$). Then define $t|\alpha$ to be

any element of the Hahn product such that for each $i \in ms(w_s)$ $t+x_{d'} \geq 0$ iff $ms(u_{d'})=ms(w_s)$ and $t+x_{d'} \leq 0$ iff $ms(v_{d'})=ms(w_s)$.
This can be done without contradiction because by the definition of δ and δ', the formula is satisfied in G_1.
Now ψ is true in G'_2, each finite part of Φ can be realised in G'_2, and since G'_2 is ω_1-saturated, Φ can be realized.□

We can obtain Theorem 2 as a corollary:

Theorem 2: If $(I,\leq)$ and $(J,\leq)$ are root systems and for all $i \in I$, $j \in J$, G_i and G_j are all divisible, and $(I,\leq)\equiv(J,\leq)$ then $\Lambda(I,(G_i)_{i\in I})\equiv\Lambda(J,(G)_{j\in J})$.

Proof: If a root system I is not dense branching then we can define an equivalence relation iRj iff $\forall k$ $k\leq i \Leftrightarrow k \leq j$. The quotient I/R is always dense branching and if $G_{\hat{\imath}}=\Lambda(\hat{\imath},(G_i)_{i\in\hat{\imath}})$ the Hahn product of the G_i for i in the equivalence class $\hat{\imath}$ then $G_{\hat{\imath}}$ is a divisible totally ordered abelian group and $\Lambda(I,(G_i)_{i\in I})=\Lambda(I/R,(G_{\hat{\imath}})_{\hat{\imath}\in I/R})$.
If $(I,\leq)\equiv(J,\leq)$ then $(I/R,\leq)\equiv(J/R,\leq)$. Since each G_i is divisible, each $G_{\hat{\imath}}$ is also divisible. To end we use theorem [G] of the introduction: $(I/R,\leq)\equiv(J/R,\leq)$ implies $(A(I/R),\ll)\equiv(A(J/R),\ll)$.□

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Low dimensional sections of basic semialgebraic sets

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Abstract

Let $X \subset \mathbf{R}^n$ be an irreducible algebraic set. A semialgebraic subset $S \subset X$ is called m-basic if it can be written as $S = \{x \in X \mid f_1(x) > 0, \ldots, f_m(x) > 0\}$ for some polynomial functions $f_1, \ldots, f_m$. The Bröcker-Scheiderer criterion asserts that a semialgebraic S is s-basic if and only if it satisfies the boundary condition $S \cap \overline{\overline{S} \setminus S}^Z = \emptyset$ and $S \cap Y$ is s-generically basic for all irreducible algebraic sets $Y \subset X$. In this paper we show that it suffices to check this condition for $\dim(Y) = s+1$, which is, in fact, the lowest dimension at which it may fail.

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Introduction

Let $X \subset \mathbf{R}^n$ be a real algebraic set, and let $\mathcal{P}(X)$ denote the ring of polynomial functions on X. Recall that a subset $S \subset X$ is called *semialgebraic* if there exist polynomials $f_{ij}, g_i \in \mathcal{P}(X)$ such that

$$S = \bigcup_{i=1}^{p} \{x \in X : f_{i1}(x) > 0, ..., f_{ir_i}(x) > 0, g_i(x) = 0\}.$$

As is well known, if S is open the g_i's in this expression can be omitted. Recall also that an open semialgebraic set is called *basic open* if furthermore $p = 1$. These basic open sets

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have attracted a lot of interest in recent times, till the proof of the beautiful theorem that states that a basic open set S has always a description

$$S = \{x \in X : f_1(x) > 0, \dots, f_s(x) > 0\}$$

with $s \leq \dim(X)$, see [Br2,3,4], [Sch], [Mh], [AnBrRz1]. However, the problem of understanding when a given semialgebraic set is basic open and, in that case, how many inequalities are needed to generate it is wide open. An immediate remark is that $S \cap \overline{(\overline{S} \setminus S)}^Z = \emptyset$ if S is basic open, where $\overline{}$ stands for the usual closure, and $\overline{}^Z$ for the Zariski closure. The only full characterization available is due to Bröcker and Scheiderer. To state it properly, let us say that a semialgebraic set S is *s-basic* if there are s polynomials $f_1, \dots, f_s \in \mathcal{P}(X)$ such that $S = \{f_1 > 0, \dots, f_s > 0\}$, and that S is *generically s-basic* if it is s-basic up to codimension 1, that is, there are s polynomials $f_1, \dots, f_s \in \mathcal{P}(X)$ and a nowhere dense algebraic subset $Z \subset X$ such that $S \setminus Z = \{f_1 > 0, \dots, f_s > 0\} \setminus Z$. Now let S be an open semialgebraic set such that $S \cap \overline{(\overline{S} \setminus S)}^Z = \emptyset$. The Bröcker-Scheiderer criterion for the generation of basic sets reads as follows:

Theorem 1. *The set S is s-basic if and only if for every irreducible subset $Y \subset X$ the intersection $S \cap Y$ is generically s-basic.*

Since the dimension bounds the number of inequalities needed to generate any basic set, we see that in every subset of X, being basic is equivalent to being d-basic, where $d = \dim(X)$. Hence the previous theorem has the following corollary:

Corollary 2. *The set S is basic if and only if for every irreducible subset $Y \subset X$ the intersection $S \cap Y$ is generically basic.*

Thus it appears the problem of whether it exists a distinguished family of subvarieties which suffices to characterize basicness. In fact, in [AnRz1] we proved:

Theorem 3. *The set S is basic if and only if for every irreducible* surface *$Y \subset X$ the intersection $S \cap Y$ is basic.*

Since in dimension 1 every semialgebraic set is 1-basic, this was the best possible result concerning dimension, and the first suggestion that obstructions to the generation of basic sets should appear in the smallest predictable dimension. According to this idea, if a basic open set requires s inequalities, we should recognize it exactly in dimension $s+1$, because in dimension $\leq s$ it certainly can be generated by s inequalities. The goal of this paper is the confirmation of this conjecture. We will prove:

Theorem 4. *Suppose that S is basic. Then S is s-basic if and only if for every irreducible subset $Y \subset X$ of dimension $s+1$ the intersection $S \cap Y$ is generically s-basic.*

The proofs of these results are always a combination of the theory of fans in spaces of orderings of function fields and the theory of the real spectrum. Fans are special sets of orderings of the field which quite surprisingly play a dramatic role in the previous questions and results. Its definition and basic properties are collected in Section 1. What makes possible the improvements concerning dimension in Theorems 3 and 4 is a better analysis of the valuation theory behind the scene. For instance, in [AnRz1] we dealt with rank 1 valuations to obtain Theorem 3 and as a byproduct we also got Theorem 4 in the simplest case $s = 1$; now, to study the general case, we have to deal with valuations of arbitrary rank. The first step is, in both cases, the reduction to discrete valuations. Then if the rank is 1, we obtain through completion power series in one variable. This is not so if the rank is arbitrary. However it can be done in a suitable way, so that the final consequence is the substitution of arbitrary fans by a very geometric type of them, defined through power series in several variables. This is a consequence of what we call the *approximation theorem for fans:* any fan can be arbitrarily approximated by what we call an algebroid fan, that is, one coming from a power series field and associated to a discrete valuation of maximum rank.

The interest in valuations is not new in real algebraic geometry, see [An], [BrSch], [Rb], [Rz1] and the forthcoming [AnRz2]. Here we exploit sistematically the notion of compatibility of a fan with a valuation, that is, the simultaneous compatibility of several different orderings, as well as the general interplay between valuations and fans. Two essential tools in our proofs are resolution of singularities and Bertini's theorem. As a matter of fact the failure of the latter in the Nash or analytic category is the reason why our results do not extend to those categories (see the counterexample in [AnRz1]). Despite this failure, many interesting things can be said in the Nash and analytic case using the techniques of this paper. However, here we work only in the algebraic case and refer the reader to [AnBrRz1], [Rz2] and the forthcoming [AnRz3] for the other two.

The paper is organized as follows. Section 1 contains the definitions and some general facts concerning fans needed later. Section 2 describes the trivializacion of fans through real valuations and the connection with power series. Section 3 is devoted to the approximation theorem for fans of function fields over the reals, which is the key step towards Theorem 4. In Section 4 we review the theory of the real spectrum that makes the connection between spaces of orderings and algebraic varieties. Finally, Section 5 contains the proof of Theorem 4.

1 Fans and basic sets

The abstract theory of spaces of orderings was developed by Marshall in the series of papers [Mr1-5]. A self-contained new presentation will appear in [AnBrRz2]. Here we only outline some basic facts.

Let K be a field, and consider its space of orderings $\Sigma = Spec_r(K)$. Given $f \in K$ and $\sigma \in \Sigma$, we can see σ as a signature $\sigma : K \to \{-1, +1\}$ which maps the element f to $+1$ or -1 according to whether f is positive or negative in the ordering σ. To keep a geometrical meaning in the notation we will write $f(\sigma) > 0$ instead of $\sigma(f) = +1$ and $f(\sigma) < 0$ instead of $\sigma(f) = -1$. A *constructible* subset of Σ is a set of the form

$$C = \bigcup_{i=1}^{p} \{\sigma \in \Sigma : f_{i1}(\sigma) > 0, \dots, f_{ir_i}(\sigma) > 0\},$$

where $f_{ij} \in K$. Such a set C is called *basic* if $p = 1$. The basic sets form a basis of *the Harrison topology* of Σ.

A *(finite) fan of* K is a finite set $F \subset \Sigma$ such that for any three orderings $\sigma_1, \sigma_2, \sigma_3 \in F$, their product $\sigma_4 = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$ is a well-defined ordering and belongs to F (we multiply orderings as signatures). Thus subsets consisting of one or two orderings are always fans and are called *trivial fans*. A basic fact is that a fan F has a structure of affine space over the field of two elements $\mathbf{F}_2 = \{-1, +1\}$, or equivalently, for any $\sigma_0 \in F$, the set σF is a vector space over $\mathbf{F}_2 = \{-1, +1\}$ with the product of signatures as inner operation and the natural scalar multiplication. In particular, it follows that $\#(F) = 2^k$, where k is the *affine dimension* of F, that is, $k+1$ is the minimal number of elements $\sigma_0, \ldots, \sigma_k \in F$ such that any $\sigma \in F$ is the product of an *odd* number of σ_i's. An important property is that if F' is an affine subspace of F then F' is again a fan.

In connection with basic sets, let us remark the immediate fact that for every basic set $C \subset \Sigma$, the intersection $F' = F \cap C$ is again a fan, and so $\#(F') = 2^l$ for some $l \leq k$.

The fundamental result concerning our problem is:

Theorem 1.1 *Let C be a constructible subset of Σ. The following assertions are equivalent:*

(a) There are s elements $f_1, \ldots, f_s \in K$ such that $C = \{f_1 > 0, \ldots, f_s > 0\}$.

(b) For every fan $F \subset \Sigma$ with $\#(F) = 2^k$ and $F \cap C \neq \emptyset$ we have $\#(F \cap C) = 2^l$ with $0 \leq k - l \leq s$.

Somehow surprisingly, 4-element fans are enough to check whether or not a set is basic:

Theorem 1.2 *Let C be a constructible subset of Σ. The following assertions are equivalent:*

(a) C is basic.

(b) For every fan $F \subset \Sigma$ with $\#(F) = 4$ we have $\#(F \cap C) \neq 3$.

We will not use Theorem 1.2 here, since we are interested in the quantitative question. Let us remark that Theorem 1.1 is only a reformulation of the usual statement, and we still need a further modification:

Corollary 1.3 *Let C be a basic constructible subset of Σ. The following assertions are equivalent:*

(a) There are s elements $f_1, \ldots, f_s \in K$ such that $C = \{f_1 > 0, \ldots, f_s > 0\}$.

(b) For every fan $F \subset \Sigma$ with $\#(F) = 2^k$ and $\#(F \cap C) = 1$ we have $k \leq s$.

Proof: Since (b) is a particular case of Theorem 1.1 (b) we only must prove (b) $\Rightarrow$ (a). For this, suppose $F \subset \Sigma$ is a fan. Since C is basic, the intersection $F' = F \cap C$ is a fan, say generated by $\sigma_1, \ldots, \sigma_{l+1}$, and $\#(F') = 2^l$. Now we can add to these σ_i's some others to get generators $\sigma_1, \ldots, \sigma_{k+1}$ of F. Finally consider the fan F'' generated by $\sigma_{l+1}, \ldots, \sigma_{k+1}$, with $\#(F'') = 2^{k-l}$. Clearly $F'' \cap C = F'' \cap F' = \{\sigma_{l+1}\}$ because they are affine subspaces of F of complementary dimensions which generate F and containing the point σ_{k+1}. Hence $\#(F'' \cap C) = 1$ and by (b), $k - l \leq s$. Now the result follows from Theorem 1.1. □

2 Fans and valuations

Let K be field and Σ its space of orderings as in Section 1. Let A be a subring of K and $\mathfrak{p}$ an ideal of A. An ordering $\sigma \in \Sigma$ *makes* $\mathfrak{p}$ *convex* if from $0 < f < g, f \in A, g \in \mathfrak{p}$ it follows $f \in \mathfrak{p}$. This implies that σ induces a unique ordering τ in the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ such that $f \bmod \mathfrak{p} >_\tau 0$ if $f >_\sigma 0$ for every $f \in A \setminus \mathfrak{p}$. The notation for this is $\sigma \to \tau$, and we say that σ *specializes to* τ or that τ is *a specialization of* σ. The proper setting for this specialization relation is the theory of the real spectrum as we will see in Section 1. However this notion was first studied in the context of valuation theory which we discuss here. A valuation ring V of K is *compatible with* an ordering $\sigma \in \Sigma$ if σ makes convex the maximal ideal $\mathfrak{m}_V$ of V. Then σ specializes to an ordering τ in the residue field k_V of V: $\sigma \to \tau$. This kind of specializations are well understood by means of the Baer-Krull theorem ([BCR]):

Theorem 2.1 *Let Γ denote the value group of V, and τ an ordering of k_V. Then there is a bijection between the set of orderings of K compatible with V and specializing to τ and the set of group homomorphisms $\phi : \Gamma \to \{+1, -1\}$.*

Note that this implies that V is compatible with some ordering if and only if its residue field is formally real. In that case we will say that V is a *real valuation ring.*

A particular situation in which Baer-Krull theorem can be seen the following: **Example 2.2.** Let B be a local regular ring with residue field L and quotient field K. Suppose $\dim(B) = m$ and consider a system of parameters $x_1, \ldots, x_m$. By induction on m, we define a valuation v_m in the quotient field K of B which has residue field L and value group $\mathbf{Z}^m$.

Indeed, if $m = 1$, then B is a discrete valuation ring and we have the corresponding discrete rank one valuation v_1. For $m > 1$, we consider the discrete valuation ring $W = B_{(x_m)}$, whose valuation we denote by w. The residue field K' of W is the quotient field of the local regular ring $B' = B/(x_m)$. By induction we have in K' a valuation v_{m-1} with residue field L and value group $\mathbf{Z}^m$. Then v_m is the composite of w and v_{m-1}. We denote its valuation ring by V_m.

Now we fix an ordering τ in L and look for the set F of orderings σ of K compatible with V_m and specializing to τ. We claim that there are 2^m, and every one is completely determined by the signs of the variables $x_1, \ldots, x_m$. Indeed, by the Baer-Krull theorem (Theorem 2.1), we only have to exhibit 2^m orderings specializing to τ and having different signs at the parameters. Again this follows by induction (we use the notations introduced above). If γ is an ordering of K' compatible with V_{m-1}, we can lift it to two orderings γ_+, γ_- of K compatible with $B_{(x_m)}$ as follows: every $f \in B_{(x_m)}$ can be written as $f = ux_m^n$, where u is a unit of $B_{(x_m)}$, and we define

$$\gamma_+(f) = \gamma(\overline{u}),$$

$$\gamma_-(f) = \gamma(\overline{u})(-1)^n,$$

(here $\overline{u}$ stands for the residue class of u in K'). Since V_m is the composite of V_{m-1} and $B_{(x_m)}$, γ_+ and γ_- are compatible with V_m and specialize to γ.

In can be checked directly that F is a fan, which can be identified with the affine space whose associated vector space is $\{-1,+1\}^m$. In fact, since any $\sigma \in F$ is completely determined by the values $\sigma(x_1),\dots,\sigma(x_m)$, if we denote by σ_0 the ordering defined by $\sigma_0(x_1)=\dots=\sigma_0(x_m)=+1$, then

$$\varphi:\sigma_0 F \to \{-1,+1\}^m; \quad \sigma_0\sigma \mapsto (\sigma(x_1),\dots,\sigma(x_m))$$

is an isomorphism. With this identification, the elements $\sigma_0,\sigma_1,\dots,\sigma_m$ defined by the following table form a minimal system of generators of F. Here $*$ can be either $+1$ or -1.

	x_1	$\cdots$	x_{m-2}	x_{m-1}	x_m
σ_0	$+1$	$\cdots$	$+1$	$+1$	$+1$
σ_1	$+1$	$\cdots$	$+1$	-1	$*$
σ_2	$+1$	$\cdots$	-1	$*$	$*$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
σ_m	-1	$\cdots$	$*$	$*$	$*$

Table 1

In other words, keeping in mind that in $\mathbf{F}_2=\{+1,-1\}$, $+1$ is the zero and -1 is the unit, geometrically we are taking σ_0 as the origin of F and the matrix of coordinates of $\sigma_0\sigma_1,\dots,\sigma_0\sigma_m$ is triangular, so that they are a basis of $\sigma_0 F$. All this can be seen as a particular case of a general situation which we describe now very briefly.

We say that the valuation ring V is *compatible with* a fan $F \subset \Sigma$ if V is compatible with every ordering $\sigma \in F$. It is easily checked that the specializations of the orderings of F form a fan in k_V, possibly trivial. In fact, the main result concerning fans and valuations is the so-called *trivialization* theorem ([Br1], [AnBrRz2]):

Theorem 2.3 *Let F be a fan of K. Then there exists a valuation ring V of K compatible with F such that the orderings of F have at most 2 distinct specializations in the residue field k_V o V.*

Conversely, given a fan $\bar{F}$ in k_V, the set of orderings of K wich are compatible with V and specialize to an ordering of $\bar{F}$ is a fan some times called the *pull-back* of $\bar{F}$. This is extremely useful for the construction of fans starting from trivial ones. For instance, Example 2.2 shows the construction of the pull-back of the trivial fan consisting of the ordering τ of L. We develope now a second example which will be of utmost importance in the rest of the paper.

Example 2.4. Let L be a field and $x_1,\dots,x_m$ indeterminates. Consider the ring of formal power series $L[[x_1,\dots,x_m]]$ and its quotient field $L((x_1,\dots,x_m))$. We set $\mathfrak{m}=(x_1,\dots,x_m)$. Let V_m the valuation ring of $L((x_1,\dots,x_m))$ constructed as in Example 2.2, that is, V_m is the composite of the discrete valuation ring $L[[x_1,\dots,x_m]]_{(x_1)}$ with the valuation ring V_{m-1} of the residue field $L((x_2,\dots,x_m))$.

a) Fix an ordering τ in L and let F_τ be the set of orderings σ of K compatible with V_m. We mentioned in Example 2.2 that these σ's are completely determined by τ

and the signs $\sigma(x_1), \ldots, \sigma(x_m)$. Indeed, let us make this precise in our situation. Let $f \in L[[x_1, \ldots, x_m]]$. We look at f as a series in x_1 with coefficients in $L[[x_2, \ldots, x_m]]$, say

$$f = x_1^{\nu_{10}}(g_{10} + \sum_{\ell \geq 1} g_\ell x_1^\ell),$$

with $0 \neq g_{10} \in L[[x_2, \ldots, x_m]]$. In particular $g_{10} + \sum g_\ell x_1^\ell$ is a unit in $L[[x_1 \ldots, x_m]]_{(x_1)}$, namely coincides with g_{10} (mod x_1). It follows that

$$\sigma(f) = \sigma(x_1)^{\nu_{10}}\gamma(g_{10})$$

where γ is the specialization of σ in $L((x_2, \ldots, x_m))$. Now, to determine $\gamma(g_{10})$ we look at it as a series in x_2 an proceed as above. This way, by induction it is clear that

$$\sigma(f) = \sigma(x_1)^{\nu_{10}}\sigma(x_2)^{\nu_{20}} \cdots \sigma(x_m)^{\nu_{m0}}\tau(u_{\nu_0})$$

where $u_{\nu_0}x_1^{\nu_{10}} \cdots x_m^{\nu_{m0}}$ is the initial form of f when we consider in $\mathbf{N}^m$ the *lexicographic ordering*. In other words, the sign of f is completely determined by the sign of its initial form. It follows that if $h \equiv f \pmod{\mathfrak{m}^\nu}$, for ν high enough so that they have the same initial form, then $\sigma(f) = \sigma(h)$.

b) Set now $m = k - 1$ and fix two distinct orderings γ_1, γ_2 in L. Let $\mathfrak{F}$ stand for the set of all orderings of $L((x_1, \ldots, x_m))$ which are compatible with V_m and specialize to either of the γ_i's. Then $\mathfrak{F}$ is a fan with $2 \cdot 2^m = 2^k$ elements. In fact $\mathfrak{F}$ is the union of the two fans F_{γ_1} and F_{γ_2} described in *a)*. In particular it follows that if F_{γ_1} is generated by $\sigma_0, \ldots, \sigma_{k-1}$ and $\sigma_k \in F_{\gamma_2}$, then $\mathfrak{F}$ is generated by $\sigma_0, \ldots, \sigma_{k-1}, \sigma_k$.

After this preparation we introduce a key notion for our work.

Definition 2.5 *Let $A \subset K$ a subring of K and F a fan of K with $\#(F) = 2^k$. We say that F is algebroid if there is an embedding $K \hookrightarrow L((x_1, \ldots, x_{k-1}))$ into a power series field such that F is the restriction to K of the fan $\mathfrak{F}$ of Example 2.4 b). We also say that F is parametrized over γ_1, γ_2 in L. Finally, we say that F is finite on A if $A \subset L[[x_1, \ldots, x_{k-1}]]$ under the above embedding.*

A typical situation where we can construct algebroid fans is the following:

Example 2.6. Let B be a local regular ring with residue field L and quotient field K. Suppose $\dim(B) = k - 1$ and consider two orderings γ_1, γ_2 in L. Fix any system of parameters $x_1, \ldots, x_{k-1}$. Then the adic completion $\hat{B}$ of B is isomorphic to $L[[x_1, \ldots, x_{k-1}]]$ and this gives an embedding $K \hookrightarrow L((x_1, \ldots, x_{k-1}))$. In the latter field we have the fan $\mathfrak{F}$ of Example 2.4 *b)* and its restriction F to K is obviously an algebroid fan parametrized over γ_1, γ_2 in L. Clearly F is finite on B.

3 Approximation of fans

Again, let K be a formally real field and $\Sigma = Spec_r(K)$. Fix an integer $k \geq 0$. Any fan of K with 2^k elements can be seen as a 2^k-tuple in the product $\Sigma_k = \overbrace{\Sigma \times \cdots \times \Sigma}^{k \text{ times}}$. Now the set Φ_k of all fans of K with 2^k elements can be seen as a subset of Σ_k. This identification is not bijective, unless we identify the tuples in Σ_k up to permutations, but we will not care about this technicality, because it is irrelevant for our purposes. Anyway, the set Σ_k carries the product topology of the Harrison topology of each factor space and, under our identification, the set Φ_k is endowed with the corresponding subset topology, which we still call *the Harrison topology*. Thus fans with 2^k elements form a topological space, namely Φ_k, and we can discuss approximation properties. In this paper we mainly deal with the case K is a finitely generated extension of $\mathbf{R}$; then we call K *a function field*, and its transcendence degree over $\mathbf{R}$ is called *dimension*. These are exactly the fields of rational functions of real algebraic varieties: a *model of* K is an irreducible real algebraic set X such that K is the field $\mathcal{K}(X)$ of rational functions of X, and K has a model if and only if it is a function field. As is well known, since K is formally real, the dimension of K as defined above coincides with the topological dimension of every model of K. Another useful fact is that we can always find *compact* models of K. This is immediate by taking the projective closure of any given model; another way to see it is to take the one-point compactification, which is possible in the real case ([BCR]).

In this Section we will show the following:

Theorem 3.1 *Let K be a function field of dimension n and X a compact model of K. Let $k \geq 2$ and $F \in \Phi_k$ be a fan of K with 2^k elements. Then F can be arbitrarily approximated in the Harrison topology by an algebroid fan F' finite on $\mathcal{P}(X)$ and parametrized over a function field of dimension $n-k+1$.*

Proof: Since X is compact, every polynomial is bounded on X, from which it follows that every real valuation ring of K contains the ring $\mathcal{P}(X)$ of polynomial functions of X. Let $F = (\sigma_i : 1 \leq i \leq 2^k)$ be the given fan, and $U = U_1 \times \cdots \times U_{2^k}$ an open neighborhood of F in Φ_k, with $U_i = \{f_{i1} > 0, \ldots, f_{ir_i} > 0\}$, $f_{ij} \in \mathcal{P}(X)$. After shrinking the U_i's we may assume that they are pairwise disjoint, and we will say that the f_{ij}'s *separate* the orderings of F. By Theorem 2.3 the fan F trivializes along a valuation ring V of K: the σ_i's are compatible with V and induce two orderings τ_1, τ_2 in the residue field k_V of V (possibly $\tau_1 = \tau_2$); as remarked before, $V \supset \mathcal{P}(X)$.

Now we apply resolution of singularities I and II ([Hk]), so that after finitely many blowings-up we may assume that X is non-singular and all the f_{ij}'s are normal crossings. Let $\mathfrak{p} \subset \mathcal{P}(X)$ be the center of V in $\mathcal{P}(X)$: $\mathfrak{p} = \mathfrak{m}_V \cap \mathcal{P}(X)$, where $\mathfrak{m}_V$ is the maximal ideal of V. Then $A = \mathcal{P}(X)_\mathfrak{p}$ is a regular local ring of dimension say d, and has a regular system of parameters $x_1, \ldots, x_d$ such that for all i, j

$$f_{ij} = u_{ij} x_1^{\alpha_{ij1}} \cdots x_d^{\alpha_{ijd}}$$

where the u_{ij} are units of A and the α_{ijk} are non-negative integers.

In this situation the residue field $\kappa(\mathfrak{p})$ of A is a subfield of the residue field k_V of V, and we denote also by τ_1, τ_2 the restriction to $\kappa(\mathfrak{p})$ of τ_1, τ_2. Notice that as above it can happens that $\tau_1 = \tau_2$, and that for each $p = 1, 2$ the signs of the elements f_{ij} in an ordering $\sigma \to \tau_p$ are completely determined by the signs of the parameters x_l in σ and the signs of the units (or more properly of their residue classes) in τ_p.

Next we split F into two disjoint sets F_1, F_2 as follows:

• If $\tau_1 = \tau_2$, pick generators $\sigma_0, \dots, \sigma_k$ of F, and choose as F_1 the fan generated by $\sigma_0, \dots, \sigma_{k-1}$, and $F_2 = F \setminus F_1$. Note that $\#(F_1) = 2^{k-1} = \frac{1}{2}\#(F)$.

• If $\tau_1 \neq \tau_2$, take as F_1 the fan F_{τ_1} consisting of all orderings of F specializing to τ_1 and $F_2 = F_{\tau_2}$. By the Baer-Krull theorem (Theorem 2.1) there are as many orderings specializing to τ_1 as specializing to τ_2, so that $\#(F_1) = \#(F_2) = \frac{1}{2}\#(F) = 2^{k-1}$. As above we may assume that F_1 is generated by $\sigma_0, \dots, \sigma_{k-1}$, and that $\sigma_k \in F_2$, so that $\sigma_0 \dots, \sigma_{k-1}, \sigma_k$ generate the whole F.

CLAIM: After some additional blowing-ups, we find a regular local ring B dominating A, with the same residue field and a system of parameters $y_1, \dots, y_d$ of B such that all f_{ij}'s are normal crossings in B with respect to them and for all $i = 0, \dots, k$ it holds

$$\sigma_i(y_j) = \begin{cases} +1 & \text{for} \quad 1 \leq j \leq d-i \\ -1 & \text{if} \quad j = d-i+1 \end{cases}$$

(compare Table 1).

In fact, first, after changing x_j by $-x_j$ if necessary, we may assume that $\sigma_0(x_j) = +1$ for all j. Now, notice that since the functions f_{ij} separate the orderings of F_1 and all these orderings specialize to τ_1, two different $\sigma, \sigma' \in F_1$ cannot have the same sign at all the parameters $x_1, \dots, x_d$. In other words, the map

$$\varphi : \sigma_0 F_1 \longrightarrow \{+1, -1\}^m$$

defined by

$$\sigma_0\sigma \longmapsto (\sigma(x_1), \dots, \sigma(x_d)),$$

is a monomorphism of $\mathbf{F}_2$-vector spaces.

Thus, there is some j such that $\sigma_1(x_j) \neq \sigma_0(x_j)$. We reorder the parameters so that for $\sigma_1(x_l) = +1$ for $1 \leq l < r$ and $\sigma_1(x_l) = -1$ for $r \leq l \leq d$. Consider the extension

$$A^{(1)} = A[x_r/x_d, \dots, x_{d-1}/x_d]_{(x_1, \dots, x_{r-1}, x_r/x_d, \dots, x_{d-1}/x_d, x_d)}.$$

We set $x_j^{(1)} = x_j$ for $1 \leq j \leq r-1$, $x_j^{(1)} = x_j/x_d$ for $r \leq j \leq d-1$ and $x_d^{(1)} = x_d$. Then $A^{(1)}$ is a regular ring dominating A, the residue fields of both rings coincide and $x_1^{(1)}, \dots, x_{d-1}^{(1)}, x_d^{(1)}$ is regular system of parameters of $A^{(1)}$. Furthermore the expression

$$f_{ij} = u_{ij} x_1^{\alpha_{ij1}} \cdots x_d^{\alpha_{ijd}}$$

can also be written

$$f_{ij} = u_{ij}(x_1^{(1)})^{\alpha_{ij1}} \cdots (x_{d-1}^{(1)})^{\alpha_{ij(d-1)}} (x_d^{(1)})^{\alpha_{ijr} + \cdots + \alpha_{ijd}}.$$

This means that the f_{ij} are still normal crossings in $A^{(1)}$, so that all conditions verified by A are similarly verified by $A^{(1)}$. Moreover, we have $\sigma_1(x_l^{(1)}) = +1$ for $1 \leq l < d-1$ and $\sigma_1(x_d^{(1)}) = -1$, so that we have constructed the first step in the induction proccess. Assume now that we have already found a local regular ring $A^{(\ell)}$ dominating A with the same residue field that the latter and with a system of parameters $x_1^{(\ell)}, \dots, x_d^{(\ell)}$ such that the f_{ij} are normal crossings for them in $A^{(\ell)}$ and for all $0 \leq i \leq \ell$ it holds $\sigma_i(x_j^{(\ell)}) = +1$ for all $1 \leq j \leq d-i$ and $\sigma_i(x_{d-i+1}^{(\ell)})$. We construct $A^{(\ell+1)}$ as follows:

Consider $\sigma_{\ell+1}$. We claim that there is $j \leq d-\ell$ such that $\sigma_{\ell+1}(x_j) = -1$. For otherwise, a look at the Table 1 shows at once that $\sigma_0\sigma_{\ell+1}$ would be in the subspace generated by $\sigma_0\sigma_1, \ldots, \sigma_0\sigma_\ell$, against our assumption that $\sigma_0, \ldots \sigma_k$ were affine independent. Then after reordering $x_1^{(\ell)}, \ldots, x_{d-\ell}^{(\ell)}$ we may assume that $\sigma_\ell(x_j) = +1$ for $1 \leq j < r$ and $\sigma_1(x_j) = -1$ for $r \leq j \leq d-\ell$. Consider the extension

$$A^{(\ell+1)} = A^{(\ell)}[x_r^{(\ell)}/x_d^{(\ell)}, \ldots, x_{d-\ell-1}^{(\ell)}/x_{d-\ell}^{(\ell)}]_{(x_1^{(\ell)}, \ldots, x_{r-1}^{(\ell)}, x_r^{(\ell)}/x_{d-\ell}^{(\ell)}, \ldots, x_{d-\ell-1}^{(\ell)}/x_d^{(\ell)}, x_{d-\ell}^{(\ell)}, \ldots, x_d^{(\ell)})},$$

and set $x_j^{(\ell+1)} = x_j^{(\ell)}$ for $1 \leq j \leq r-1$, $x_j^{(\ell+1)} = x_j^{(\ell)}/x_{d-\ell}$ for $r \leq j \leq d-\ell-1$ and $x_j^{(\ell+1)} = x_j^{(\ell)}$ for $d-\ell \leq j \leq d$. Then an immediate computation shows that for $0 \leq i \leq \ell+1$ it holds $\sigma_i(x_j^{(\ell+1)}) = +1$ for all $1 \leq j \leq d-i$ and $\sigma_i(x_{d-i+1}^{(\ell+1)})$, so that we have done the step $\ell+1$ and therefore the claim is complete.

Once this is done, consider any $\sigma \in F$. There are two possibilities:

- $\sigma \in F_1$. Then $\sigma = \sigma_{i_1} \cdots \sigma_{i_s}$ with $0 \leq i_1 < \ldots < i_s \leq k-1$ and necessarily s is odd. Let $1 \leq l \leq d-k+1$; since $\sigma_{i_1}(y_l) = \cdots = \sigma_{i_s}(y_l) = +1$ we get $\sigma(y_l) = +1$.
- $\sigma \in F_2$. Then $\sigma = \sigma_{i_1} \cdots \sigma_{i_s} \cdot \sigma_k$ with $0 \leq i_1 < \ldots < i_s \leq k-1$ and necessarily s is even. Let $1 \leq l \leq d-k+1$; we get $\sigma(y_l) = \sigma_{i_1}(y_l) \cdots \sigma_{i_s}(y_l) \cdot \sigma_k(y_l) = \sigma_k(y_l)$.

In conclusion, for all $\sigma \in F_1$ we have $\sigma(y_j) = +1$ for $1 \leq j \leq d-k+1$, while for all $\sigma \in F_2$ we have $\sigma(y_j) = \sigma_k(y_j)$. This implies that we have two bijections $\varphi_p : F_p \to \{-1, +1\}^{k-1}$ given by $\sigma \mapsto (\sigma(y_{d-k+2}), \ldots, \sigma(y_d))$, $p = 1, 2$. In fact, since the functions f_{ij} separate the orderings of each F_p, the argument above shows that φ_p is injective, and since all sets involved have 2^{k-1} elements, the mappings are bijective.

Now we consider the following diagram

$$\begin{array}{ccc} B & \longrightarrow & C = B_{(y_{d-k+2}, \ldots, y_d)} \subset K \\ \downarrow & & \downarrow \\ B/(y_{d-k+2}, \ldots, y_d) & \longrightarrow & k_C \\ \downarrow & & \\ k_A = k_B & & \end{array}$$

where k_B, k_C stand for the residue fields of B, C respectively. By construction, these two residue fields are finitely generated over $\mathbf{R}$. Now let k'_B be a quasicoefficient field of B, that is a subfield $k'_B \subset B$, such that the extension $k'_B \subset k_B$ induced by the canonical homomorphism $B \to k_B$ is algebraic (even finite in our case). Then, since k_C is the quotient field of the ring $B/(y_{d-k+2}, \ldots, y_d)$, which is local regular of dimension $d-k+1$ and $k_B = k_A = \kappa(\mathfrak{p})$, we get

$$\begin{gathered} \text{tr.deg.}[k_C : \mathbf{R}] = \text{tr.deg.}[k_C : k'_B] + \text{tr.deg.}[k'_B : \mathbf{R}] \geq \\ (d-k+1) + \text{tr.deg.}[\kappa(\mathfrak{p}) : \mathbf{R}] = (d-k+1) + \dim(\mathcal{P}(X)/\mathfrak{p}) = \\ (d-k+1) + \dim(\mathcal{P}(X)) - \text{ht}(\mathfrak{p}) = (d-k+1) + \dim(K) - \dim(B) = n-k+1. \end{gathered}$$

Now, we chase orderings through the diagram, starting in $k_B = \kappa(\mathfrak{p})$ with our τ_1, τ_2:

- Since $B/(y_{d-k+2},\dots,y_d)$ is local regular with parameters $y_1,\dots,y_{d-k+1}$, we can lift τ_1 to an ordering γ_1 of k_C such that $\gamma_1(x_l)=+1$ for $1\le l\le d-k+1$ (Example 2.2). Also we can lift τ_2 to an ordering γ_2 of k_C such that $\gamma_2(x_l)=\sigma_k(x_l)$ for $1\le l\le d-k+1$.

- Since C is local regular with parameters $y_{d-k+1},\dots,y_d$, we can built up an algebroid fan F' of K parametrized over the two orderings γ_1,γ_2 in k_C (Example 2.6). Let F'_p be the set of orderings of F' specializing to γ_p, for $p=1,2$. Now we also have two bijections $\varphi'_p : F'_p \to \{-1,+1\}^{k-1} : \sigma' \mapsto (\sigma'(y_{d-k+2}),\dots,\sigma'(y_d))$, $p=1,2$, and consequently we obtain bijections $F_p\to F'_p : \sigma\mapsto\sigma'$ such that:

  (a) $\sigma(y_l)=\sigma'(y_l)$ for $d-k+1<l\le d$,

  (b) If $p=1$, then for $\sigma\in F_1$ we have $\sigma(y_l)=+1=\gamma_1(y_l)=\sigma'(y_l)$ for $1\le l\le d-k$. If $p=2$, then for $\sigma\in F_2$ we have $\sigma(y_l)=\sigma_k(y_l)=\gamma_1(y_l)=\sigma'(y_l)$ for $1\le l\le d-k+1$.

  (c) $\sigma,\sigma'\to\gamma_p$.

  This gives another bijection $\psi : F\to F' : \sigma\mapsto\sigma'$, such that $\sigma(y_l)=\sigma'(y_l)$ for $1\le l\le d$ and $\sigma(u)=\sigma'(u)$ for any unit $u\in B$. Consequently, $\sigma(f_{ij})=\sigma'(f_{ij})$ for all i,j, and since the f_{ij}'s define the neighborhood $U\subset\Phi_k$ of F fixed at the beginning, we conclude $F'\in U$, which completes the proof.

□

We finish this section by pointing out that the restriction to compact models in the last theorem is essencial, as Example 4.5 will show.

4 Review on real spectra

In order to progress further we need the theory of the real spectrum. Here we just review the most basic facts, relying on [BCR] as general reference.

Let A be any commutative ring with unit. The real spectrum $Spec_r(A)$ of A is the set of all pairs $\alpha=(\mathfrak{p}_\alpha,\le_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of A and $\le_\alpha$ is an ordering in the residue field $\kappa(\mathfrak{p}_\alpha)$. We denote by $\kappa(\alpha)$ the real closure of $\kappa(\mathfrak{p}_\alpha)$ with respect to $\le_\alpha$, and can view α as a homomorphism $\alpha : A\to A/\mathfrak{p}_\alpha\subset\kappa(\mathfrak{p}_\alpha)\subset\kappa(\alpha) : f\mapsto f(\alpha)$. Now let $\alpha,\beta\in Spec_r(A)$. We say that α *specializes to* β or that β is *a specialization of* α and write $\alpha\to\beta$ if $f(\beta)>0$ implies $f(\alpha)>0$ for $f\in A$; more algebraically, $\alpha\to\beta$ if and only if $\mathfrak{p}_\alpha\subset\mathfrak{p}_\beta$ and the canonical map $A/\mathfrak{p}_\alpha\to A/\mathfrak{p}_\beta$ sends elements $\ge_\alpha 0$ to elements $\ge_\beta 0$. Of course, this is the same specialization introduced earlier in Section 2.

In the setting of the real spectrum we can impose sign conditions on the elements of A and use notations like $\{f_1>0,\dots,f_s>0\}\subset Spec_r(A)$ for $\{\alpha\in Spec_r(A) : f_1(\alpha)>$

$0, \ldots, f_s(\alpha) > 0\}$. Then we define in the obvious way *constructible sets*

$$C = \bigcup_{i=1}^{p} \{f_{i1} > 0, ..., f_{ir_i} > 0, g_i = 0\},$$

basic open sets

$$C = \{f_{i1} > 0, ..., f_{ir_i} > 0\},$$

and *the Harrison topology* generated by these basic open sets. In terms of this topology, the specialization relation introduced above behaves as a limit. For instance, if C is open, $\beta \in C$ and $\alpha \to \beta$, then $\alpha \in C$.

We also define *the Zariski topology* by analogy with the Zariski prime spectrum: a subbasis consists of all sets of the form $\{f \neq 0\}$; we distinguish the operations in this topology with an index Z.

If A is a field we find again the space of orderings described in Section 1. It is clear from the definitions that

$$Spec_r(A) = \bigcup_{\mathfrak{p}} Spec_r(\kappa(\mathfrak{p})),$$

where the $\mathfrak{p}$'s run among the prime ideals of A. This simple remark supports the idea of *patching* the informations obtained from the residue fields of A to learn about A itself. Actually, it works to prove:

Theorem 4.1 *Let A be a commutative ring with unit and C an open constructible subset of $Spec_r(A)$ such that $S \cup \overline{(\overline{C} \setminus C)}^Z = \emptyset$. Let s be a positive integer. Suppose that for every prime ideal $\mathfrak{p}$ of A there are $g_1, \ldots, g_s \in A$ such that $C \cap Spec_r(\kappa(\mathfrak{p})) = \{g_1 > 0, \ldots, g_s > 0\} \cap Spec_r(\kappa(\mathfrak{p}))$. Then there are $f_1, \ldots, f_s \in A$ such that $C = \{f_1 > 0, \ldots, f_s > 0\}$.*

This theorem has a long history. It was first obtained by Bröcker, [Br1], in case A was an algebra finitely generated over a real closed field R, but he could not control completely the number of equations involved. This was solved by Scheiderer in [Sch], who already remarked that the argument worked for any excellent ring A. At the same time Bröcker found a proof that only required A to be noetherian, [Br3]. Finally, Marshall discovered how to modify all those proofs to obtain the result for arbitrary A, [Mr6].

Now let $A = \mathcal{P}(X)$ be the ring of polynomial functions of a real algebraic set $X \subset \mathbf{R}^n$. Then we define the tilda operation $S \mapsto \tilde{S}$. It maps a semialgebraic set $S \subset X$ to the constructible set $\tilde{S} \subset Spec_r(A)$ defined by any formula that also defines S. This definition is consistent and gives a bijection that preserves inclusions and topological operations (by the Tarski principle). This tilda operator is main tool to translate semialgebraic problems and statements in terms of the real spectra.

Finally, suppose that X is irreducible and let $K = \mathcal{K}(X)$. Then $Spec_r(K) \subset Spec_r(A)$, and the tilda operation induces a mapping $S \mapsto \tilde{S} \cap Spec_r(K)$, which is *generically* injective: if $S, T \subset X$ are semialgebraic sets such that $\tilde{S} \cap Spec_r(K) = \tilde{T} \cap Spec_r(K)$, then $S \setminus Z = T \setminus Z$ for some nowhere dense algebraic set $Z \subset X$. In this way we can mix the *geometric* and the *algebraic* settings to study our problem. For all of this we refer to [BCR], [Br4], [AnBrRz1,2]. For instance, Theorem 1 of the introduction is just a translation of Theorem 4.1. We also use this strategy to deduce directly from Theorem 1.2 the following statement:

Corollary 4.2 *Let S be a semialgebraic subset of an irreducible real algebraic set $X \subset \mathbf{R}^n$. Then the following assertions are equivalent:*

(a) S is generically basic.

(b) For every fan F of the field $\mathcal{K}(X)$ with $\#(F) = 4$ we have $\#(F \cap \tilde{S}) \neq 3$.

This was our starting point in [AnRz1] to prove Theorem 3 of the introduction. Here we will work similarly to prove Theorem 4, using the following consequence of Corollary 1.3:

Corollary 4.3 *Let S be a generically basic semialgebraic subset of an irreducible real algebraic set $X \subset \mathbf{R}^n$. Then the following assertions are equivalent:*

(a) S is generically s-basic.

(b) For every fan F of the field $\mathcal{K}(X)$ with $\#(F) = 2^k$ and $\#(F \cap \tilde{S}) = 1$ we have $k \leq s$.

The next step towards the proof of Theorem 4 of the introduction is:

Proposition 4.4 *Let S be a generically basic semialgebraic subset of a compact irreducible real algebraic set $X \subset \mathbf{R}^n$ of dimension d. Then the following assertions are equivalent:*

(a) S is generically s-basic.

(b) For every algebroid fan F of the field $\mathcal{K}(X)$ finite over $\mathcal{P}(X)$ and parametrized over a function field of dimension $d-k+1$ such that $\#(F) = 2^k$ and $\#(F \cap \tilde{S}) = 1$ we have $k \leq s$.

Proof: We only have to prove (b) $\Rightarrow$ (a). So, suppose S is not generically s-basic. By Corollary 4.3 there is a fan F of the field $\mathcal{K}(X)$ with $\#(F) = 2^k$ and $\#(F \cap \tilde{S}) = 1$, but $k > s$. Now let $f_1, \ldots, f_r$ be the functions appearing in a description of S. For every $\sigma \in F$ we put $\varepsilon_{\sigma i} = \sigma(f_i), 1 \leq i \leq r$, and $U_\sigma = \{\varepsilon_{\sigma 1} f_1 > 0, \ldots, \varepsilon_{\sigma r} f_r > 0\}$. Then $U = \prod_{\sigma \in F} U_\sigma$ is a neighborhood of F and by Theorem 3.1 there is an algebroid fan $F' \in U$ finite over $\mathcal{P}(X)$ and parametrized over a function field of dimension $d-k+1$. It is obvious from our definition of U that $\#(F' \cap \tilde{S}) = \#(F \cap \tilde{S}) = 1$. Since $k > s$, we are done. $\square$

Example 4.5. We construct a semialgebraic set $S \subset \mathbf{R}^2$ which is not generically basic, but the obstruction can only be read through fans compatible with valuations of $\mathcal{K}(\mathbf{R}^2) = \mathbf{R}(x, y)$ that are not finite on $\mathcal{P}(\mathbf{R}^2) = \mathbf{R}[x, y]$. Consequently those fans cannot be approximated by others finite on $\mathbf{R}[x, y]$.

To define S consider the sets (*Figure 1*)

$$S_1 = \{xy \geq 1\},\ S_1^+ = S_1 \cap \{x \geq 0\},\ S_1^- = S_1 \cap \{x \leq 0\},$$
$$S_2 = \{xy \leq -1\},\ S_2^+ = S_2 \cap \{x \geq 0\},\ S_2^- = S_2 \cap \{x \leq 0\}.$$

and put $S = S_1 \cup S_2^+$. We will denote by $\tilde{S}_1$ the constructible subset of the space of orderings of $\mathcal{K}(\mathbf{R}^2)$ defined by the same equations as S_1, and similarly $\tilde{S}_1^+, \tilde{S}_1^-$, etc.

Now to prove our previous claim, let $F = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a fan with $\#(F \cap \tilde{S}) = 3$. Since S_1 is generically 1-basic, $\#(F \cap \tilde{S}_1) = 0$ or 2; analogously, $\#(F \cap \tilde{S}_2) = 0$ or 2. Since S_2^+ is generically basic, $\#(F \cap \tilde{S}_2^+) = 0, 1$ or 2. Hence $\#(F \cap \tilde{S}_1) = 2$ and $\#(F \cap \tilde{S}_2^+) = \#(F \cap \tilde{S}_2^-) = 1$, say $\sigma_1, \sigma_3 \in \tilde{S}_1$, $\sigma_2 \in \tilde{S}_2^+$ and $\sigma_4 \in \tilde{S}_2^-$. Suppose now that there is a valuation V of $\mathbf{R}(x, y)$ compatible with F such that $\mathbf{R}[x, y] \subset V$. Then the maximal ideal $\mathfrak{m}_V$ of V lies over a real prime ideal $\mathfrak{p}$ of $\mathbf{R}[x, y]$, and the σ_i's make $\mathfrak{p}$ convex and specialize to at most two orderings τ_1, τ_2 in the residue field $\kappa(\mathfrak{p})$. Now we will argue using the real spectrum of the ring $\mathbf{R}[x, y]$. We distinguish two possible cases:

- $\mathrm{ht}(\mathfrak{p}) = 2$. Then $\mathfrak{p}$ is a maximal ideal, that is, the ideal of a point $z \in \mathbf{R}^2$. If $z \notin S_j$, since S_j is closed, no σ_i would be in S_j. Hence $z \in S_1 \cap S_2 = \emptyset$, which is absurd. Thus this case is impossible.

- $\mathrm{ht}(\mathfrak{p}) = 1$. Then $\mathfrak{p}$ is the ideal of an irreducible curve Z. Then suppose that, say, $\sigma_4 \to \tau_2$. Then τ_2 is not an inner point of $\tilde{S}$, for otherwise, since the interior of $\tilde{S}$ is an open constructible set, the generization σ_4 would belong to S too. Also, we have $\sigma_i \to \tau_2$ for some other σ_i, say σ_1. Since $\sigma_1 \in \tilde{S}_1$ and $\tilde{S}_1$ is closed, it follows that $\tau_2 \in \tilde{S}_1$. Altogether we have that τ_2 belongs to the boundary $\partial \tilde{S}_1$ of $\tilde{S}_1$, which by construction is the hyperbola $xy - 1 = 0$. This means that $xy - 1 \in \mathfrak{p}$, or equivalently that $Z \subset \{xy - 1 = 0\}$. Now we have $\sigma_2 \to \tau_1$, and arguing as above we get $\tau_1 \in \partial \tilde{S}_2$, or equivalently $Z \subset \{xy + 1 = 0\}$. Since the two hyperbolas are disjoint we get a contradiction. The rest of the cases are treated similarly.
In conclusion there is not such a $\mathfrak{p}$, what shows that there is not a valuation V compatible with F and finite over $\mathbf{R}[[x, y]]$.

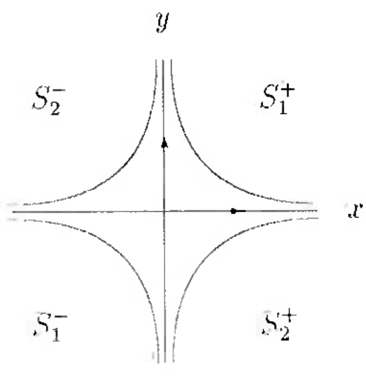

Figure 1

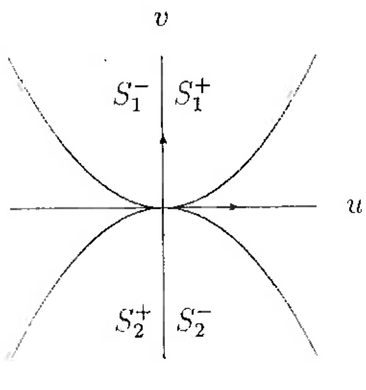

Figure 2

Finally we have to prove that F does exist if we do not require the finiteness condition. To do this we work in the projective plane with coordinates $(x_0 : x_1 : x_2)$.

where $x = x_1/x_0, y = x_2/x_0$. Actually, we work at the point $(0:0:1)$, or better in the affine chart $x_2 \neq 0$. We put $u = x_0/x_2, v = x_1/x_2$ and our sets are given (birationally) by the sign conditions that follow:

$$S_1 = \{v \geq u^2\},\ S_1^+ = S_1 \cap \{u \geq 0\},\ S_1^- = S_1 \cap \{u \leq 0\},$$
$$S_2 = \{v \leq -u^2\},\ S_2^+ = S_2 \cap \{u \leq 0\},\ S_2^- = S_2 \cap \{u \geq 0\},$$

and of course $S = S_1 \cup S_2^+$ (*Figure 2*). Now we obtain the fan F starting with a valuation compatible with it. Namely, the discrete rank 1 valuation $\mathbf{R}[u,v]_{(u)}$, whose residue field $\mathbf{R}(v)$ has two orderings: τ_1, with v positive and infinitesimal with respect to $\mathbf{R}$, and τ_2, with v negative and infinitesimal with respect to $\mathbf{R}$. Then F will consist of the four liftings σ_1, σ_3 and σ_2, σ_4 of τ_1 and τ_2 defined by $\sigma_1(t) = \sigma_4(t) = +1, \sigma_3(t) = \sigma_2(t) = -1$. Clearly $\sigma_1, \sigma_2, \sigma_3 \in \tilde{S}$ and $\sigma_4 \notin \tilde{S}$. Furthermore, there are two valuations compatible with F. One is $\mathbf{R}[u,v]_{(u)}$, and the other is the composite of this with $\mathbf{R}[v]_{(v)}$. The centers of this valuations are, respectively, the line $u = 0$ and the point $u = v = 0$. In projective coordinates they are the line $x_0 = 0$ and the point $(0:0:1)$, in both cases infinite when we work in the affine (x,y)-plane.

5 Proof of the main result

Theorem 4 of the introduction will be an immediate consequence of the following:

Theorem 5.1 *Let S be a generically basic semialgebraic subset of an irreducible real algebraic set $X \subset \mathbf{R}^n$. Let $Z \subset X$ be any proper algebraic subset containing the singular locus of X and the boundary of S, $\partial S = \overline{S} \setminus S^\circ$. Then the following assertions are equivalent:*
(a) S is generically s-basic.
(b) For any irreducible algebraic set $Y \subset X$ of dimension $s+1$, not contained in Z, the intersection $S \cap Y$ is generically s-basic.

Proof: Assume first that S is generically s-basic, and let $Y \subset X$ be an irreducible subset not contained in Z. Denote by $\mathfrak{p} \subset \mathcal{P}(X)$ the ideal of Y. Since Y is not contained in the singular locus of X the localization $\mathcal{P}(X)_\mathfrak{p}$ is a local regular ring of dimension say d whose residue field is $L = \mathrm{qf}(\mathcal{P}(X)/\mathfrak{p}) = \mathcal{K}(Y)$. Now suppose that $S \cap Y$ is not generically s-basic. Then by Corollary 1.3 there is a fan $F = (\sigma_i : 1 \leq i \leq 2^k)$ of L such that $\#(F \cap \tilde{S}) = 1$ and $k > s$. This F lifts to a fan $F' = (\sigma_i' : 1 \leq i \leq 2^k)$ of $\mathcal{K}(X)$ with $\sigma_i' \to \sigma_i$. Indeed, as we explained in Example 2.2, using a regular system of parameters $x_1, \ldots, x_d$ of $\mathcal{P}(X)_\mathfrak{p}$ we can lift any ordering σ of L to 2^d different orderings of K, each corresponding to a choice of signs for the given parameters. Hence we fix all parameters positive and lift every σ_i to σ_i'. It is very easy to check that the σ_i''s form a fan and of course $\#(F') = 2^k$. We claim that $\#(F' \cap \tilde{S}) = 1$. Indeed, suppose $\sigma_i \in \tilde{S}$. Since Z does not contain Y we have $\sigma_i \notin \tilde{Z}$, and therefore $\sigma_i \notin \widetilde{\partial S}$, because $\partial S \subset Z$. Thus we have $\sigma \in \tilde{S}^\circ$, which is constructible and open. Since $\sigma_i' \to \sigma_i$, we get $\sigma' \in \tilde{S}^\circ \subset \tilde{S}$. On the other hand, suppose $\sigma_i \notin \tilde{S}$. Again we have $\sigma_i \notin \tilde{Z}$. Then $\sigma_i \notin \tilde{Z} \cup \tilde{S}$, and we get $\sigma_i \notin \tilde{\tilde{S}}$, which is constructible and closed. It follows that $\sigma_i' \notin S$ either. Whence, $\#(F \cap \tilde{S}) = 1$ and S is not generically s-basic, as claimed. Note that for this implication we do not need any special type of fan.

In order to prove the converse implication, we can substitute X by its one-point compactification or, in other words, assume that X is compact. Now suppose S is not generically s-basic. By Proposition 4.4 there is an algebroid fan F of the field $K = \mathcal{K}(X)$ finite on $\mathcal{P}(X)$ and parametrized over a function field L of dimension $d-k+1$ such that $\#(F) = 2^k$, $\#(F \cap \tilde{S}) = 1$ and $k > s$. This means that F is defined through an embedding $\phi : K \hookrightarrow L((x_1, \ldots, x_{k-1}))$ and two orderings γ_1, γ_2 in L, and that the ring $\mathcal{P}(X)$ of polynomial functions of X is contained in the ring $L[[x_1, \ldots, x_{k-1}]]$ via ϕ. Now, since L is a function field there is an irreducible algebraic set $W \subset \mathbf{R}^m$ whose field of rational functions $\mathcal{K}(W)$ is L, that is, L is the quotient field of the ring $\mathcal{P}(W)$ of polynomial functions on W.
Now let H stand for a generic hyperplane section of W and $\mathfrak{p}$ for the ideal of H in $\mathcal{P}(W)$. By Bertini's theorem ([Jn], [BCR]), H is a nonsingular irreducible subset of W, and $\mathfrak{p}$ a real prime ideal. Note that the field $\mathcal{K}(H)$ of rational functions of H is the residue field of $\mathfrak{p}$, that is, the quotient field of $\mathcal{P}(W)/\mathfrak{p}$. With all of this we have the following diagram

$$\begin{array}{ccc} \mathcal{P}(X) \subset V \xrightarrow{\phi} \mathcal{K}(W)[[x_1, \ldots, x_{k-1}]] & & \\ \cup & & \\ \mathcal{P}(W)_{\mathfrak{p}}[[x_1, \ldots, x_{k-1}]] & \xrightarrow{\varphi} & \mathcal{K}(H)[[x_1, \ldots, x_{k-1}]] \end{array}$$

where the homomorphism φ is the obvious extension of the canonical mapping $\mathcal{P}(W)_{\mathfrak{p}} \to \mathcal{K}(H)$.
Since the ring $\mathcal{P}(X)$ is an algebra finitely generated over $\mathbf{R}$ we can pick finitely many generators $f_1, \ldots, f_q$ in $\mathcal{P}(X)$; we add to these the equations, say $f_{q+1}, \ldots, f_s$, involved in a description of the semialgebraic set S and an equation of Z. All these functions f_i are in $\mathcal{K}(W)[[x_1, \ldots, x_{k-1}]]$, and so they have power expansions $f_i = f_i(x) = \sum_\nu (g_{i\nu}/h_{i\nu}) x^\nu$, where $\nu \in \mathbf{N}^{k-1}$ and $g_{i\nu}, h_{i\nu} \in \mathcal{P}(W)$. As our hyperplane section H is generic, we can suppose no $g_{i\nu}, h_{i\nu}$ vanishes on H (although there are infinitely many $g_{i\nu}, h_{i\nu}$'s, their number is countable, and working over the reals we can use Baire's theorem). In particular $h_{i\nu} \notin \mathfrak{p}$ implies that the $f_i(x)$'s are well defined elements of $\mathcal{P}(W)_{\mathfrak{p}}[[x_1, \ldots, x_{k-1}]]$. Finally, since the f_i's generate $\mathcal{P}(X)$ we get $\phi(\mathcal{P}(X)) \subset \mathcal{P}(W)_{\mathfrak{p}}[[x_1, \ldots, x_{k-1}]]$ and consequently we have the map

$$\psi = \varphi\phi : \mathcal{P}(X) \to \mathcal{K}(H)[[x_1, \ldots, x_{k-1}]].$$

Moreover $g_{i\nu} \notin \mathfrak{p}$ implies that the coefficients of the $f_i(x)$'s are units in $\mathcal{P}(W)_{\mathfrak{p}}$ and so $\psi(f_i) = \varphi(f_i(x)) = \sum_l (\overline{g_{i\nu}/h_{i\nu}}) x^\nu$ is a non-zero element of $\mathcal{K}(H)[[x_1, \ldots, x_{k-1}]]$ (here $\overline{}$ stands for the residue class mod $\mathfrak{p}$).
Now we complete the choice of the generic hyperplane section H. To do it, we set $F = F_1 \cup F_2$, where F_p contains the orderings of F that specialize to γ_p, $p = 1, 2$. Then every ordering $\sigma \in F_p$ is determined by a sign condition $\varepsilon : \{x_1, \ldots, x_{k-1}\} \to \{+1, -1\}$. Also we know from Example 2.4 *a)* that the sign of f_i in any such orderinf is completely determined by its initial form (with respect to the lexicographic

ordering in the exponents), say $g_{i\nu_{0i}}/h_{i\nu_{0i}}x^{\nu_{0i}}$. Let $\tilde{G}_p \subset Spec_r(\mathcal{K}(W))$ be the open neighborhood of γ_p defined by $\{g_{1\nu_{01}}/h_{1\nu_{01}} > 0, \ldots, g_{s\nu_{0s}}/h_{s\nu_{0s}} > 0\}$. Then we have that for any ordering $\gamma'_p \in G_p$ its lifting σ' corresponding to the sign condition ε has at the f_i's the same signs that σ. This implies that for any two orderings $\gamma'_1 \in \tilde{G}_1$ and $\gamma'_2 \in \tilde{G}_2$ the fan F' parametrized over them verifies also $\#(F' \cap \tilde{S}) = 1$ and $k > s$ (cf. Examples 2.2 and 2.6).

Now, we denote by G_1, G_2 the two open semialgebraic subsets of W corresponding to the neigborhoods just constructed. These semialgebraic sets are Zariski dense in W, which guarantees that we can choose the generic hyperplane section H to meet both of them. This implies that there are $\gamma'_1 \in G_1$ and $\gamma'_2 \in G_2$ which make the ideal $\mathfrak{p}$ of H convex. In other words, γ'_1 and γ'_2 induce two orderings τ_1 and τ_2 in the residue field of $\mathfrak{p}$, which is $\mathcal{K}(H)$. Then we parametrize over τ_1 and τ_2 a fan F'' of $\mathcal{K}(H)[[x_1, \ldots, x_{k-1}]]$. We have bijections $F \to F' \to F'' : \sigma \mapsto \sigma' \mapsto \sigma''$ such that σ, σ' and σ'' are all defined by the same sign condition $\varepsilon : \{x_1, \ldots, x_{k-1}\} \to \{+1, -1\}$.

After this preparation, we have the following diagram

$$\begin{array}{ccc} & \mathcal{P}(X) & \\ & {}^{\varphi}\swarrow \quad \searrow^{\psi} & \\ \mathcal{P}(W)_{\mathfrak{p}}[[x_1, \ldots, x_{k-1}]] & \longrightarrow & \mathcal{K}(H)[[x_1, \ldots, x_{k-1}]] \\ \downarrow & & \downarrow \\ \mathcal{P}(W)_{\mathfrak{p}} & \longrightarrow & \mathcal{K}(H) \end{array} \qquad \begin{array}{ccc} \sigma' & \longrightarrow & \sigma'' \\ \downarrow & & \downarrow \\ \gamma'_p & \longrightarrow & \tau_p \end{array}$$

where σ' and σ'' are defined by the same sign condition ε_σ, as explained above.

Now consider the kernel $\mathfrak{q}$ of the homomorphism ψ. Its zero set is an algebraic set $Y \subset X$ with $\mathcal{P}(Y) = \mathcal{P}(X)/\mathfrak{q}$, and $\dim(Y) = \dim(X) - \mathrm{ht}(\mathfrak{q})$. Furthermore, the fan F'' consisting of the σ'''s restricts to a fan F^* in $\mathcal{K}(Y)$ such that $\#(F^* \cap \tilde{S}) = 1$, because by construction the signs of σ'' at the $\psi(f_j)$'s coincide with those of σ'. Consequently, the semialgebraic set $S \cap Y$ is not generically s-basic. Furthermore, since among the f_i's there is an equation of Z, and no $\psi(f_i)$ is zero, Y is not contained in Z. Hence it only remains to show that we can impose the further condition $\dim(Y) < \dim(X)$ and from that the proof will end by induction.

For that we will approximate ψ algebraically. Roughly speaking notice that since $f_1, \ldots, f_s$ generate $\mathcal{P}(X)$, the homomorphism ψ is completely determined by the images $\psi(f_1), \ldots, \psi(f_s)$. Let $a_1(x), \ldots, a_r(x) \in \kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]$ be series such that $a_i(x) \equiv \psi(f_i) \pmod{\mathfrak{m}^n}$ for a suitable $n \in \mathbf{N}$ and suppose that we may define a homomorphism $\psi' : \mathcal{P}(X) \to \kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]$ by $\psi'(f_i) = a_i(x)$. Then, if n is large enough so that the initial forms of the $\psi(f_i)$'s coincide with the initial forms of the $\psi'(f_i)$'s, since these initial forms determine the signs of these elements, it follows readily that the fan F^* induced in $\mathcal{P}(X)/\mathfrak{q}$ by the fan F'', verifies that $\#(F^* \cap \tilde{S}) = 1$. In other words the aproximation ψ' of ψ gives rise to a subvariety Y' in which $S \cap Y'$ is not basic. We will see that this ψ' can be constructed so that the a_i's are algebraic series, what will imply that $\dim(Y') < \dim(X)$.

To do that let $\mathfrak{n} = \psi^{-1}(x_1, \ldots, x_{k-1})$ and consider the localization $A = \mathcal{P}(X)_{\mathfrak{n}}$. The homomorphism ψ extends to the henselization A^h. Now, $\mathcal{P}(X)$ is a quotient of a polynomial ring, say $\mathcal{P}(X) = \mathbf{R}[T_1, \ldots, T_m]/\mathfrak{J}$. We denote by $\mathfrak{N}$ the ideal of $\mathbf{R}[T_1, \ldots, T_m]$ corresponding to $\mathfrak{n}$, and assume that $Z_1, \ldots, Z_r$ is a minimal system of generators of $\mathfrak{N}$. It follows that $A^h = (\mathbf{R}[T_1, \ldots, T_m]_{\mathfrak{N}})^h / \mathfrak{J}^h$. Thus, we have $(\mathbf{R}[T_1, \ldots, T_m]_{\mathfrak{N}})^h = \kappa(\mathfrak{n})[[Z_1, \ldots, Z_r]]_{\text{alg}}$, so that A^h is a quotient of an algebraic power series ring. In fact this can be seen as follows: By Noether's normalization Lemma, we may assume that $\mathbf{R}[T_1, \ldots, T_m]/\mathfrak{N}$ is finite over $\mathbf{R}[T_1, \ldots, T_{m-r}]$, so that the field $\kappa(\mathfrak{n})$ is an algebraic extension of $k = \mathbf{R}(T_1, \ldots, T_{n-r})$. Thus we have

$$k[Z_1, \ldots, Z_r] \subset \mathbf{R}[T_1, \ldots, T_m]_{\mathfrak{N}} \subset (\mathbf{R}[T_1, \ldots, T_m]_{\mathfrak{N}})^{\wedge} = \kappa(\mathfrak{n})[[Z_1, \ldots, Z_r]]$$

where the first extension is algebraic, and

$$k[Z_1, \ldots, Z_r] \subset \kappa(\mathfrak{n})[Z_1, \ldots, Z_m]_{(Z_1, \ldots, Z_r)} \subset \kappa(\mathfrak{n})[[Z_1, \ldots, Z_r]]$$

Thus using the characterization of the henselization as the algebraic closure of the ring in its completion, it follows that

$$(\mathbf{R}[T_1, \ldots, T_m]_{\mathfrak{N}})^h = \kappa(\mathfrak{n})[[Z_1, \ldots, Z_r]]_{\text{alg}}$$

as claimed.

Let $\bar{\psi} : \kappa(\mathfrak{n})[[Z_1, \ldots, Z_r]]_{\text{alg}} \to \kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]$ be the composition of ψ and the canonical epimorphism $\kappa(\mathfrak{n})[[Z_1, \ldots, Z_r]]_{\text{alg}} \to A^h$, and let $g_1, \ldots, g_t$ be a system of generators of the ideal $\mathfrak{J}^h$. We follow the method of [Tg, Chap. III, section 5, page 64]. Set $z_i(x) = \bar{\psi}(Z_i)$. Then we have

$$g_i(z(x)) = 0 \qquad \text{for all} \quad i = 1, \ldots, m.$$

By M. Artin's approximation theorem, cf. [BCR, Theorem 8.3.1, page 154], there a re $y_1(x), \ldots, y_m(x) \in \kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]$ arbitrarily close to $z_1(x), \ldots, z_m(x)$ in the $\mathfrak{m}$-adic topology of $\kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]$ such that

$$g_i(y(x)) = 0 \qquad \text{for all} \quad i = 1, \ldots, m.$$

This means that the homomorphism $\bar{\psi}' : \kappa(\mathfrak{n})[[Z_1, \ldots, Z_r]]_{\text{alg}} \to \kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]$ defined by $Z_i \mapsto y_i(x)$ factors through A^h. This way we can approximate arbitrarily ψ by $\psi' : A^h \to \kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]_{\text{alg}}$ as claimed. Hence, substituting ψ by ψ' we may suppose $\psi(\mathcal{P}(X)) \subset \kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]_{\text{alg}}$.

It follows that ψ induces an embedding $\mathcal{P}(Y) \hookrightarrow \kappa(\mathfrak{p})[[x_1, \ldots, x_{k-1}]]_{\text{alg}}$, which extends to the quotient fields $\mathcal{K}(Y) \hookrightarrow \kappa(\mathfrak{p})((x_1, \ldots, x_{k-1}))_{\text{alg}}$. Counting transcendence degrees over $\mathbf{R}$ we find

$$\begin{aligned} \dim(Y) &= \text{tr.deg.}[\mathcal{K}(Y) : \mathbf{R}] \leq \text{tr.deg.}[\kappa(\mathfrak{p})((x_1, \ldots, x_{k-1}))_{\text{alg}} : \mathbf{R}] = \\ &(k-1) + \text{tr.deg.}[\kappa(\mathfrak{p}) : \mathbf{R}] = (k-1) + \dim(H) < (k-1) + \dim(W) = \dim(X) \end{aligned}$$

as wanted. □

We finish the paper with the following:

Proof of Theorem 4: It is clear that if S is s-basic any intersection $S \cap Y$ with an irreducible subset $Y \subset X$ is also s-basic, and so generically s-basic. Conversely, suppose S is not s-basic. By the Bröcker-Scheiderer criterion (Theorem 1) there is an irreducible subset $X' \subset X$ such that $S \cap X'$ is not generically s-basic. Then, by Theorem 5.1 there is an irreducible subset $Y \subset X'$ of dimension $s + 1$ such that $S \cap Y$ is not gen rically s-basic, and we are done.

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Measuring similarity of models*

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Abstract

We describe an approach, introduced in [HV90], to measuring similarities and differences between uncountable models. A central concept in this approach is the concept of a transfinite Ehrenfeucht-Fraïssé game together with the concept of approximating these games with trees. The study of such games is closely linked with infinitely deep languages, introduced in [HR76], a generalization of the usual infinitary languages $L_{\kappa\lambda}$. We give an overview of recent work on the Ehrenfeucht-Fraïssé game and on the infinitely deep languages.

1 Introduction

The so called *finite quantifier* languages $L_{\kappa\omega}$ and their fragments have given rise to a rich and interesting *definability theory*. This theory works particularly nicely on countable structures and in the case $\kappa = \omega_1$. The obvious generalisation, the *infinite quantifier* languages $L_{\kappa\lambda}$, have given rise to almost no interesting mathematics at all. In particular, the generalisation $L_{\omega_2\omega_1}$ of $L_{\omega_1\omega}$ has led to no general theory of models of cardinality ω_1.

Hintikka and Rantala introduced a different approach to generalizing $L_{\kappa\omega}$ [HR76]. They considered so called *constituents* of mathematical structures and were led to the following idea: Rather than allowing transfinite sequences of strings of existential quantifiers and transfinite sequences of universal quantifiers, one should allow transfinite sequences of quantifier and connective alternations. This leads to powerful logics which extend not only the infinitary languages $L_{\kappa\lambda}$ but also extensions of $L_{\kappa\lambda}$ by the usual game-quantifier.

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Karttunen realized that while it is essential that the new infinitary expressions of [HR76] have infinite descending sequences of subformulas, an important distinction is made, if no *uncountable* descending sequences of subformulas are allowed [Kar84]. This distinction is of the same nature as the distinction between a game-quantified sentence of $L_{\omega_1 G}$ and its approximations in $L_{\infty\omega}$.

Most of the work on the new infinitary languages has centered around the problem of distinguishing models with infinitary sentences. This problem can be formulated in terms of a transfinite Ehrenfeucht-Fraïssé game. In Section 2 of this paper we describe the relevant notions related to this game. A central concept in this approach to infinitary logic is the concept of a tree with no uncountable branches. These trees are used as measures of similarity of two structures. We find strong parallels between the role of such trees in the study of uncountable models and the role of ordinals in the study of countable models. Section 3 is devoted to a survey of the structure of such trees. Section 4 builds on the contention that the most fundamental mathematical properties of classes of models of cardinality ω_1 are really topological properties of ${}^{\omega_1}\omega_1$ viewed as a generalized Baire space. We survey the basics of descriptive set theory in the space ${}^{\omega_1}\omega_1$. Section 5 gives an account of the analysis of isomorphism-types of uncountable models using trees. Finally, in Section 6 we introduce the infinitely deep languages and survey their basic properties.

2 The Ehrenfeucht-Fraïssé-game

To see how the new powerful infinitary logics behave and help us study uncountable models, it is not necessary to introduce the languages themselves at all. We can go a long way by studying *Ehrenfeucht-Fraïssé*-games only. This is also in line with the approach of [HR76], since consituents are descriptions of positions in Ehrenfeucht-Fraïssé-games. The new feature, analogous to allowing transfinite sequences of quantifier alternations, is that we study Ehrenfeucht-Fraïssé-games of length $> \omega$. We use

$$EF_\alpha(\mathcal{A}, \mathcal{B})$$

to denote the Ehrenfeucht-Fraïssé-game of length α between $\mathcal{A}$ and $\mathcal{B}$, which we now define. There are two players, called $\exists$ and $\forall$. During a round of the game $\forall$ first picks an element of one of the models and then $\exists$ picks an element of the other model. Let a_i be the element of A and b_i the element of B picked during round i of the game. There are altogether α rounds. Finally, $\exists$ wins the game if the resulting mapping $a_i \mapsto b_i$ is a partial isomorphism and otherwise $\forall$ wins. We say that a player *wins* $EF_\alpha(\mathcal{A}, \mathcal{B})$ if he has a winning strategy in it.

A trivial but fundamental observation is:

Lemma 2.1 *If $\mathcal{A}$ and $\mathcal{B}$ have cardinality $\leq \kappa$, then*

1. $\exists$ *wins* $EF_\kappa(\mathcal{A},\mathcal{B})$ *if and only if* $\mathcal{A} \cong \mathcal{B}$.

2. $\forall$ *wins* $EF_\kappa(\mathcal{A},\mathcal{B})$ *if and only if* $\mathcal{A} \not\cong \mathcal{B}$.

One consequence of the above Lemma is that $EF_\kappa(\mathcal{A},\mathcal{B})$ is determined whenever $\mathcal{A}$ and $\mathcal{B}$ have cardinality $\leq \kappa$. For models of cardinality $> \kappa$ the game $EF_\kappa(\mathcal{A},\mathcal{B})$ need not be determined, as the following result shows:

Theorem 2.2 ([MSV]) *There are models* $\mathcal{A}$ *and* $\mathcal{B}$ *of cardinality* ω_3 *so that the game* $EF_{\omega_1}(\mathcal{A},\mathcal{B})$ *is non-determined. It is consistent relative to the consistency of a measurable cardinal, that* $EF_{\omega_1}(\mathcal{A},\mathcal{B})$ *is determined for all models of cardinality* $\leq \omega_2$. *It is consistent relative to the consistency of* ZFC, *that* $EF_{\omega_1}(\mathcal{A},\mathcal{B})$ *is non-determined for some models of cardinality* $\leq \omega_2$.

In the case $\kappa = \omega$ we have the notion of a *ranked* game. To see what this means, suppose τ is a winning strategy of $\forall$ in $EF_\omega(\mathcal{A},\mathcal{B})$. Every round of the game, $\forall$ playing τ, ends after a finite number of moves at the victory of $\forall$. So we can put an ordinal rank on the moves of $\forall$ and demand that the rank goes down on each move. In this way we get a rank on the triple $(\mathcal{A},\mathcal{B},\tau)$. The *Scott rank* of $\mathcal{A}$ is the smallest α such that if $\mathcal{B} \not\cong \mathcal{A}$ then for some winning strategy τ of $\forall$ in $EF_\omega(\mathcal{A},\mathcal{B})$, the rank of $(\mathcal{A},\mathcal{B},\tau)$ is at most α.

We shall now introduce a similar concept for $EF_\kappa(\mathcal{A},\mathcal{B})$. Of course we cannot use ordinals to rank the moves of $\forall$ since the rank may have to decrease transfinitely many times in succession. Instead we take an arbitrary winning strategy τ of $\forall$ and form the tree

$$S_{\mathcal{A},\mathcal{B},\tau}$$

of all possible sequences of successor length of moves of $\exists$ against τ so that $\exists$ has not yet lost the game. We get a tree with no branches of length κ and we use this tree itself as a rank for $(\mathcal{A},\mathcal{B},\tau)$.

Rather than taking first a winning strategy of $\forall$ and then the tree of all plays of $\exists$, we may also directly consider winning strategies of $\exists$ in *short* games ([Hyt87]). Let

$$K_{\mathcal{A},\mathcal{B}}$$

be the set of winning strategies of $\exists$ in the games $EF_\alpha(\mathcal{A},\mathcal{B})$, where $\alpha < \kappa$ is a successor ordinal. We order the strategies as follows. Suppose σ is a winning strategy of $\exists$ in $EF_\alpha(\mathcal{A},\mathcal{B})$ and τ is a winning strategy of $\exists$ in $EF_\beta(\mathcal{A},\mathcal{B})$. Then $\sigma \leq \tau$ if $\alpha \leq \beta$ and τ agrees with σ for the first α moves of $EF_\beta(\mathcal{A},\mathcal{B})$. This ordering makes $K_{\mathcal{A},\mathcal{B}}$ a tree. If this tree has a branch of length κ, then $\exists$ can follow the strategies on the branch and win $EF_\kappa(\mathcal{A},\mathcal{B})$.

Starting from the concept of Scott rank, we have introduced two different measures of similarity of structures. Before we can compare these two measures to each other and to other trees, we have to develop tools for comparing trees. The big difference in using (non-well-founded) trees to estimate structural differences, rather than ordinals is that the structure of ordinals is well-understood but the structure of trees is not. This explains why we have to investigate structural properties of the class of all trees before we can proceed in our study of the transfinite Ehrenfeucht-Fraïssé-game.

3 Structure of trees

A *tree* is a partially ordered set with a smallest element (*root*) in which the set of predecessors of every element is well-ordered by the partial ordering.

We can think of ordinals as *well-founded* trees, i.e., trees with no infinite branches. For example, we may identify an ordinal α with the tree B_α of sequences $(\alpha, \alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_n < ... < \alpha_1 < \alpha$ and the sequences are ordered by end-extension. It is easy to see that if we assign ordinals to nodes of B_α in such a way that extensions of nodes get smaller ordinals, then α is the smallest ordinal that can be assigned in this process to the root of B_α. In this way we can assign an ordinal $o(T)$ to any well-founded tree T. So there is a nice correspondence between ordinals and well-founded trees. On the other hand, we can think of an ordinal α as a one-branch (non-wellfounded, if $\alpha \geq \omega$) tree. We use α itself to denote this linear tree.

We order the family of all trees as follows: $T \leq T'$ if there is an order-preserving $f : T \to T'$ (i.e. $x < y$ implies $f(x) < f(y)$). Note that this f need not be one-one. The strict ordering $T < T'$ is defined to hold if $T \leq T'$ and $T' \not\leq T$. Finally, $T \equiv T'$ if $T \leq T'$ and $T' \leq T$. We use σT to denote the tree of all ascending chains from T. Kurepa observed that $T < \sigma T$. With the σ-operation we define a stronger ordering of trees: $T \ll T'$ iff $\sigma T \leq T'$. The following properties of these orderings are fairly easy to prove:

Lemma 3.3 ([HV90]) *1. $\sigma T \not\leq T$, i.e., if $T \ll T'$, then $T < T'$.*

2. $<$ and $\ll$ are transitive relations.

3. $T \ll \sigma T$ but there is no T' with $T \ll T' \ll \sigma T$

4. The relation $\ll$ is well-founded.

5. For well-founded trees both $T < T'$ and $T \ll T'$ are equivalent to $o(T) < o(T')$.

The reason for introducing the relation $\ll$ is that it comes up very naturally in applications. Also, proving $T \ll T'$ is a handy direct way of achieving $T' \not\leq T$.

The ordering of trees can be defined also in terms of a *comparison game* $G(T,T')$. There are two players $\exists$ and $\forall$. Player $\forall$ starts and moves an element of T'. Then player $\exists$ responds with an element of T. The game goes on, $\forall$ playing elements of T' and $\exists$ playing elements of T, both in a strictly ascending order. The first player unable to move loses.

Lemma 3.4 ([HV90]) *1. $T' \leq T$ if and only if $\exists$ wins $G(T,T')$.*

2. $T \ll T'$ if and only if $\forall$ wins $G(T,T')$.

We need some operations on trees. Let T and T' be trees. The tree $T \oplus T'$ consists a disjoint union of T and T' identified at the root. So $T \oplus T'$ is the *supremum* of T and T' relative to $\leq$. The tree $T \otimes T'$ consists of pairs (t,t'), where $t \in T$, $t' \in T'$ and t has the same height in T as t' has in T'. The elements of $T \otimes T'$ are ordered coordinatewise. Clearly, $T \otimes T'$ is the *infimum* of T and T' relative to $\leq$. The operations $\bigoplus_{i \in I}$ and $\bigotimes_{i \in I}$ are defined similarly. We can also define "arithmetic" operations on trees. The tree $T + T'$ is obtained from T by adding a copy of T' at the end of each maximal branch of T. With this definition, $B_\alpha + B_\beta \equiv B_{\beta+\alpha}$. The product $T \cdot T'$ consists of triples (g,t,t'), where $t \in T$, $t' \in T'$ and g is a mapping which associates every predecessor of t' with a maximal branch of T. We set $(g,t,t') \leq (g_1,t_1,t_1')$ if ($t' = t_1'$ and $t \leq t_1$) or ($t' < t_1'$, g coincides with g_1 on predecessors of t' and $t \in g'(t')$). Again, $B_\alpha \cdot B_\beta \equiv B_{\alpha \cdot \beta}$. Intuitively, $T \cdot T'$ is obtained from T' by replacing every element by a copy of T. Since T is likely to have branching, there are different ways of progressing from a node of T' to its successor through the copy of T. This is why the elements of $T \cdot T'$ have the g-component. If we limit the way a branch of $T \cdot T'$ can pass through T', we arrive at the following variant $T \cdot T'$. Let G be a set of maximal branches of T. The tree $T \cdot_G T'$ consists of triples $(g,t,t') \in T \cdot T'$ such that, if $t'' < t'$, then $g(t'') \in G$. The ordering is defined as in $T \cdot T'$.

A tree T is *reflexive* if $T \leq \{s \in T : t \leq_T s\}$ for every $t \in T$. Every tree T can be extended to a reflexive tree in the following way ([Huu91, HT91]): Let $R(T)$ be the set of finite sequences $(t_0,...,t_n)$ of elements of T. We can think of this sequence as a linear ordering which starts with $\{t \in T : t \leq t_0\}$, continues with $\{t \in T : t \leq t_1\}$, then with $\{t \in T : t \leq t_2\}$, etc. until t_n comes in the end. In this way $R(T)$ gets a natural tree-ordering: if s and s' are elements of $R(T)$, then we define $s \leq s'$ to mean that as linear orderings, s is equal to s' or is an initial segment of s'. It is easy to see that $T \leq R(T)$ and that $R(T)$ is reflexive. It is also interesting to note that if T has no branches of length $\kappa > \omega$, then neither has $R(T)$. We can split $R(T)$ into parts that are called *phases* in [HT91]. Namely, if $s = (t_0,...,t_n) \in R(T)$, we call the number n the phase of s and denote it by $p(n)$. Elements of phase 0 form an isomorphic copy of T. Each element $(t_0,...,t_n)$ of phase n extends to an isomorphic copy $\{(t_0,...,t_{n+1}) : t_{n+1} \in T\}$ of T.

We can picture the mutual ordering of the two types of trees that arise from ordinals as follows:

$$B_0 < B_1 < ... < B_\omega < ... < B_{\omega_1} < ... < \omega < \omega + 1 < ... < \omega_1 < ...$$

Note that ω has a proper class $\{B_\alpha : \alpha \in On\}$ of predecessors. The predecessors of ω_1 are all the various trees without uncountable branches. An interesting example is the tree $T_p = (\bigoplus_{\alpha<\omega_1} \alpha) \cdot \omega$, introduced in [Huu91]. This tree has the remarkable property that

$$T_p \leq T \text{ or } T \ll T_p$$

for any tree T of height ω_1 ([Huu91]). So T_p has a very special place among predecessors of ω_1. The whole picture of the ordering of all trees is quite complicated. We shall now show that some trees are mutually $\leq$-incomparable.

Let $A \subseteq \omega_1$. Recall that A is *closed unbounded* if it is uncountable and contains the supremum of each of its proper initial segments. We say that A is *stationary*, if it meets every closed unbounded subset of ω_1. The complement of a stationary set is *co-stationary*. Finally, a stationary and co-stationary set is called *bistationary*. It is a not-too-hard consequence of the Axiom of Choice that there are bistationary subsets of ω_1. In fact, there are ω_1 disjoint stationary subsets of ω_1 and hence 2^{ω_1} bistationary subsets A_α of ω_1 such that $A_\alpha \setminus A_\beta$ is bistationary whenever $\alpha \neq \beta$. Bistationary sets can be used to construct interesting trees without uncountable branches. If A is a bistationary subset of ω_1, let $T(A)$ be the tree of sequences of elements of A that are ascending, continuous and have a last element.

Lemma 3.5 ([HV90, Tod81])

1. *If A is bistationary, then $T(A)$ is a tree of height ω_1 with no uncountable branches.*
2. *If A, B and $B \setminus A$ are bistationary, then $T(B) \not\leq T(A)$. If also $A \subset B$, then $T(A) < T(B)$.*
3. *If A and B are bistationary, then $T(A) \not\ll T(B)$.*
4. *If T is an Aronszajn tree and A is bistationary, then $T \not\leq T(A)$.*

Proof. Every stationary set has closed subsets of all order-types $< \omega_1$. This implies that $T(A)$ has height ω_1. An uncountable branch in $T(A)$ would give rise to a closed unbounded subset of A contrary to the co-stationarity of A. The first claim is proved. For the second claim, suppose $f : T(B) \to T(A)$ is order-preserving. For countable α, let F_α be a function on $T(B)$ so that $F_\alpha(s)$ is some $s' > s$ with $\max(s') > \alpha$. For any countable limit ordinal α, let S_α be a countable subset of $T(B)$ containing $\emptyset$ and closed under every F_β, where $\beta < \alpha$. Let C be the closed unbounded set of countable α such that if $s \in S_\alpha$, then $\max(s) < \alpha$ and $\max(f(s)) < \alpha$. Let $\alpha \in C \cap (B \setminus A)$. Let (s_n) be an ascending sequence in S_α with $\alpha = \sup_n \max(t_n)$. Then $\sup_n \max(f(t_n)) = \alpha$. Since $\alpha \in B \setminus A$, we have a contradiction. The second claim is proved. The third and fourth claims are proved similarly. **Q.E.D.**

By combining the above lemma and the fact that there are 2^{ω_1} bistationary subsets A_α of ω_1 such that $A_\alpha \setminus A_\beta$ is bistationary whenever $\alpha < \beta$, we get the following result:

Proposition 3.6 ([HV90]) *There is a set of trees $\{T_\alpha : \alpha < 2^{\omega_1}\}$ such that for all $\alpha < \beta$:*

(1) *T_α has height ω_1 and cardinality 2^ω.*

(2) *T_α has no uncountable branches.*

(3) *T_α and T_β are incomparable by $\leq$.*

The claim remains true if condition (3) above is replaced by one of the following:

(3') $T_\alpha < T_\beta$.

(3") $T_\alpha > T_\beta$.

So there is an explosion in the hierarchy of trees between the trees of countable height and the one-branch tree ω_1. This is in sharp contrast with the situation between trees of finite height and the one-branch tree ω, where we have all the well-founded trees in nice linear order one after another.

We have observed that the class of trees with no uncountable branches has ascending chains, descending chains and antichains of cardinality 2^{ω_1}. All these chains arise from the trees $T(A)$, A bistationary. Several questions suggest themselves. Maybe these trees are essentially all there is in this family. Or maybe there is some relatively small number of "representatives" of these trees into which everything else can be reduced. As to the first question, H. Tuuri has pointed out, that if T is the tree of one-one sequences of rationals such that the sequence has a last element, then $T \not\leq T(A)$ (proved like Lemma 3.5 (3)) and $T(A) \not\leq T$ for bistationary A (as $T(A)$ is non-special by [Tod81]). So this T is an example of a tree substantially different from the trees $T(A)$.

We approach the question of "representatives" with the notion of a universal family of trees. A family $\mathcal{U}$ of trees is *universal* for a class $\mathcal{V}$ of trees if $\mathcal{U} \subseteq \mathcal{V}$ and

$$\forall T \in \mathcal{V} \exists S \in \mathcal{U}(T \leq S).$$

If we want to find a universal family for the class of all trees with no uncountable branches, there is an obstacle: If the universal family is a set, as it is reasonable to assume, we can apply the σ-operation to its supremum, and obtain a tree which contradicts the universality of the family. So we can only hope to find universal families for restricted classes of trees.

Let $\mathcal{T}_{\omega_1}$ be the class of trees of cardinality ω_1 and with no uncountable branches. If CH holds, then there cannot be a universal family of size $\leq \omega_1$ for $\mathcal{T}_{\omega_1}$, because of the function σ. On the other hand, Hella observed that if $2^\omega = 2^{\omega_1}$, then an upper bound for $\mathcal{T}_{\omega_1}$ is obtained from the full binary tree of height ω by simply extending all its branches by different elements of $\mathcal{T}_{\omega_1}$. The resulting tree has cardinality 2^ω. It follows from $\neg CH + MA$ that there is a single tree $T \in \mathcal{T}_{\omega_1}$ so that $T' \leq T$ holds for all $T' \in \mathcal{T}_{\omega_1}$ ([MV]). So here we have a universal family of cardinality 1.

Theorem 3.7 ([MV]) *The statement "There is a universal family of cardinality ω_2 for $\mathcal{T}_{\omega_1}$" is independent of ZFC+CH+$2^{\omega_1} \geq \omega_3$.*

We may also ask whether the trees $T(A)$ can be majorized by one single tree. In [MS] a tree T is called a *Canary tree* if it has cardinality 2^ω, has no uncountable branches, and in any extension of the universe in which no new reals are added and in which some stationary subset of ω_1 is destroyed, T has an uncountable branch. This is equivalent to saying that T has cardinality 2^ω, has no uncountable branches, and satisfies $T(A) \leq T$ for each bistationary A ([MV]).

Theorem 3.8 ([MS]) *The statement "There is a Canary tree" is independent from $ZFC + GCH$.*

The structure of trees with no uncountable branches is far from being understood even in the light of the above results. More investigation is needed. It is now quite clear that ZFC alone is not sufficient for deciding questions about these trees. The Continuum Hypothesis, for example, makes a big difference. It would be interesting to find new axioms which would fix the structure of trees more or less completely.

4 Topology of the space $\mathcal{N}_1$

There are properties of countable models and infinitary formulas which are so basic that they can be formulated in purely topological terms. To arrive at these one identifies countable models with elements of the *Baire space* $\mathcal{N} = {}^\omega\omega$, whereby classes of countable models are identified with subsets of $\mathcal{N}$. D. Scott established the basic relation between the space $\mathcal{N}$ and $L_{\omega_1\omega}$: An invariant subset of $\mathcal{N}$ is Borel iff it is (in this identification) the class of countable models of a sentence of $L_{\omega_1\omega}$ ([Sco65]). R. Vaught developed further the connection between model theoretic properties of $L_{\omega_1\omega}$ and topological properties of the Baire space ([Vau73]).

A characteristic example of this connection is the *undefinability of well-order in* $L_{\omega_1\omega}$, proved in [LE66], which can be seen as a consequence of the relatively simple topological property of $\mathcal{N}$, that the codes of well-orderings is a non-analytic set. Similarly the interpolation theorem of $L_{\omega_1\omega}$ may be thought of as a logical version of the topological fact that disjoint $\mathbf{\Sigma}^1_1$ sets can be separated by a Borel set. Finally, the basic topological property of the Baire space, that every closed set is the disjoint union of a countable set and a perfect set, and its elaboration that the cardinality of an analytic set is either $\leq \omega_1$ or 2^ω, appear behind many results of model theory. We have in mind examples such as the result in [Kue68] that the number of automorphisms of a countable structure is ω or 2^ω, and the result in [Mor70] that the number of non-isomorphic countable models of a sentence of $L_{\omega_1\omega}$ is either $\leq \omega_1$ or 2^ω. In such cases as the above we feel that the underlying topological fact reveals the *actual mathematical construction* behind the logical result.

We may analogously identify models of cardinality ω_1 with elements of a *generalized Baire space* $\mathcal{N}_1 = {}^{\omega_1}\omega_1$. A basic neighborhood of an element $f \in \mathcal{N}_1$ is a set of the form

$$N(f,\alpha) = \{g \in \mathcal{N}_1 : g(\beta) = f(\beta) \text{ for } \beta < \alpha\},$$

where $\alpha < \omega_1$. Note that the intersection of a countable family of basic neighborhoods is still a basic neighborhood, and that there is a dense set of the cardinality of the continuum, namely the set of eventually constant functions. The space $\mathcal{N}_1$ is what Sikorski calls ω_1*-metrizable* space ([Sik49]).

In this context we are mostly interested in properties of analytic and co-analytic sets of this space. These concepts are defined in the standard way, which we now recall: A set $A \subseteq \mathcal{N}_1$ is *analytic* or $\mathbf{\Sigma}^1_1$, if there is a closed set $B \subseteq \mathcal{N}_1 \times \mathcal{N}_1$ such that for all f: $f \in A$ if and only if $\exists g((f,g) \in B)$. A set is *co-analytic* or $\mathbf{\Pi}^1_1$ if its complement is $\mathbf{\Sigma}^1_1$, and $\mathbf{\Delta}^1_1$ if it is both $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$.

The standard example of a co-analytic non-analytic subset of $\mathcal{N}$ is the set of codes of well-orderings of ω. This may be rephrased as the statement that the set of codes of countable trees with no infinite branches is a co-analytic non-analytic subset of $\mathcal{N}$. Analogously, the set of "codes" of trees of cardinality ω_1 with no uncountable branches is a prime candidate for a co-analytic non-analytic subset of $\mathcal{N}_1$. To arrive at this set, we introduce some notation. Let π be a bijection from $\omega_1 \times \omega_1$ onto ω_1. If $f \in \mathcal{N}_1$, let $\leq_f = \{(\alpha,\beta) : f(\pi(\alpha,\beta)) = 0\}$. We may think that f "codes" the binary relation $\leq_f$. Clearly, every binary relation on ω_1 is coded by some $f \in \mathcal{N}_1$ in this way. Let $T_f = (\omega_1, \leq_f)$ and

$$TO = \{f \in \mathcal{N}_1 : T_f \text{ is a tree with no uncountable branches}\}.$$

Lemma 4.9 (**[MV]**)

1. *The set* TO *is co-analytic.*

2. *If $A \subseteq TO$ is analytic, then there is a tree W of cardinality $\leq 2^\omega$ with no uncountable branches such that $T_g \leq W$ holds for all $g \in A$.*

3. *If CH holds, TO is non-analytic.*

Proof. The first claim is trivial, so we move to the second claim. If $f \in \mathcal{N}_1$ and $\alpha < \omega_1$, let $\overline{f}(\alpha)$ be the sequence $(f(\beta))_{\beta<\alpha}$. Let R be a closed set such that $f \in A$ holds if and only if $\exists g(f,g) \in R$. Let $U(f)$ be the set of sequences $\overline{g}(\alpha) = (g(\xi))_{\xi<\alpha}$ such that $N((f,g),\alpha) \cap R \neq \emptyset$. Now $U(f)$ is a tree and it is easy to see, that

$$f \in A \iff U(f) \text{ has an uncountable branch.}$$

Let W be the tree of triples $(\overline{f}(\alpha), t, \overline{h}(\alpha))$, where $f \in \mathcal{N}_1$ so that T_f is a tree, t is an element of T_f of height α and $\overline{h}(\alpha) \in U(f)$. Any uncountable branch of W would give rise to an element f of $A \setminus TO$. Hence W cannot have uncountable branches. Suppose now $f \in A$ is arbitrary. Let $(\overline{h}(\alpha))_{\alpha<\omega_1}$ be uncountable branch in $U(f)$. If $t \in T_f$ has height α, let $\phi(t)$ be the triple $(\overline{f}(\alpha), t, \overline{h}(\alpha))$. The mapping ϕ shows that $T_f \leq W$. This ends the proof of the second claim. For the third claim, we assume that TO were analytic, and derive a contradiction. We consider the second claim with the choice $A = TO$. Since we assume CH, we can find $f \in TO$ so that $\sigma(W)$ is isomorphic to T_f. We get the contradiction $\sigma(W) \leq T_f \leq W \ll \sigma(W)$. **Q.E.D.**

A subset $C \subseteq \mathcal{N}_1$ is $\mathbf{\Pi}^1_1$-*complete* if C is co-analytic and for every co-analytic set A there is a continuous mapping ϕ on $\mathcal{N}_1$ such that for all f: $f \in A$ if and only if $\phi(f) \in C$. Assuming CH, the set TO is $\mathbf{\Pi}^1_1$-complete. Without CH the set TO need not be $\mathbf{\Pi}^1_1$-complete:

Proposition 4.10 ([MV]) *If $MA + \neg CH$ holds then TO is $\mathbf{\Delta}^1_1$.*

The proof of Lemma 4.9 can be elaborated to give a more general result. Let A be a co-analytic set. If we assume CH, we can use $\mathbf{\Pi}^1_1$-completeness of TO to construct a continuous mapping ϕ so that $f \in A$ if and only if $\phi(f) \in TO$. Let

$$A_{\phi,g} = \{f \in \mathcal{N}_1 : \phi(f) \leq T_g\}.$$

The proof of the following result is essentially contained in the proof of Lemma 4.9.

Proposition 4.11 ([MV]) *Assume CH. Suppose A is co-analytic and ϕ is as above. Then:*

1. *$A_{\phi,g}$ is analytic for each $g \in TO$.*

2. *If $B \subseteq A$ is analytic, then there is a $g \in TO$ such that $B \subseteq A_{\phi,g}$ (Covering Property).*

3. A is $\mathbf{\Delta}_1^1$ if and only if there is a $g \in TO$ such that $T_{\phi(f)} \leq T_g$ for all $f \in A$.

An interesting analytic subset of $\mathcal{N}_1$ is the set CUB of characteristic functions of subsets of ω_1 which contain a closed unbounded set. Respectively, we have the co-analytic set $STAT$ of characteristic functions of stationary subsets of ω_1. The continuous mapping ϕ associated with this co-analytic set, assuming CH, can be chosen to be the following very natural mapping: If $f \in \mathcal{N}_1$ and $A = \{\alpha : f(\alpha) \neq 0\}$, let $\phi(f)$ be a canonical code of the tree $T(A)$. Now $f \in STAT$ if and only if $\phi(f) \in TO$. Hence, assuming CH, the set $STAT$ is $\mathbf{\Delta}_1^1$ if and only if there is an $f \in TO$ such that $T(A) \leq T_f$ for all co-stationary A. In Section 3 we called such a tree a *Canary tree* and we noted (Theorem 3.8) that the existence of a Canary tree is independent of $ZFC + CH$. The following Proposition follows from Proposition 4.11:

Proposition 4.12 *The following conditions are equivalent:*

1. *CUB is $\mathbf{\Delta}_1^1$.*
2. *$STAT$ is $\mathbf{\Sigma}_1^1$.*
3. *There is a Canary tree.*

So we cannot decide in $ZFC + CH$ the question whether CUB is $\mathbf{\Delta}_1^1$ or not. The best that is known at the moment is that CUB is not $\mathbf{\Sigma}_3^0$ or $\mathbf{\Pi}_3^0$ ([MV]).

Proposition 4.13 ([MV]) *Assume CH. Let A and B be disjoint analytic sets. There is a $\mathbf{\Delta}_1^1$-set C such that $A \subseteq C$ and $C \cap B = \emptyset$. (Separation Property)*

Proof. Suppose ϕ is continuous so that $f \notin B$ if and only if $\phi(f) \in TO$. By the Covering Property there is a $g \in TO$ so that $A \subseteq C$, where $C = (-B)_{\phi,g}$. Clearly $C \cap B = \emptyset$. **Q.E.D.**

The Separation Property becomes more interesting if we can generate the $\mathbf{\Delta}_1^1$-sets via a Borel type hierarchy analogously with the Borel hierarchy of the classical Baire space $\mathcal{N}$. In fact, such a generalized Borel hierarchy, called Borel* hierarchy, can be defined for $\mathcal{N}_1$ ([MV]). Then $\mathbf{\Delta}_1^1$-subsets of $\mathcal{N}_1$ will be exactly the so called *determined* Borel*-sets ([Tuu, MV]).

The *Cantor-Bendixson Theorem* says that any closed subset of $\mathcal{N}$ can be divided into a perfect part and a scattered part. The perfect part is empty or of the cardinality of the continuum. The scattered part is countable. The corresponding result for analytic sets says that any analytic subset of $\mathcal{N}$ contains a non-empty perfect subset or else has cardinality $\leq \omega_1$. We shall now address the question whether similar results hold for $\mathcal{N}_1$.

It is easy to see that every closed subset of $\mathcal{N}_1$ can be represented as the set of all uncountable branches of a subtree of $\mathcal{N}_1$. So the possible cardinalities of closed subsets of $\mathcal{N}_1$ are limited to the possible numbers of uncountable branches of trees of height ω_1. There are trivial examples of trees where the number of uncountable branches is any number $\leq \omega_1$, 2^ω or 2^{ω_1}. Nothing more can be said on the basis of ZFC or even $ZFC + CH$, alone. An analysis of the Cantor-Bendixson Theorem for $\mathcal{N}_1$ is contained in [Vää91]. The implication to the question of cardinality of closed subsets of $\mathcal{N}_1$ is:

Proposition 4.14 ([Vää91]) *The statement "Every closed subset of $\mathcal{N}_1$ has cardinality $\leq \omega_1$ or $= 2^{\omega_1}$" is independent of $ZFC + CH+$ there is an inaccessible cardinal.*

A similar result holds for analytic sets ([MS]).

5 Measuring similarity of models

In this Section we return to the idea introduced in Section 2 of using trees to measure similarity of models of cardinality ω_1. For this purpose we introduced the trees $S_{\mathcal{A},\mathcal{B},\tau}$ and $K_{\mathcal{A},\mathcal{B}}$. We are now ready to compare these trees to each other. Let κ be the common cardinality of $\mathcal{A}$ and $\mathcal{B}$ and

$$S_{\mathcal{A},\mathcal{B}} = \bigotimes \{S_{\mathcal{A},\mathcal{B},\tau} : \tau \text{ is a winning strategy of } \forall \text{ in } EF_\kappa(\mathcal{A},\mathcal{B})\}.$$

We let $S_{\mathcal{A},\mathcal{B}}$ consist of just one branch of length κ in the special case that $\mathcal{A} \cong \mathcal{B}$.

Proposition 5.15 *Let $\mathcal{A}$ and $\mathcal{B}$ be two structures of cardinality κ and of the same vocabulary. Then $K_{\mathcal{A},\mathcal{B}} \leq S_{\mathcal{A},\mathcal{B}}$. If $K_{\mathcal{A},\mathcal{B}}$ is well-founded, then $K_{\mathcal{A},\mathcal{B}} \equiv S_{\mathcal{A},\mathcal{B}}$.*

Proof. Suppose $\sigma \in K_{\mathcal{A},\mathcal{B}}$. If $\mathcal{A} \cong \mathcal{B}$, then $S_{\mathcal{A},\mathcal{B}}$ has a κ-branch and $K_{\mathcal{A},\mathcal{B}} \leq S_{\mathcal{A},\mathcal{B}}$ holds trivially. Suppose then τ is a winning strategy of $\forall$ in $EF_\kappa(\mathcal{A},\mathcal{B})$. Let $f(\sigma)$ be the sequence of moves in $EF_\kappa(\mathcal{A},\mathcal{B})$ when $\forall$ plays τ and $\exists$ plays σ. Clearly, $f(\sigma) \in S_{\mathcal{A},\mathcal{B},\tau}$ and f is order-preserving. Suppose then $K_{\mathcal{A},\mathcal{B}}$ is well-founded but there is no winning strategy τ of $\forall$ such that $S_{\mathcal{A},\mathcal{B},\tau} \leq K_{\mathcal{A},\mathcal{B}}$. Note that $\mathcal{A} \ncong \mathcal{B}$, for otherwise $K_{\mathcal{A},\mathcal{B}}$ has a branch of length κ. Let $S_{\mathcal{A},\mathcal{B},\tau}(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$ be the tree of all possible sequences of successor length of moves of $\exists$ against τ so that $\exists$ has not yet lost the game, and the first n moves of the game have been $(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})$. Let $I(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$, $n \geq 0$, be the set of such winning strategies τ of $\forall$ in $EF_\kappa(\mathcal{A},\mathcal{B})$ that the sequence of first n moves $(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})$ in $EF_\kappa(\mathcal{A},\mathcal{B})$ is consistent with τ. To derive a contradiction, we describe a winning strategy of $\exists$ in $EF_\omega(\mathcal{A},\mathcal{B})$. Suppose $\forall$ starts this game with a_0. If there is no b_0 such that for all $\tau \in I(a_0, b_0)$ we have $S_{\mathcal{A},\mathcal{B},\tau}(a_0, b_0) \not\leq K_{\mathcal{A},\mathcal{B}}$, then there is $\tau \in I()$ such that $S_{\mathcal{A},\mathcal{B},\tau} \leq K_{\mathcal{A},\mathcal{B}}$, contrary to our assumption. Hence $\exists$ must have a move b_0 with the property that for all

$\tau \in I(a_0, b_0)$ we have $S_{\mathcal{A},\mathcal{B},\tau}(a_0, b_0) \not\leq K_{\mathcal{A},\mathcal{B}}$. Next $\forall$ plays (e.g.) b_1. As above, we may infer that there has to be a move a_1 for $\exists$ so that for all $\tau \in I(a_0, b_0, a_1, b_1)$ we have $S_{\mathcal{A},\mathcal{B},\tau}(a_0, b_0, a_1, b_1) \not\leq K_{\mathcal{A},\mathcal{B}}$. Going on in this manner yields the required winning strategy of $\exists$ in $EF_\omega(\mathcal{A}, \mathcal{B})$. **Q.E.D.**

So if the difference between $\mathcal{A}$ and $\mathcal{B}$ is so easy to detect that $K_{\mathcal{A},\mathcal{B}}$ is even well-founded, which is the case if $\mathcal{A} \not\equiv_{L_{\infty\omega}} \mathcal{B}$, then $K_{\mathcal{A},\mathcal{B}} \equiv S_{\mathcal{A},\mathcal{B}}$. We shall see below (Proposition 5.21) that for non-isomorphic models $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \equiv_{L_{\infty\omega}} \mathcal{B}$, there may be a huge gap between $K_{\mathcal{A},\mathcal{B}}$ and $S_{\mathcal{A},\mathcal{B}}$.

A basic concept in our closer analysis of similarity of models is the following *approximated* Ehrenfeucht-Fraïssé-game: Let T be a tree. The game $EF_\alpha(\mathcal{A}, \mathcal{B}, T)$ is like $EF_\alpha(\mathcal{A}, \mathcal{B})$ except that $\forall$ has to go up the tree T move by move. Thus there are two players, $\exists$ and $\forall$. During a round of the game $\forall$ first picks an element of one of the models and an element of T, and then $\exists$ picks an element of the other model. Let a_i be the element of A, b_i the element of B and t_i the element of T picked during round i of the game. There are altogether α rounds. Finally, $\exists$ wins the game if the resulting mapping $a_i \mapsto b_i$ is a partial isomorphism or the sequence of elements t_i does not form an ascending chain in T. Otherwise $\forall$ wins.

Proposition 5.16 *1. $\exists$ wins $EF_\kappa(\mathcal{A}, \mathcal{B}, T)$ if and only if $T \leq K_{\mathcal{A},\mathcal{B}}$.*

2. $\forall$ wins $EF_\kappa(\mathcal{A}, \mathcal{B}, T)$ with strategy τ if and only if $S_{\mathcal{A},\mathcal{B},\tau} \ll T$.

Proof. The point here is that while $\forall$ goes up the tree $K_{\mathcal{A},\mathcal{B}}$, he reveals longer and longer strategies for $\exists$. Player $\exists$ can simply use these strategies against $\forall$. At limits we envoke the fact that strategies in $K_{\mathcal{A},\mathcal{B}}$ are of successor length. The strategy of $\forall$ in $EF_\kappa(\mathcal{A}, \mathcal{B}, \sigma S_{\mathcal{A},\mathcal{B},\tau})$ is to play in $\sigma S_{\mathcal{A},\mathcal{B},\tau}$ the sequence of previous moves of $\exists$, and otherwise follow τ. **Q.E.D.**

We call a tree T of height α an *equivalence-tree* of $(\mathcal{A}, \mathcal{B})$ if $\exists$ wins the game $EF_\alpha(\mathcal{A}, \mathcal{B}, T)$, and a *non-equivalence tree* of $(\mathcal{A}, \mathcal{B})$ if $\forall$ wins the game $EF_\alpha(\mathcal{A}, \mathcal{B}, T)$. Proposition 5.16 above implies that $K_{\mathcal{A},\mathcal{B}}$ is the largest equivalence tree of $(\mathcal{A}, \mathcal{B})$. The tree $K_{\mathcal{A},\mathcal{B}}$ is unsatisfactory in one respect, though: there is no reason to believe that it has cardinality $\leq \omega_1$ even if CH is assumed. A tree $T \in \mathcal{T}_{\omega_1}$ is a *Karp tree* of $(\mathcal{A}, \mathcal{B})$ if it is an equivalence tree of $(\mathcal{A}, \mathcal{B})$ but σT is not. Respectively, a tree $T \in \mathcal{T}_{\omega_1}$ is a *Scott tree* of $(\mathcal{A}, \mathcal{B})$ if σT is a non-equivalence tree of $(\mathcal{A}, \mathcal{B})$ but T is not.

Theorem 5.17 ([HV90]) *Every pair of models $(\mathcal{A}, \mathcal{B})$ has a Karp tree T_1 and a Scott tree T_2, and $T_1 \leq T_2$*

The structure of Karp trees and Scott trees of pairs of structures is not fully understood yet. For rather trivial reasons, the families of Karp trees and Scott trees of a given pair of structures are closed under supremums. The following theorem contains some less obvious results that have been obtained about the ordering of Scott or Karp trees of a pair of models.

Theorem 5.18 *1. There are models $\mathcal{A}$ and $\mathcal{B}$ of cardinality ω_1 such that the pair $(\mathcal{A}, \mathcal{B})$ has 2^{ω_1} Scott trees which are mutually non-comparable by $\leq$. ([HV90])*

2. There are models $\mathcal{A}$ and $\mathcal{B}$ of cardinality ω_1 such that the pair $(\mathcal{A}, \mathcal{B})$ has two Scott trees the infimum of which is not a Scott tree. ([Huu91])

3. There are models $\mathcal{A}$ and $\mathcal{B}$ of cardinality ω_1 such that the pair $(\mathcal{A}, \mathcal{B})$ has two Karp trees the infimum of which is not a Karp tree. ([Huu91])

A tree $T \in \mathcal{T}_{\omega_1}$ is a *universal equivalence tree* of a model $\mathcal{A}$ of cardinality ω_1 if $\mathcal{A} \cong \mathcal{B}$ holds for every $\mathcal{B}$ of cardinality ω_1 for which T is an equivalence tree of $(\mathcal{A}, \mathcal{B})$. If

$$K_{\mathcal{A}} = \bigoplus \{K_{\mathcal{A},\mathcal{B}} : |B| \leq \omega_1, \mathcal{B} \not\cong \mathcal{A}\}$$

and $T \equiv \sigma K_{\mathcal{A}}$ with $|T| \leq \omega_1$, then T is a universal equivalence tree of $\mathcal{A}$. A tree $T \in \mathcal{T}_{\omega_1}$ is a *universal non-equivalence tree* of a model $\mathcal{A}$ of cardinality ω_1 if $\mathcal{A} \not\cong \mathcal{B}$ implies T is a non-equivalence tree of $(\mathcal{A}, \mathcal{B})$ for every $\mathcal{B}$ of cardinality ω_1. This is equivalent to the claim that for every $\mathcal{B} \not\cong \mathcal{A}$ of cardinality ω_1 there is some winning strategy τ of $\forall$ in $EF_\kappa(\mathcal{A}, \mathcal{B})$ so that $S_{\mathcal{A},\mathcal{B},\tau} \ll T$.

Note that a universal non-equivalence tree is necessarily also a universal equivalence tree. Thus having a universal non-equivalence tree is a stronger property than having a universal equivalence tree. Every countable model has universal non-equivalence trees. This tree is the canonical tree arising from the Scott rank of the model. The concepts of universal non-equivalence tree and universal equivalence tree are attempts to find an analogue of Scott rank for uncountable models.

It is clear that many models of cardinality ω_1 do have universal non-equivalence trees. Let us consider an example. Let T be an ω-stable first order theory with $NDOP$ (or countable superstable with $NDOP$ and $NOTOP$, see [SB89]). By [She90, Chapter XIII Section 1], any two $L_{\infty\omega_1}$-equivalent models of T of cardinality ω_1 are isomorphic. There is a back-and-forth characterisation of $L_{\infty\omega_1}$-equivalence which, from the point of view of $\forall$, is a special case of $EF_{\omega\cdot\omega}(\mathcal{A}, \mathcal{B})$. Hence every model of T of cardinality ω_1 has a universal non-equivalence tree of height $\leq \omega \cdot \omega$.

Theorem 5.19 ([HT91]) *Let $\kappa = \kappa^{<\kappa} > \omega$. There is a model $\mathcal{A}$ of cardinality κ with the following property: For any tree T such that $|T| = \kappa$ and T has no branches of length κ there is a model $\mathcal{B}$ of cardinality κ so that $\mathcal{A} \not\cong \mathcal{B}$ but $\exists$ has a winning startegy in $EF_\kappa(\mathcal{A}, \mathcal{B}, T)$. Thus $\mathcal{A}$ has no universal equivalence tree.*

Proof. Note that $\kappa = \kappa^{<\kappa}$ implies κ is regular. The models $\mathcal{A}$ and $\mathcal{B}$ are constructed using the reflexivity operation R introduced in Section 3. Let T_0 be $\kappa^{<\kappa}$ as a tree of sequences of ordinals. We let $\mathcal{A}$ be the tree-ordered structure $(R(T_0), \leq)$. Let

$$T_1 = ((\bigoplus_{\alpha<\kappa} \alpha) \cdot T) + 1$$

and $T_2 = T_1 \otimes T_0$. Let f be the canonical projection $T_2 \to T_1$. We can extend f to $R(T_2)$ by letting $f((s_0, ..., s_n)) = f(s_n)$. Let $\mathcal{B}$ be the tree-ordered structure $(R(T_2), \leq)$. Now $\mathcal{A}$ has branches of length κ but $\mathcal{B}$ has none, so $\mathcal{A} \not\cong \mathcal{B}$. To finish the proof we have to describe the winning strategy of $\exists$ in $EF_\kappa(\mathcal{A}, \mathcal{B}, T)$. Because of the special relation between T and T_1, it suffices to show that $\exists$ wins the game $EF'_\kappa(\mathcal{A}, \mathcal{B}, T_1)$ which differs from $EF_\kappa(\mathcal{A}, \mathcal{B}, T)$ by allowing $\forall$ to play only elements of $\mathcal{A}$ and $\mathcal{B}$ the predecessors of which have been played already.

Recall that elements of $R(T_0)$ and $R(T_2)$ come in different phases. An element $(s_0, \ldots, s_n)$ of phase n may have extensions $(s_0, \ldots, s'_n)$ inside phase n but it also has extensions $(s_0, \ldots, s_n, \ldots, s_m)$ of higher phase. During the game elements a_α of $R(T_0)$, elements b_α of $R(T_2)$ and elements t_α of T_1 are played. Here α refers to the round of the game. The strategy of $\exists$ is to play in the obvious way but taking care that he never increases phase by more that 1, and making sure that when $p(b_\alpha) = p(a_\alpha) + 1$, then $f(b_\alpha) \leq t_\alpha$.

Suppose now $\forall$ plays a_α of limit height. There is a chain of predecessors a_β of a_α converging to a_α. The corresponding elements b_β will eventually be inside one phase and because of the "+1" in the definition of T_1, will converge to some element b_α. This is the response of $\exists$.

Suppose then $\forall$ plays a_α of successor height and a_β is the immediate predecessor of a_α. If $p(b_\beta) = p(a_\beta) + 1$, then $f(b_\beta) \leq t_\beta < t_\alpha$, so $f(b_\beta)$ is not maximal in T_1. Then $\exists$ can let b_α be a successor of b_β in $R(T_2)$ so that $p(b_\beta) = p(b_\alpha)$ if and only if $p(a_\beta) = p(a_\alpha)$ and $f(b_\beta) \leq t_\alpha$. If $p(b_\beta) = p(a_\alpha)$, then $f(b_\beta)$ may be maximal in T_1. In that case $\exists$ lets b_α be a successor of b_β in $R(T_2)$ of the next phase. Then $f(b_\alpha)$ is the root of T_1, so $f(b_\alpha) \leq t_\alpha$. Additionally, $\exists$ has to avoid the $< \kappa$ elements played already during the game, but this is not a problem because of the "$\otimes T_0$" part of the definition of T_2.

The case that $\forall$ plays b_α rather than a_α is similar, only easier. **Q.E.D.**

The models constructed in the above theorem are unstable. This is not an accident, as the following result shows:

Theorem 5.20 (**[HT91]**) *(CH) If T is a countable unstable first order theory, then there is a model $\mathcal{A}$ of T of cardinality ω_1 so that $\mathcal{A}$ has no universal equivalence tree.*

On the other hand, it is not just the unstable theories that have models with no universal equivalence tree. The paper [HT91] has results about models without universal equivalence tree of certain stable theories. Also, there is a p-group of cardinality ω_1 without universal equivalence tree ([MO]).

The situation is more complicated with universal non-equivalence trees. We know already that models of ω-stable theories with $NDOP$ do have universal non-equivalence trees.

Theorem 5.21 (A. Mekler) *(CH) Let F be the free abelian group of cardinality $\aleph_1$. Suppose $A \subseteq \omega_1$ is bistationary. There is an $\aleph_1$-free group H so that $\exists$ does not win $EF_{\omega\times 3}(F, H)$, and $\forall$ wins the game $EF_{\omega_1}(F, H, \sigma T(A) + \omega \cdot 2)$ but not the game $EF_{\omega_1}(F, H, T(A) + \omega \cdot 2)$.*

Note that for F and H as above, the tree $K_{F,H}$ has height $\leq \omega \cdot 3$, but $S_{F,H}$ has height ω_1.

Corollary 5.22 ([MS]) *(CH) There is a universal non-equivalence tree for the free abelian group of cardinality $\aleph_1$ if and only if there is a Canary tree.*

Proof. Suppose there is a Canary tree T. We show that $T_1 = \sigma T + \omega \cdot 2$ is a universal non-equivalence tree for F. Suppose H is an abelian group of cardinality $\aleph_1$. We may safely assume H is $\aleph_1$-free, for otherwise $\forall$ wins easily. Hence we may as well assume H arises from a bistationary set A as in the proof above. Now $T(A) \leq T$. By the previous Theorem, $\forall$ wins $EF_{\omega_1}(F, H, T_1)$. Suppose then T is a universal non-equivalence tree of F. To show that T is a Canary tree, let A be bistationary. Let H arise from A as above. Now $\forall$ has a winning strategy τ in $EF_{\omega_1}(F, H, T)$. Let us then work in a generic extension of the universe, where A contains a cub set but no new reals are introduced. In that extension $F \cong H$, but τ still applies to any sequence of moves of $\exists$, whence T contains an uncountable branch. So T is a Canary tree. **Q.E.D.**

So the statement that the abelian group F does not have a universal non-isomorphism tree is independent of $ZFC + CH$. This is not an accident, as the following general result demonstrates:

Theorem 5.23 ([HT91]) *If ZFC is consistent, then the following statement is consistent with CH: Every countable non-superstable first order theory has a model of cardinality ω_1 without a universal non-equivalence tree.*

If we give up CH, the situation changes again dramatically. In [HST] it is proved consistent relative to the consistency of an inaccessible cardinal, that ($\neg CH$ and) every linear ordering of cardinality ω_1 has a universal equivalence tree which is of the form $T + 1$, where T has cardinality ω_1.

The orbit $orb(R)$ of a relation on κ is the set $\{S \subseteq \kappa^n : (\kappa, R) \cong (\kappa, S)\}$. D. Scott ([Sco65]) proved that the orbit of a relation on ω is a $\mathbf{\Delta}^1_1$-subset of $\mathcal{N}$. For orbits of relations on ω_1 the corresponding question is tied up with the problem of the existence of universal equivalence and non-equivalence trees. Implication (2)→(1) in the following Proposition together with a model-theoretic argument for its proof were suggested by H. Tuuri.

Proposition 5.24 **([MV])** *The following two conditions are equivalent:*

(1) *(ω_1, R) has a universal non-equivalence tree.*

(2) *$orb(R)$ is $\mathbf{\Delta}^1_1$.*

Proposition 5.24 shows that the question, whether a model of cardinality ω_1 can be assigned a tree-invariant via the Ehrenfeucht-Fraïssé game, which is in close relation with stability-properties of the first order theory of the model, has also a topological formulation.

We end this Section with a result which further emphasizes the relationship between properties of trees and properties of models:

Theorem 5.25 **([STV])** *The following two conditions are equivalent:*

(1) *There is a tree of cardinality and height ω_1 with exactly λ uncountable branches.*

(2) *There is a model of cardinality ω_1 with exactly λ automorphisms.*

Note that the set of uncountable branches of a tree of cardinality and height ω_1 is (up to some identification) a closed subset of $\mathcal{N}_1$. It is consistent relative to the consistency of an inaccessible cardinal, that there are no closed subsets C of $\mathcal{N}_1$ with $\omega_1 < |C| < 2^{\omega_1}$. On the other hand, a Kurepa tree satisfies (1) with $\lambda = \omega_2$ and it is possible to have a Kurepa tree with ω_2 uncountable branches while $2^{\omega_1} > \omega_2$. So there is a lot of freedom for the number of automorphisms of a model of cardinality ω_1. For comparison, recall that the number of automorphisms of a countable model is $\leq \omega$ or $= 2^\omega$.

6 Infinitely deep languages

Let $\mathcal{A}$ be a fixed structure. The property of another structure $\mathcal{B}$ that $\exists$ wins $EF_\kappa(\mathcal{A}, \mathcal{B}, T)$ can be expressed by an infinitary game sentence which imitates the progress of the game $EF_\kappa(\mathcal{A}, \mathcal{B}, T)$. These infinitary game sentences are the origin of what we call *infinitely deep languages.*

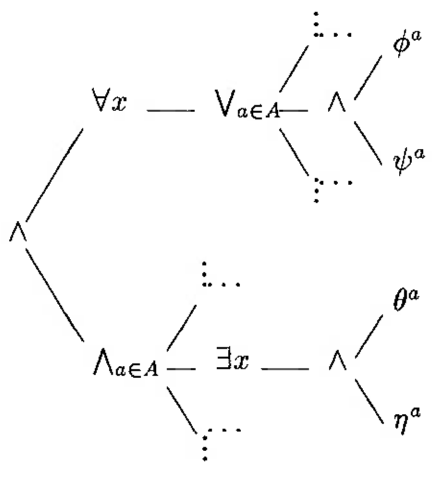

Figure 1: Formula F_A.

Let $\mathcal{A}$ be a structure of cardinality ω_1. We assume the language of $\mathcal{A}$ to be finitary and of cardinality $\leq \omega_1$. let T be a tree of height ω_1 in which every node has at most ω_1 successors, there is no branching at limits, and there are no maximal branches of limit length. The universe of $\mathcal{A}$ is denoted by A. We shall define an infinitary formula $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$ by describing its syntax-tree. We think of syntax-trees of formulas as labelled trees. Figure 1 shows a syntax-tree F_A that we shall use to build up $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$.

Let us consider an arbitrary maximal branch C of F_A. The branch C ends in ϕ^a, ψ^a, θ^a or η^a for some $a = a(C) \in A$. Let G be the set of branches C which end in ψ^a or η^a. Let us consider the tree $F_A \cdot_G T$. To make $F_A \cdot_G T$ a syntax-tree, we assign lables $l(g, w, t)$ to nodes (g, w, t) of $F_A \cdot_G T$ as follows. Only nodes $\forall x$, $\exists x, \phi^a, \psi^a, \theta^a$ and η^a of the various copies of F_A are given a label. For other nodes the label is as in the picture of F_A. Suppose we are at a node $(g, \forall x, t)$ of $F_A \cdot_G T$. Let $(t_\xi)_{\xi\leq\alpha}$ be the sequence of $\{s \in T : s \leq t\}$ in ascending order. We let $l(g, \forall x, t) = \forall x_\alpha$. Staying in the same copy of F_A we let $l(g, \exists x, t) = \exists x_\alpha$. If t is maximal in T, we let $l(g, \psi^a, t) = l(g, \eta^a, t) = \forall x_0(x_0 = x_0)$. If t is not maximal in T, we let $l(g, \psi^a, t) = l(g, \eta^a, t) = \wedge$. Let $a_\xi = a(g(t_\xi))$ for $\xi \leq \alpha$. We let $l(g, \phi^a, t)$ and $l(g, \theta^a, t)$ be the conjunction of atomic and negated atomic formulas $\phi(x_\xi)_{\xi\leq\alpha}$ such that $\mathcal{A} \models \phi(a_\xi)_{\xi\leq\alpha}$. This ends the definition of the labelling of nodes of $F_A \cdot_G T$. The labelled tree $(F_A \cdot_G T, l)$ is our $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$.

The formula $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$ can be given semantics by means of the obvious semantic game. The *dual* formula $\psi^T_{\mathcal{A},\vec{a}}(\vec{z})$ of $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$ is obtained by replacing in the lables of $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$ everywhere $\wedge$ by $\vee$, $\vee$ by $\wedge$, $\forall$ by $\exists$, $\exists$ by $\forall$ and the labels $l(g, \phi^a, t), l(g, \psi^a, t), l(g, \theta^a, t)$ and $l(g, \eta^a, t)$ by their negations.

The formulas $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$ and $\psi^T_{\mathcal{A},\vec{a}}(\vec{z})$ are taylor-made so that player $\exists$ has a winning strategy in the game $EF_{\omega_1}((\mathcal{B},\vec{b}),(\mathcal{A},\vec{a}),T)$, if and only if $\mathcal{B} \models \phi^T_{\mathcal{A},\vec{a}}(\vec{b})$, and player $\forall$

has a winning strategy in $EF_{\omega_1}((\mathcal{B},\vec{b}),(\mathcal{A},\vec{a}),T)$ if and only if $\mathcal{B} \models \psi^T_{\mathcal{A},\vec{a}}(\vec{b})$. We shall now define a general concept of which formulas $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$ and $\psi^T_{\mathcal{A},\vec{a}}(\vec{z})$ are examples.

A *quasiformula* is a labelled tree (T, l), where T is a tree with no maximal branches of limit length and no branching at limits, and $l(t)$ is

1. a countable conjunction of atomic and negated atomic formulas, if t is maximal in T.

2. $\wedge$ or $\vee$, if t has more than one successor in T.

3. $\exists u$ or $\forall u$, where u is a variable symbol, otherwise.

Definition 6.26 ([Kar84]) *The infinitary language $M_{\omega_2\omega_1}$ consists of quasiformulas (T, l), where T is a tree of height ω_1 in which every node has at most ω_1 successors, and there is no u and no branch b of T such that $l(t)$ alternates infinitely many times between the values $\forall u$ and $\exists u$ on b.*

The semantics of $M_{\omega_2\omega_1}$ is defined via a semantic game, exactly as for any game formulas. A formula is *determined* if this semantic game is always determined. The formulas $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$ and $\psi^T_{\mathcal{A},\vec{a}}(\vec{z})$ are clearly examples of formulas of $M_{\omega_2\omega_1}$. These formulas need not be determined, but they are determined in models of cardinality $\leq \omega_1$.

The *quantifier-rank* of a formula (T, l) of $M_{\omega_2\omega_1}$ is the subtree T' of T which consists of nodes t with $l(t) = \forall u$ or $l(t) = \exists u$, where u is a variable symbol. The tree T' may not have a unique root, but relations like $T' \leq T$ still make sense.

The Ehrenfeucht-Fraïssé games $EF_\kappa(\mathcal{A},\mathcal{B},T)$ have dominated our discussion all the way from the beginning. The special connection between $EF_\kappa(\mathcal{A},\mathcal{B},T)$ and $M_{\omega_2\omega_1}$ is revealed by the following easy fact:

Proposition 6.27 ([Kar84]) *Let $\mathcal{A}$ and $\mathcal{B}$ be two models of the same similarity type and T a tree of height ω_1 in which every node has at most ω_1 successors, there is no branching at limits, and there are no maximal branches of limit length. Then the following two conditions are equivalent:*

(1) *$\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of $M_{\omega_2\omega_1}$ of quantifier-rank $\leq T$.*

(2) *Player $\exists$ has a winning strategy in the game $EF_{\omega_1}(\mathcal{A},\mathcal{B},T)$.*

Note that $M_{\omega_2\omega_1}$ is, up to logical equivalence, closed under conjunctions and disjunctions of length $\leq 2^\omega$ and universal and existential quantification over countable sequences of variables. Although $M_{\omega_2\omega_1}$ is closed under *dual* in the obvious sense, there is no trivial reason for it to be closed under negation, because the relevant semantic games need not be determined, as the example below shows. In fact, Tuuri showed that a sentence of $M_{\omega_2\omega_1}$ has a negation in $M_{\omega_2\omega_1}$ *if and only if* it is definable by a sentence whose semantic game is determined ([Tuu]).

Example 6.28 *Let $A \subseteq \omega_1$ be bistationary. Let ϕ_A be the following sentence of $M_{\omega_2\omega_1}$:*

$$\bigwedge_{\alpha_0<\omega_1} \bigvee_{\alpha_1>\alpha_0} \cdots \bigwedge_{\alpha_{2n+2}>\alpha_{2n+1}} \bigvee_{\alpha_{2n+3}>\alpha_{2n+2}} \cdots \phi_{(\alpha_0\ldots\alpha_n\ldots)}$$

where

$$\phi_{(\alpha_0\ldots\alpha_n\ldots)} = \begin{cases} \exists x(x=x) & \text{if } \sup_{n<\omega}\alpha_n \in A \\ \exists x\neg(x=x) & \text{if } \sup_{n<\omega}\alpha_n \notin A \end{cases}$$

Neither ϕ_A nor the dual of ϕ_A is true in any model. In this case the semantic game is non-determined. We still have a negation for ϕ_A in the semantic sense, for example $\exists x(x=x)$.

A $PC(M_{\omega_2\omega_1})$-sentence consists of a sequence of $\leq \omega_1$ existential second-order quantifiers followed by an $M_{\omega_2\omega_1}$-sentence. The existentially quantified predicates are allowed to have any countable ordinal as their arity. The $PC(L_{\omega_2\omega_1})$-sentences are defined analogously. It is easy to see that every $PC(M_{\omega_2\omega_1})$-sentence can be defined by a $PC(L_{\omega_2\omega_1})$-sentence. This observation combined with a standard Skolemization argument gives:

Proposition 6.29 ([Kar84]) *Suppose Φ is a $PC(M_{\omega_2\omega_1})$-sentence and $\mathcal{A}$ is a model of Φ. Then there is a submodel $\mathcal{B}$ of $\mathcal{A}$ so that $|\mathcal{B}| \leq 2^\omega$ and $\mathcal{B} \models \Phi$.*

Proposition 6.30 *If CH holds and there is a Kurepa tree, then some sentence of $M_{\omega_2\omega_1}$ does not have a negation.* [1]

So, what can we express in the language $M_{\omega_2\omega_1}$? We have already pointed out that the formulas $\phi^T_{\mathcal{A},\vec{a}}(\vec{z})$ and $\psi^T_{\mathcal{A},\vec{a}}(\vec{z})$ are in $M_{\omega_2\omega_1}$. This immediately gives the following nice characterisation of rigidity. Recall that a countable model is rigid if and only if all its elements are definable in $L_{\omega_1\omega}$, and a relation on a countable model is invariant if and only if it is definable by a formula of $L_{\omega_1\omega}$.

Proposition 6.31 *Suppose $\mathcal{A}$ is a model of cardinality ω_1. The following conditions are equivalent:*

1. *$\mathcal{A}$ is rigid.*
2. *Every element of $\mathcal{A}$ is definable by a determined $M_{\omega_2\omega_1}$-formula.*

Proof. Suppose $\mathcal{A}$ is rigid. If $b \in A$, we can find a tree T_b with no uncountable branches so that $a \neq b$ if and only if $\mathcal{A} \models \psi^{T_b}_{\mathcal{A},b}(a)$. Now

$$\mathcal{A} \models \forall x(x = a \leftrightarrow \bigwedge_{b\neq a} \psi^{T_b}_{\mathcal{A},b}(x)).$$

Q.E.D.

[1]Recently T. Huuskonen proved this without assuming CH or a Kurepa tree.

Proposition 6.32 ([Hyt90]) *The following conditions are equivalent for any relation R on $\mathcal{A}$:*

(1) *R is invariant (i.e., fixed by all automorphisms of $\mathcal{A}$).*

(2) *R is definable on $\mathcal{A}$ by a determined $M_{\omega_2\omega_1}$- formula.*

Proof. Suppose R is invariant. If $b \in R$ and $a \notin R$, we can find a tree $T_{b,a}$ with no uncountable branches such that $\mathcal{A} \models \psi^{T_{b,a}}_{\mathcal{A},a}(b)$. Now

$$\mathcal{A} \models \forall x (x \in R \leftrightarrow \bigvee_{b \in R} \bigwedge_{a \notin R} \psi^{T_{b,a}}_{\mathcal{A},a}(x)).$$

Q.E.D.

If $\mathcal{A}$ is a model of cardinality ω_1, let $I(\mathcal{A})$ denote the class $\{\mathcal{B} : \mathcal{B} \cong \mathcal{A}\}$. That is, $I(\mathcal{A})$ is the isomorphism type of $\mathcal{A}$. We say that $I(\mathcal{A})$ is *(determinedly) $M_{\omega_2\omega_1}$-definable* if there is a sentence ϕ in $M_{\omega_2\omega_1}$ so that $I(\mathcal{A})$ is the class of models of ϕ of cardinality $\leq \omega_1$ (and ϕ is determined in models of power $\leq \omega_1$).

Proposition 6.33 *Let $\mathcal{A}$ be a model of cardinality ω_1.*

(1) *$\mathcal{A}$ has a universal equivalence tree if and only if $I(\mathcal{A})$ is $M_{\omega_2\omega_1}$-definable.*

(2) *$\mathcal{A}$ has a universal non-equivalence tree if and only if $I(\mathcal{A})$ is determinedly $M_{\omega_2\omega_1}$-definable.*

Proof. (1) If T is a universal equivalence tree of $\mathcal{A}$, then $\phi^T_{\mathcal{A}}$ defines $I(\mathcal{A})$ among models of cardinality $\leq \omega_1$. Conversely, assume $\phi = (T, l)$ defines $I(\mathcal{A})$ among models of cardinality $\leq \omega_1$. To prove that T is a universal equivalence tree of $\mathcal{A}$, suppose $\exists$ wins $EF_{\omega_1}(\mathcal{A}, \mathcal{B}, T)$. Since $\mathcal{A} \models \phi$, we have by Proposition 6.27 that $\mathcal{B} \models \phi$. Hence $\mathcal{A} \cong \mathcal{B}$.

(2) If T is a universal non-equivalence tree of $\mathcal{A}$, then first of all, $\phi^T_{\mathcal{A}}$ defines $I(\mathcal{A})$ among models of cardinality $\leq \omega_1$. Moreover, $\phi^T_{\mathcal{A}}$ is determined in models of cardinality $\leq \omega_1$, for if $\mathcal{B} \not\models \phi^T_{\mathcal{A}}$, then $\forall$ wins $EF_{\omega_1}(\mathcal{A}, \mathcal{B}, T)$, and hence $\mathcal{B} \models \psi^T_{\mathcal{A},\mathcal{B}}$. Conversely, assume a determined $\phi = (T, l)$ defines $I(\mathcal{A})$ among models of cardinality $\leq \omega_1$. To prove that T is a universal non-equivalence tree of $\mathcal{A}$, suppose $\mathcal{B} \not\cong \mathcal{A}$. So $\mathcal{B}$ satisfies the dual of ϕ. Now $\forall$ wins $EF_{\omega_1}(\mathcal{A}, \mathcal{B}, T)$ by following ϕ in $\mathcal{A}$ and the dual of ϕ in $\mathcal{B}$.**Q.E.D.**

So whenever we can find a universal equivalence tree for a model $\mathcal{A}$ of cardinality ω_1, we can find an $M_{\omega_2\omega_1}$-sentence which is an *invariant* of $\mathcal{A}$, i.e., identifies the isomorphism type of $\mathcal{A}$.

Let us now turn to the question, what *cannot* be expressed in $M_{\omega_2\omega_1}$. The most interesting concept undefinable in $L_{\omega_1\omega}$ is the notion of well-ordering. The analogous result for $M_{\omega_2\omega_1}$ is that the class of trees with no uncountable branches is undefinable in $M_{\omega_2\omega_1}$. This fact alone is as central in the study of $M_{\omega_2\omega_1}$ as undefinability of well-order is in the study of $L_{\omega_1\omega}$. The proof we present for this fact is topological. For this it is useful to observe that if Φ is a $PC(M_{\omega_2\omega_1})$-sentence, then the set $\{R \subseteq \omega_1 : (\omega_1, R) \models \Phi\}$ is a $\mathbf{\Sigma}^1_1$-subset of $\mathcal{N}_1$.

Proposition 6.34 (**[Hyt87, Oik88]**) *(CH) The class of trees $(T,<)$ of cardinality ω_1 with no uncountable branches is not $PC(M_{\omega_2\omega_1})$-definable.*

Proof. Suppose Φ is a $PC(M_{\omega_2\omega_1})$-sentence whose models are exactly the trees $(T,\leq)$ which have no uncountable branches. Let $A = \{f \in \mathcal{N}_1 : (\omega_1, \leq_f) \models \Phi\}$. Since Φ is $PC(M_{\omega_2\omega_1})$, A is a $\mathbf{\Sigma}^1_1$-subset of TO. By Proposition 4.11 there is a tree W of cardinality ω_1 with no uncountable branches so that $T_f \leq W$ for all $f \in A$, contradiction. **Q.E.D.**

Proposition 6.35 (**[Hyt87]**) *(CH) For any $PC(M_{\omega_2\omega_1})$-sentence Φ there is a mapping $T \mapsto \Phi^T$ from $\mathcal{T}_{\omega_1}$ to $M_{\omega_2\omega_1}$ so that*

(1) $\models \Phi \rightarrow \bigwedge\{\Phi^T : T \in \mathcal{T}_{\omega_1}\}$.

(2) $\mathcal{A} \models \bigwedge\{\Phi^T : T \in \mathcal{T}_{\omega_1}\} \rightarrow \Phi$ *if $\mathcal{A}$ has cardinality $\leq \omega_1$.*

Proof. The analog of the classical game-representation of $PC(L_{\omega_1\omega})$-sentences or $\mathbf{\Sigma}^1_1$-sets, deriving from Svenonius and Moschovakis, is a game G of length ω_1 of the following kind. If $\mathcal{A} \models \Phi$, then $\exists$ wins G. If $\mathcal{A} \not\models \Phi$ and $\mathcal{A}$ has cardinality $\leq \omega_1$, then $\forall$ wins G. Let G^T be obtained from G by demanding $\forall$ to go move by move up the tree T. If $T \in \mathcal{T}_{\omega_1}$, then the property that $\exists$ wins G^T can be expressed by an $M_{\omega_2\omega_1}$-sentence Φ^T. If $\mathcal{A} \not\models \Phi$, $\mathcal{A}$ has cardinality $\leq \omega_1$, and τ is a winning strategy of $\forall$ in G, then τ gives a winning strategy for $\forall$ even in the game G^T, where T is the tree of all possible sequences (of successor length) of moves of $\exists$ against τ such that $\exists$ has not lost yet. **Q.E.D.**

Proposition 6.36 (**[Hyt90]**) *(CH) Suppose Φ and Ψ are $PC(M_{\omega_2\omega_1})$-sentences so that $\Phi \wedge \Psi$ has no models. Then there is an $M_{\omega_2\omega_1}$-sentence θ so that $\Phi \models \theta$ and $\Psi \wedge \theta$ has no models. (Craig Interpolation Theorem for $M_{\omega_2\omega_1}$)*

Proof. Let $T \mapsto \Phi^T$ be the mapping given by Proposition 6.35. If $\Phi^T \wedge \Psi$ has no models for some $T \in \mathcal{T}_{\omega_1}$, we are done. So let us assume $\Phi^T \wedge \Psi$ has a model for all each $T \in \mathcal{T}_{\omega_1}$. By Proposition 6.29, we may assume these models have cardinality $\leq \omega_1$. But this means that the class of trees $(T,<)$ of cardinality ω_1 with no uncountable branches is $PC(M_{\omega_2\omega_1})$-definable as the class of trees $(T',<')$ of cardinality ω_1 for which there is a tree $(T,<)$, an order-preserving mapping $T' \rightarrow T$, and a model of $\Phi^T \wedge \Psi$. This contradicts Proposition 6.34. **Q.E.D.**

A logic $\mathcal{L}$ satisifes the *Souslin-Kleene Interpolation Theorem* if every $PC(\mathcal{L})$-expression, the negation of which is also definable by a $PC(\mathcal{L})$-expression, is actually explicitly definable in $\mathcal{L}$. It is well-known that $L_{\omega_1\omega}$ satisfies the Souslin-Kleene Interpolation Theorem but $L_{\omega_2\omega_1}$ does not.

Theorem 6.37 ([Hyt90]) *(CH) The smallest extension of $L_{\omega_2\omega_1}$ to a logic which satisfies the Souslin-Kleene interpolation theorem is the largest fragment of $M_{\omega_2\omega_1}$ which is closed under negation.*

One interpretation of Theorem 6.37 is that $L_{\omega_2\omega_1}$ has implicit expressive power which the syntax of the logic is not able to express explicitly. This emphasizes the naturalness of $M_{\omega_2\omega_1}$ as an extension of $L_{\omega_2\omega_1}$. Various extensions of Craig interpolation theorem for $M_{\omega_2\omega_1}$ have been proved in [Tuu] and [Oik].

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Representation of Curves in the Real Plane, and Construction of Curves with Given Topology.

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Abstract.- *We are interested in the following problem: if we are given a topological model for an algebraic curve in the real plane (i.e. something which is isotopic to a certain algebraic curve), what is the minimum degree of a polynomial which 'realizes' it?*

In the particular case of the model being compact and with only double points, a superior bound for the needed degree is $4N + 2K$*, where* N *and* K *represent the numbers of double points and connected components respectively (*[**Sa2**]*), and in the other hand for any* N *and* K *we show examples not realizable with degree lower than* $2N + 2K$.

Here we claim that this later is actually the worst-case optimal superior bound, and we show a method to construct the polynomial with this degree from the topological model, although the proof is not complete.

We introduce the notion of 'prime factors' of a curve (which are the essential components in which the curve can be decomposed) and show that these prime factors have good geometrical properties, which we enclose under the name of 'quasiconvexity'. We also study the problem of combinatorially characterizing the topology of a plane curve, and show a data structure appropiate for this characterization, based on the so-called 'Gauss codes'.

1. Introduction.

If we have two subsets A and B in a topological space X, and a global homeomorphism which sends A to B, we say that (A, X) and (B, X) are *topologically equivalent* or that A and B have the same *topological shape* in X. In the context of real algebraic geometry an interesting question is knowing which are the possible pairs $(V, \mathbb{R}^n)$, or $(V, \mathbb{RP}^n)$ up to topological equivalence, with V an algebraic set.

The answer to this question is far from trivial in the general case (see for example [**BCR**], or [**AK**]), but simple if we restrict ourselves to the real (affine or projective) plane: any imbedded graph in $\mathbb{RP}^2$ or $\mathbb{R}^2$ with even order (possibly zero) in every vertex has the shape of an algebraic set, and conversely any algebraic set $V \subsetneq \mathbb{RP}^2$ has the same shape that an imbedded graph with even order. For $\mathbb{R}^2$ the characterization is the same except that there can be a certain number (finite and even) of branches going to infinity, and thus the algebraic set can be noncompact.

Nevertheless, the classical proofs of this characterization normally use polynomial approximation of C^∞ functions ([**AK**]), and thus say nothing about the degree needed to 'realize' a given topological shape by an algebraic curve.

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In **[Sa2]** we show a construction which works well (both in the projective and the affine plane) if the topological model we want to realize is compact and has only double singular points, and in this case the bound obtained is that every topological model can be realized with degree

$$d \leq 4N + 2K,$$

where N is the number of singular points and K the number of connected components in the topological model. For nonsingular curves this bound gives $d \leq 2K$, which is trivial (for we can construct any non singular model as a product of circles, plus may be a line), but also optimal (if the model consists on K nested ovals it can not be 'realized' by an algebraic curve of degree lower than $2K$, because any line crossing the most inner oval intersects the model $2K$ times).

In the other hand, for any N and K there exist examples of singular curves with N double points which cannot be realized with lower degree then $2N + 2K$, due to topological obstructions:

Let us see first the case of a connected curve, and let N be an arbitrary number of double points. If we construct $N + 1$ circles one inside the next one, two consecutive ones being tangent, the resulting topological model has N double points, and cannot be realized with degree lower than $2N + 2$, because in any realization of it a line passing by the most inner region necessarily cuts the curve in at least $2N + 2$ points. The example generalizes to non connected curves with N double points and K connected components just considering $K - 1$ additional circles inside the inner region and one inside another. We could say even more: for any sequence of numbers $N_1, N_2, \ldots, N_K$, with $\sum N_i = N$, a curve can be constructed with K components each having N_i double points, and not realizable with degree lower than $2N + 2K$ (see figure 1 for an example with 2 connected components and $2 + 3$ double points).

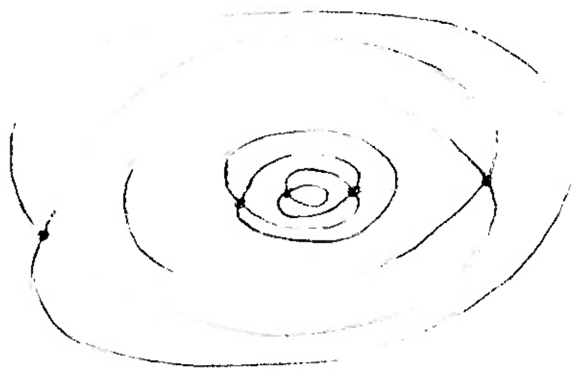

Figure 1

The question is whether these examples are the worst case for each pair of numbers N, K or not. The method discribed in this paper makes us think that they are, i.e. that any compact topological model in the plane with N double points and K connected components can be realized with degree at most $2N + 2K$ (this will be our corollary 7.8); the construction we show would give such a realization, except for some detail that we will remark in section 7 (see conjecture 7.5).

Moreover our results indicate why the examples we have mentioned as worst-cases are indeed worse than others. Proposition 7.7 says that the only connected topological models that possibly need degree $2N + 2$ to be realized are those in which every vertex disconnects the model if we delete it (as it happens in the examples). The rest of connected, double points models can always be realized with degree at most $2N$. (The converse is not true, some models in which every vertex disconnect them can be also realized with low degrees).

In the first part of the work (sections 2 and 3) we abord, as a previous question, the problem of how we can combinatorially characterize the topological shape of a diagram by means of a finite data structure. Such a characterization is necessary if we want to have an algorithm of construction of algebraic curves with given topology, because the data characterizing the shape would actually be the input for the algorithm.

What we show there is a brief summary of some parts of the coauthors respective works **[Go]** and **[Sa]**:

Section 2 is devoted to introduce our topological representations of algebraic curves, (what we call a *diagram* is in fact the topological model we will use to make the constructions), and section 3 introduces the data structure we propose to represent their topological shape, based on the so-called Gauss-codes.

Other authors have given different solutions to this question: **[Roy]**, **[AM]**, **[GT]**, work in a context closely related to ours: they are given a polynomial (or more) and they give algorithms which compute the topological shape of its real zero set, by means of a Cylindrical Algebraic Decomposition (the two formers), and seminumerical root finding methods (the later). Nevertheless they do not have a good representation of topological shapes. Both Gianni-Traverso and Arnon-McCallum represent non singular curves by some data containing the number and mutual disposition of the connected components of the curve (which are in this case either ovals or lines), but say few or nothing for the singular case, while M. F. Roy gives for the singular case a data structure which permits to recover the topological shape of the curves, but which is not an invariant of the shape (in fact it depends even on the cartesian coordinates chosen). This makes very difficult to know if two such structures correspond to the same topological shape or not.

Guibas and Stolfi (**[G-S]**) propose, in the context of Voronoi diagrams, a representation by means of what they call an algebra of edges, representation which could be applied to algebraic curves but seems less appropiate than ours.

The data structure we propose here has the following three good features:

i) it characterizes the topological shape of a diagram (two diagrams with the same code have the same topological shape).

ii) it is a topological shape invariant, up to certain basic operations roughly consisting on permutations of the symbols that compose the code. We can easily compute whether two such codes come from the same topological shape.

iii) it has a good relation with the topology of the curve, in the sense that topological manipulations are well translated to codes.

Sections 4 and 5 deal with the topological manipulations we will need in the algebraic constructions, and give a self-interesting topological result (proposition 5.4) which is that every connected diagram with only double points that cannot be disconnected by cutting only two (different) edges can be put in quasiconvex form (quasiconvexity is defined in 5.1).

Section 6 shows the algebraic realization of diagrams in the general case (in which we do not know how to bound the degrees), and section 7 in the particular case of diagrams with only double points.

2. Curves and diagrams.

In this section we use the word 'curve' in its topological meaning, a (closed) curve being then a continuous map from the standard circle into the real plane.

Definition 2.1 A *diagram* is a finite set of topological curves, i.e., a continuous map f from a topological space X into $\mathbb{R}^2$, where X is a finite, disjoint union of circles. We call *vertices* of the diagram the points of $\mathbb{R}^2$ which have more than one inverse image in X, and *order* of a vertex its number of inverse images. We pose to diagrams the following finiteness conditions: they must have a finite number of vertices, each having finite order.

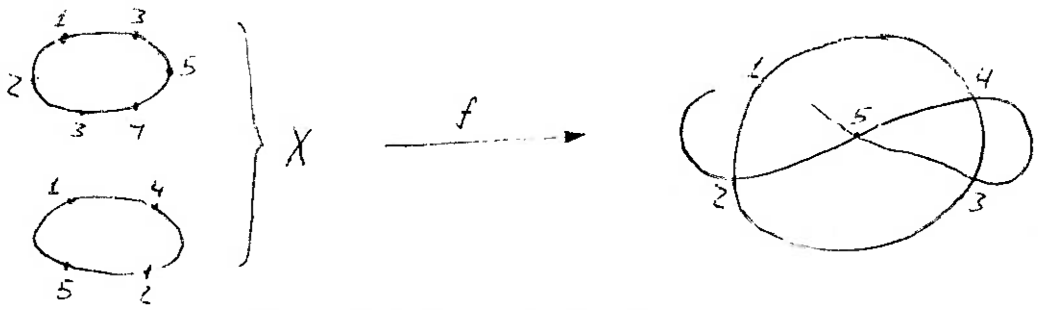

Figure 2: A diagram in the plane.

Sometimes we are going to call diagram not the continuous map but only its image in the plane. With this language flexibility a *diagram* is always a compact subset of the plane and is homeomorphic to a *graph* with even order in all its vertices. Thus every diagram is isotopic to an algebraic curve, and we can consider diagrams as being the topological models of compact algebraic curves: any compact algebraic curve will consist on a (topologically) 1-dimensional part, which is represented by a diagram, and a finite number of isolated points. Isolated points are not important for us, because any isolated point can be algebraically realized by a degree-2 polynomial, and glued into the rest of the curve without affecting the bound $2N + 2K$: each isolated point increases by 2 the degree of the curve, but it also increases by 1 the number K of connected components.

For diagrams we could give a stronger definition of shape than we gave for subsets of the plane, because the $2i$ edges that reach to a given vertex of order i are associated in pairs by the map which defines the diagram, forming what we may call the i (local) branches of the diagram at the vertex (note the analogy with the local branches of an algebraic curve at a singular point). This branches make possible to distinguish between, for example, tangencial and transversal double points, and so we can consider two diagrams whose images are isotopic, not having the same shape as diagrams if their branches do not coincide. A strong definition of shape for diagrams is:

Definition 2.2 Let $f(X)$ and $g(Y)$ be two diagrams in the real plane, f and g being their defining maps. Then we say that f and g have the same *strong topological shape* if there exists an homeomorphism h from the plane into itself such that $h(f(X)) = g(Y)$ and a new homeomorphism t from X into Y such that $h \circ f = g \circ t$.

The condition $h(f(X)) = g(Y)$ is superfluous in the definition, but we include it to make explicit that this new definition of shape is stronger than the old one. Another concept related with the local branches just mentioned is that of transversality:

Definition 2.3 Let $f(X)$ be a diagram in the plane and V be one of its vertices, of order i. We will say that the diagram is *transversal* at V if all the branches of the diagram at V have equal number of edges at each side (this number being necessarily $i - 1$). We say that a diagram is transversal if it is transversal in all its vertices.

If we consider diagrams just as imbedded graphs, thus forgetting that some edges prolong each other, we can not distinguish between transversal and non transversal diagrams, neither between weak and strong topological shape. In fact:

i) Every diagram has the same (weak) shape than one transversal diagram.

ii) Two transversal diagrams have the same weak shape if and only if they have the same strong one.

Coming back to the relation between diagrams and algebraic curves, the above considerations give us two canonical ways to associate a 'diagram structure' to a given compact plane algebraic curve without isolated points: in the first one we follow the algebraic branches of the curve to build the map f, giving a diagram which can be non transversal, and in the second one we cross all the vertices transversally, in the sense of our definition. The second procedure gives a transversal diagram which contains only the topological information of the algebraic curve as a subset of the plane (its weak shape), while the first one contains a part of the algebraic information of the curve: it says which pairs of topological half-branches form the analytical branches of the curve at each singular point (its strong topological shape).

Although the first procedure seems more natural to deal with algebraic curves in this work we are going to adopt the transversal method which has two advantages for our purposes: firstly, it is simpler to deal only with transversal diagrams, and secondly in the algebraic constructions we are going to make we obtain always nondegenerate singular points (which are topologically transversal).

3.Gauss Codes.

In this section we are going to describe the announced characterization of plane curves and diagrams, and see its properties.

The starting point is a coding method for curves described by Gauss (**[Ga]**): Gauss associated to any *normal* curve in the plane (normal means here having only double transversal vertices) the list of the double points of the curve, given in their cyclic order (and thus each double point appearing twice). If we name vertices with the numbers from 1 to N, where N is the number of vertices of the curve. the so obtained Gauss code of the curve is a list containing twice each of the symbols $1, \ldots, N$. Nevertheless it is easily seen that not every list having twice each number from 1 to N is the Gauss code of a curve in the plane (for example the list $(1,2,1,2)$ is not), so Gauss asked what were the necessary and sufficient conditions for such a list being a Gauss code. (He gave the necessary condition of every symbol from 1 to N having exactly an even number of symbols between its two appearances, but this condition proved not to be sufficient. Recent authors have given the complete solutions **[RT]**, **[Ros]**, **[LM]**, **[Go]**. See also **[KMPS]** for a recent survey on Gauss codes).

We can easily generalize Gauss codes to our diagrams considering, instead of one list, as many lists as curves form the diagram, a list consisting on the vertices one crosses when moving along a curve (see figure 3). The set of these lists is the Gauss code of the diagram. Note that one or several of the lists in the diagram can be the empty list, if the associated curve is an oval with no vertices.

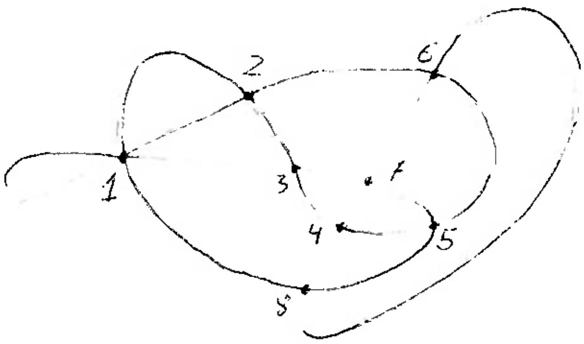

Figure 3: A diagram in the plane with Gauss code (1,2,3,4,5,6,2,1,1,3,7,5,8) (8,4,7,6).

If we want Gauss codes to be an invariant of the shape of a diagram, we must introduce some equivalence relations between codes, because of the arbitrary choices made in the construction process. We say that two codes are equivalent if we can obtain one of them from the other by a finite sequence of operations of the following types:

-Renumbering of the vertices (which corresponds with the choice of the 'names' for the double points).

-Cyclic permutation of the symbols in one of the lists (which corresponds to the choice of an initial point to start the list in each curve).

-Inversion of one of the lists (which corresponds to changing the direction to move along the curve), and

-Reordering of the lists in the code.

Note that if two codes are equivalent, then the maximum number of operations required to obtain one from the other is one of the first and forth types and one of the second and third for each list forming the code, because of the conmutativity of operations of different kinds. This is important because ensures that we can algorithmically construct all the codes which are equivalent to a given one, for example to test whether two diagrams, given by their codes, have the same shape or not.

Gauss codes are now a shape invariant of the diagram up to this equivalence relation. (A strong shape invariant, properly speaking, because diagrams with the same weak shape can have different codes depending on the transversality relations between the branches.) Nevertheless, they do not, in general, characterize the topological shape of a diagram, i.e. the same code has different-shape realizations as a diagram. We need to add some extra information to obtain a shape characterizing code.

We do it as follows: firstly, we choose one of the two possible global orientations of the real plane, and for each vertex of the diagram we number cyclically its $2i$ edges (where i is the order of the vertex), starting by an arbitrary one and following the chosen orientation. Then, we construct the Gauss code of the diagram as we did before, but we include in the code not only the vertex number, but also the edges by which we come in and out of the vertex when moving along the curve. We write the numbers corresponding to these two edges as a subscript and a superscript in the number which represents the vertex (see figure 4).

We call the so constructed code the *extended Gauss code* of the diagram. Again the extended Gauss code is a strong shape invariant up to equivalence of codes if we define two new equivalence operations with codes:

- A cyclic renumbering of the edges in one given vertex, and
- A global orientation change, i.e. an inversion on the cyclic ordering of the edges of all the vertices.

In figure 4 we have chosen clockwise orientation of the plane and we start numbering the edges from the horizontal-right position (as showed).

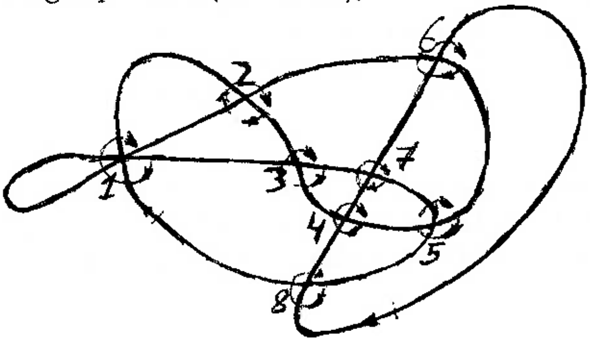

Figure 4. Extended Gauss code: $({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_45^2\ {}_18^3)\ ({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$

Obviously, the extended Gauss code contains much more topological information of the diagram than the non extended one, for it includes the local disposition of the edges of a vertex. Let us see that it permits for example to recover the cycles that form the boundary of the faces (which is not true for the usual Gauss codes). We see this in our example: firstly, we can obtain the edges of the diagram by simply breaking the code into pieces in the following way:

$$[1^5\ {}_32]\ [2^1\ {}_43]\ [3^2\ {}_34]\ [4^1\ {}_35]\ [5^1\ {}_16]\ [6^3\ {}_42]\ [2^2\ {}_61]$$

$$[1^3\ {}_41]\ [1^1\ {}_33]\ [3^1\ {}_37]\ [7^1\ {}_45]\ [5^2\ {}_18]\ [8^3\ {}_21]$$

$$[8^4\ {}_24]\ [4^4\ {}_27]\ [7^4\ {}_26]\ [6^4\ {}_28]$$

Now we can recover the cycle of the edges in the boundary of a face (in the anticlockwise sense) starting by an arbitrary edge, say $[1^5\ {}_32]$ and looking for the edge which has its second vertex (2) with the sub- or superscript which inmediately follows in the clockwise sense. In our example, we must look for a 2^4 or a ${}_42$, and that gives us the edge $[6^3\ {}_42]$. We revert this edge to $[2^4\ {}_36]$, and glue it to the first one to give $[1^5\ {}_32][2^4\ {}_36]$. We look then for a 6_4, and we find $[6^4\ {}_28]$. The cycle follows with $[8^3\ {}_21]$ and ends with $[1^3\ {}_41]$ (because the next edge to this one would be the original $[1^5\ {}_32]$. The obtained cycle is then $[1^5\ {}_32][2^4\ {}_36][6^4\ {}_28][8^3\ {}_21][1^3\ {}_41]$, which is the cycle of the exterior face. We can obtain all the cycles in the same way, the process finishing when every edge has been taken twice. The complete face cycles obtained for our example are listed below:

$$[1^5\ {}_32][2^4\ {}_36][6^4\ {}_28][8^3\ {}_21][1^3\ {}_41] \qquad [2^1\ {}_43][3^1\ {}_37][7^4\ {}_26][6^3\ {}_42]$$

$$[3^2\ {}_34][4^4\ {}_27][7^3\ {}_13] \qquad [4^1\ {}_35][5^4\ {}_17][7^2\ {}_44] \qquad [7^1\ {}_45][5^1\ {}_16][6^2\ {}_47]$$

$$[2^2\ {}_61][1^1\ {}_33][3^4\ {}_12] \qquad [5^2\ {}_18][8^2\ {}_46][6^1\ {}_15] \qquad [8^1\ {}_25][5^3\ {}_14][4^2\ {}_48]$$

$$[8^4\ {}_24][4^3\ {}_23][3^3\ {}_11][1^2\ {}_38] \qquad [1^4\ {}_31] \qquad [1^6\ {}_22][2^3\ {}_51]$$

Yet the extended Gauss code of a diagram does not characterize completely its shape, and this is for two reasons: firstly, from the extended Gauss code we can recover which are the connected components of a diagram (because that is a part of its graph structure), but not how they are mutually disposed in the plane; secondly, even for connected diagrams, the extended Gauss code does not say which are the exterior and interior parts of the diagram.

We can see this second fact more clearly if we consider the one point compactified of the real plane, which is a sphere. Every plane diagram can then be viewed at as a diagram in the sphere, the sphere having one special point which represents the infinity. The Gauss code of the diagram in the sphere can be obtained in the same way as we did in the plane, but the code does not tell us in which region of the diagram is the infinity point placed. The topological shape of the diagram in the plane depends on this isposition of the infinity point respect to it, so the extended Gauss code cannot characterize its shape. Nevertheless, we can say the following (proof can be found in **[Sa)**:]

Proposition 3.1 *The extended Gauss code of a connected diagram characterizes its strong topological shape as a diagram in the sphere, i.e. two given connected diagrams have the same strong topological shape in the sphere if and only if they have the same extended Gauss code (up to the equivalence relation for codes).* ■

What we do then to make codes characterize the strong shape for plane diagrams? First of all, we build the codes asociated to the connected components of the diagram, including in the code something which tells us which is the exterior face of each component (for example the cycle of edges of the exterior face). Then we can build a rooted tree to represent the disposition of the different components, in the same way as **[GT]** do for non singular curves (with each component represented by a node in the tree, and the components which are included in others represented below them), and add to the tree some information saying in which face of the inmediately upper component we must place a given one. This finishes the problem of characterizing the shape of diagrams in the plane .

We are going to mention finally the solution to the original Gauss problem applied to our extended Gauss codes, i.e. the decision of whether a given code is realizable as a diagram in the plane. The solution is very simple and generalizes easily to other surfaces.

Definition 3.2 We call an *(extended) gauss-like code* a sequence of lists globally containing all the symbols $1, \ldots, N$ at least twice, each of the symbols having a subscript and a superscript, and with the sub/superscripts of each symbol $k = 1, \ldots, N$ going from 1 to $2i_k$, where i_k is the number of appearences of k (its order).

First of all a gauss-like code is realizable in the plane if and only if each of its connected components is, so we can restrict ourselves to the connected case. Secondly we recall that we know how to get from an extended Gauss code the edges that form the cycle of a face in the corresponding diagram. In particular, we can find the number of faces, because each face of a connected diagram in the plane has only one cycle of edges. We claim that:

Proposition 3.3 *A connected gauss-like code is realizable in the plane if and only if it satisfies the Euler formula $F - E + V = 2$, where F is the number of faces (cycles of edges) that result from the code, V is the number of vertices, and E is the number of edges, which coincides with the total number of vertex symbols composing in the code (the 'code length').* ■

The proof can again be found in **[Sa]**; necessity of the condition is trivial once we know that the faces of a connected plane diagram are simply connected with the exception of the exterior one which is a ring, while the sufficiency is due to a more general result saying that every gauss-like code can be realized in some compact orientable surface, and the Euler characteristic of the minimal one that realizes a code is given by the stated Euler formula. Thus a code that satisfies the formula can be realized in a sphere, and deleting a point to the sphere, in the plane. Note that the formula is satisfied by our example: $11 - 17 + 8 = 2$.

For more detailed descriptions and proofs, and for a generalization of everything concerning extended Gauss codes to compact surfaces, even in the non-empty boundary and in the non-orientable cases see [**Sa**].

4 Flips and Flops. Prime Diagrams.

We are going to begin now the study of the geometrical manipulations on diagrams that will led to the algebraic constructions (and to some interesting topological results also). From now on, we are going to work only with transversal diagrams, and in some places we will demand them to be connected or to have only double vertices. Being transversal means that we can think on diagrams as being just drawings in the plane, and forget the continuous map from which they are the image (because of the equivalence between shape and strong shape for these diagrams).

Our aim is to give a method to construct any diagram from a collection of simpler diagrams and a sequence of well defined topological operations that transform this simpler diagrams into the one we had. The two basic operations we need are one to delete singular points from a diagram and other to add them. We call this operations *flips* and *flops* respectively:

Definition 4.1 To make a *flip* in a vertex of the diagram we take the $2i$ edges of the vertex and join them two by two in consecutive pairs, thus making the singular point dissappear. This can be made in two possible ways up to isotopy, shown in figure 5.a. Flips can be easily treated with extended Gauss codes: if we have the code for the original diagram, a flip is characterized by the name of the vertex in which we make the flip and some additional information distinguishing the two possible flips.

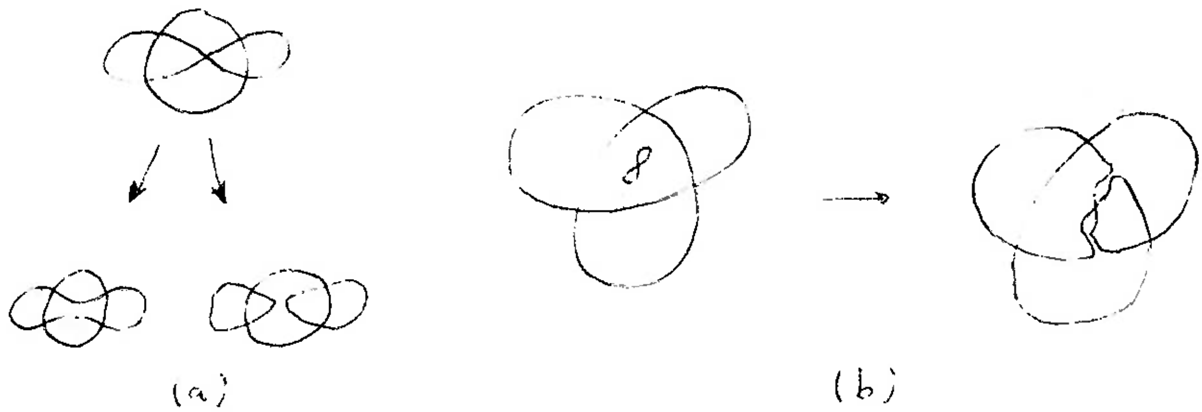

Figure 5

Definition 4.2 *Flops* are the inverse operation of flips. To make a flop we must choose one of the faces of the diagram and a list of some (at least two) of the edges which bound this face. An edge may appear more than once in the list, and the total number i of edges in the list will be called the *order* of the flop. The geometrical flop is made inserting in the chosen face an i-petals flower (as shown in figure 5.b), and then joining each petal to one of the edges.

If the face is simply connected this can be made in only one way up to shape equality; in other case we will need some extra information about the 'paths' along which we must place the 'petals' of the flower. Nevertheless we are only going to be concerned with simply connected faces; note that in a connected diagram in the plane all the faces are simply connected except for the infinity one.

Both flips and flops can be easily made in the Gauss code that represents the diagram. We show with an example the way to find the code of the resulting diagram of a flop from the old one's code. Consider the diagram of section 3, whose extended Gauss code was

$$({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_45^2\ {}_18^3)\ \ ({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$$

and suppose that we want to make a flop in the face $[7^1\ {}_45][5^1\ {}_16][6^2\ {}_47]$, joining the edges $[7^1\ {}_45]$, $[5^1\ {}_16]$, and again $[5^1\ {}_16]$ (the geometrical flop is showed in figure 6).

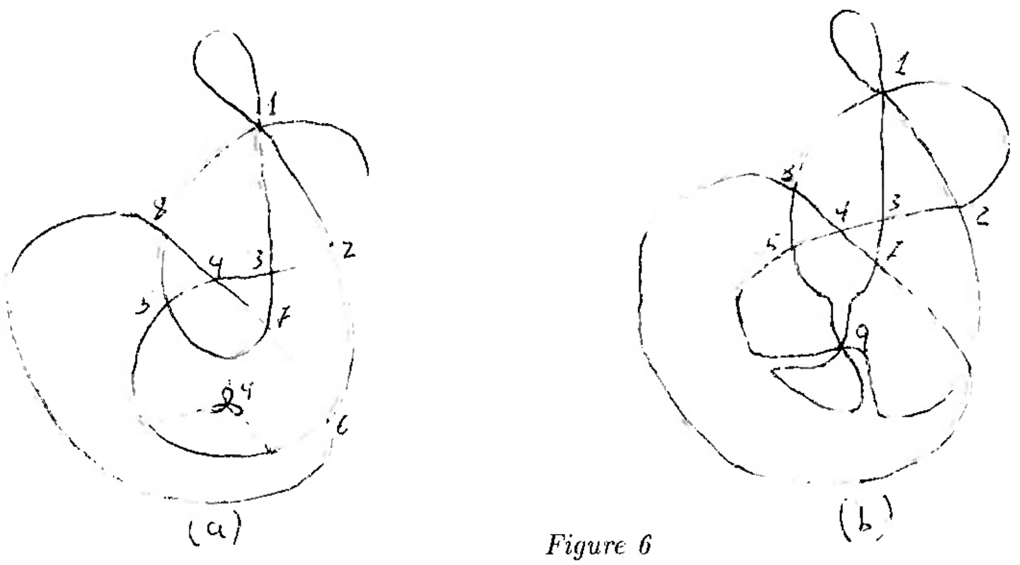

Figure 6

We give the name '9' to the new vertex, and add as many symbols 9 in each edge of the cycle as its number of appearings in the list; this gives the cycle $[7^1\ \mathbf{9}\ {}_45][5^1\ \mathbf{9}\ \mathbf{9}\ {}_16][6^2\ {}_47]$.

We then add the to the new '9' symbols the subscripts $1,\ldots,6$ in decreasing order:

$$[7^1\ {}_6\mathbf{9}^5\ {}_45][5^1\ {}_4\mathbf{9}^3\ {}_2\mathbf{9}^1\ {}_16][6^2\ {}_47],$$

and put these new 'edges' instead of the old ones in the original code:

$$({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_4\mathbf{9}^3\ {}_2\mathbf{9}^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_6\mathbf{9}^5\ {}_45^2\ {}_18^3)$$

$$({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$$

this is the extended Gauss code for a diagram having the required (weak) shape, but which is not transversal. To make it transversal we just break the code in all the appearings of the symbol '9' and reglue the pieces in such a way that each symbol '9' has as subscripts two opposite edges (i.e. two numbers whose difference is 3):

$$({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_49]\ [9^3\ {}_29]\ [9^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_69]\ [9^5\ {}_45^2\ {}_18^3),$$

$$({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$$

and regluing:

$$({}_{2}1^{5}\ {}_{3}2^{1}\ {}_{4}3^{2}\ {}_{3}4^{1}\ {}_{3}5^{1}\ {}_{4}9]\ [9^{1}\ {}_{1}6^{3}\ {}_{4}2^{2}\ {}_{6}1^{3}\ {}_{4}1^{1}\ {}_{3}3^{1}\ {}_{3}7^{1}\ {}_{6}9]\ [9^{3}\ {}_{2}9]\ [9^{5}\ {}_{4}5^{2}\ {}_{1}8^{3}),$$

$$({}_{2}8^{4}\ {}_{2}4^{4}\ {}_{2}7^{4}\ {}_{2}6^{4})$$

i.e.:

$$({}_{2}1^{5}\ {}_{3}2^{1}\ {}_{4}3^{2}\ {}_{3}4^{1}\ {}_{3}5^{1}\ {}_{4}9^{1}\ {}_{1}6^{3}\ {}_{4}2^{2}\ {}_{6}1^{3}\ {}_{4}1^{1}\ {}_{3}3^{1}\ {}_{3}7^{1}\ {}_{6}9^{3}\ {}_{2}9^{5}\ {}_{4}5^{2}\ {}_{1}8^{3}),\ ({}_{2}8^{4}\ {}_{2}4^{4}\ {}_{2}7^{4}\ {}_{2}6^{4})$$

which is the code for the new diagram.

The way in which flips and flops are used to build up diagrams is the following: we make flips to a given diagram D_0 until we arrive to a simpler one D_k, and in each step $i = 1, \ldots, k$ we compute the code of the new diagram obtained by the i^{th} flip D_i, as well as the information to recover the code of D_{i-1} from D_i (i.e. the information concerning the inverse flop of the flip). The shape of the final diagram D_k joint to the inverted sequence of flops determines the shape of D_0. The choice of the 'simpler' diagram D_k to stop the process depends on our purposes, but clearly it is always possible to arrive to a diagram without any vertices (i.e. a collection of ovals), if we want to.

For connected diagrams with only double points this flip/flop decomposition of diagrams is specially useful, because of the following result:

Proposition 4.3 *Let D be a connected diagram in the plane and let V be one of its vertices, of order 2. Then one of the two possible flips in vertex V leaves the diagram connected.*

Proof: Let '1', '2', '3' and '4' represent the four edges in vertex V in a cyclic order, and let's what happens to D when we delete the point V.

If $D \setminus \{V\}$ is connected, then both flips on V are connected. If it is not, each of the four edges at V must be connected in $D \setminus \{V\}$ to another one, because the arc beginning in an 'open edge' of $D \setminus \{V\}$ must end in an open edge.

Now suppose that one of the flips in V gives a diagram which is not connected, for example the flip which joins '1' to '2' and '3' to '4'. Then '1' can only be connected in $D \setminus \{V\}$ to '2' (for otherwise the four edges would be connected to each other in the 'flipped' diagram), and thus the other flip gives a connected diagram (because it connects '1' to '4' and '2' to '3'). A counter example for higher order points is an '*eightfigure*' with an oval crossing its double point (see figure 7). ∎

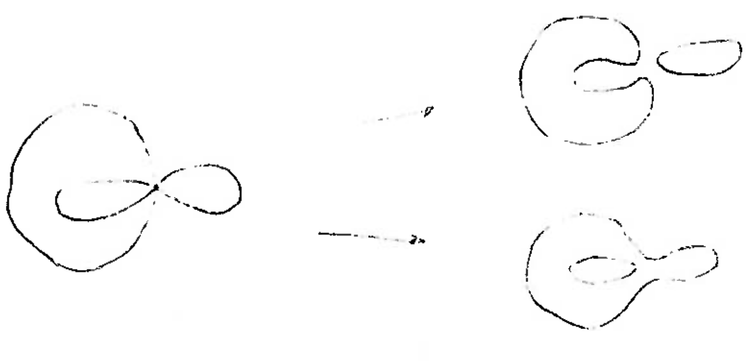

Figure 7

The proposition implies that the final diagram in the flip decomposition of a connected, double points diagram can be always chosen to be connected andnonsingular, and thus a single oval. This fact is used in the construction of algebraic curves with given topology shown in **[Sa2]** (in fact, there the final diagram can be either an oval or a pseudo-line, because the context in which diagrams are defined is more general).

Nevertheless, here we will prefer to use other diagrams instead of ovals to start the construction, and for that we need to introduce a type of connected, double points, diagrams with good decomposition properties, which we call *prime diagrams.* (Prime diagrams can be defined with vertices of higher order, but the good properties we mention are obtained only for double points).

Definition 4.4 Let D be a connected diagram with only double points. We say that D is *prime* if it cannot be disconnected by 'cutting' only two edges, or equivalently if there do not exist two adjacent faces in the diagram which have two different common edges on their boundaries.

The three main features about prime diagrams are:

Proposition 4.5 *Let Z be a (connected, transversal, double points) diagram. Then:*

i) The non-extended Gauss code of Z characterizes its shape in the sphere (compactified plane). Therefore the information added in the extended codes is irrelevant for this diagrams and their plane shape is determined by the Gauss code and the 'infinity face' additional information.

ii) Let V be an arbitrary vertex of Z. Then at least one of the two flips at V gives a new prime diagram.

iii) If a flop gives as final diagram Z, then the flop is made joining two **different** *edges of the initial diagram.*

In (ii) and (iii) the initial diagrams of both the flip and the flop are assumed to have at least one vertex.

Proof: i) The proof of this can be found in **[Go]**. It is too long to put it here, and in fact this property of prime diagrams, although it may be the main one to express the meaning of being prime, is not relevant to our purposes. We indicate just that the reason why non prime diagrams with the same non extended code can have different shapes (in the sphere) is that one of the parts of the diagram can be turned 'inside-out' as in figure 8, and that does not happen for prime diagrams (a prime diagram can be turned inside out as a whole, but that does not change its shape in the sphere).

Figure 8

ii) Consider the following sketch of the two possible flips at V (figure 9). Suppose that (a) is prime and both (b) and (c) are not prime, and we are going to arrive to a contradiction.

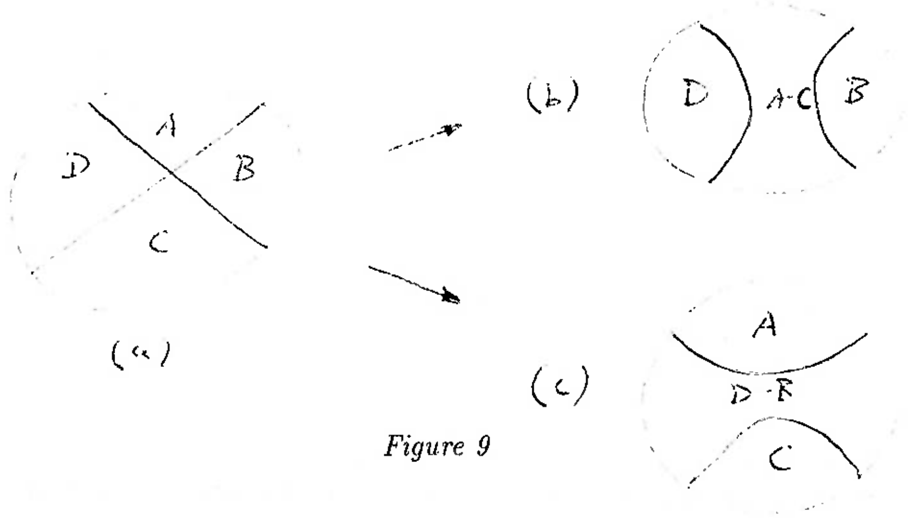

Figure 9

Diagram (b) being not prime, one of the two faces that share two edges must be the middle face ($A \cup C$), for in other case (a) would not be prime. For the same reason the other face cannot be B nor D, so we call it E, E being then a face of the initial diagram which is adjacent to both A and C. With the same considerations for the horizontal flip we obtain another face F adjacent to both D and B, and the following sketch:

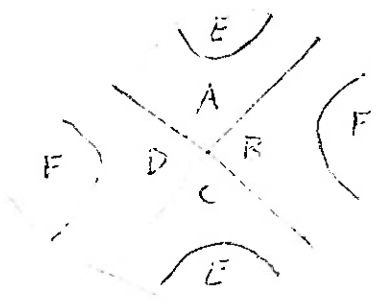

Figure 10

Now the contradiction arrives if we study whether E and F are the same face or different ones: they can not be the same, for in figure 10 we can find a line that goes from E to F crossing the diagram in exactly three points, and if they are different we can find two lines going from E to E and from F to F respectively, and crossing each other in exactly one point. Both things are impossible because transversaly crossing curves in the plane must have an even number of intersections (we recall that our diagram is a finite union of curves).

iii) It is easy to prove in its reciprocal form: an order 2 flop in the same edge of a diagram which has at least one vertex gives a diagram which is not prime. The following picture shows this. The final faces A and B share the two edges a and b, and the existence of at least one initial vertex ensures that a and b are not the same edge, so the final diagram is not prime.

Figure 11

∎

When we have a diagram which is not prime we can 'factorize' it by cutting the two edges that disconnect it and then regluing the pairs of open edges that lay in the same connected component. (This process can be described as a flop in the two edges followed by a flip in the new vertex obtained, as shown in figure 12, and gives as a result two connected diagrams each having at least one of the initial vertices.)

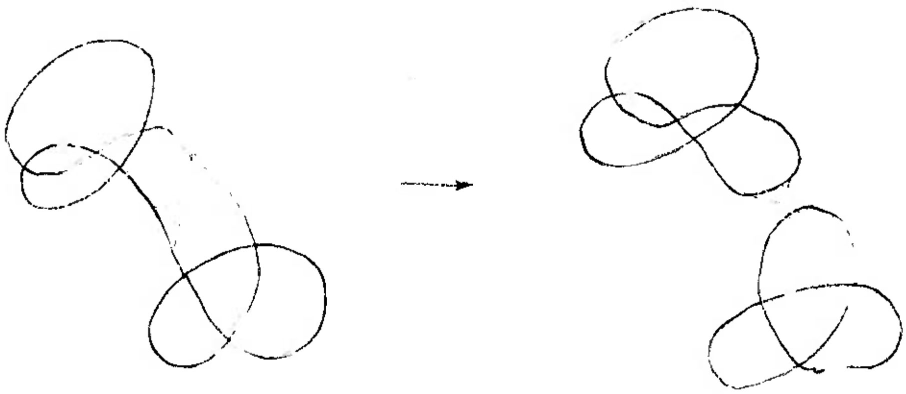

Figure 12: Decomposition of a diagram

The process can be continued with these two new diagrams if they are not prime until we have a finite collection of prime diagrams in the plane which we call the prime factors of the initial diagram.

The factorization of a diagram is not unique, because when we 'reglue' the open edges in pairs we can do it in two different ways (or, equivalently, if we make a flop in the two disconnecting edges followed by a flip on the new vertex, there are two ways to make the flop, one in each of the two faces which share the edges). Nevertheless, these different ways give diagrams which are equivalent as independent diagrams in the sphere, i.e. diagrams with the same extended Gauss codes, but possibly with different disposition respect to one another and to the infinity point. It can be also shown that the factorization does not depend on the order we choose to make the decompositions.

The important point concerning this prime-factors decomposition is that if we know how to realize by an algebraic curve each of the prime factors of the decomposition it is easy to 'reglue' the algebraic prime factors to have a realization of the whole diagram. We will come back to this point in section 7.

In the case of only double points we can refine a little the construction, and section 5 is devoted to prepare this refinement.

5. Quasiconvexity of Prime Diagrams.

Definition 5.1 Let D be a (connected, double points) diagram in the plane. We will say that D is *quasiconvex* if we can choose a point P_e in every edge e of the diagram in such a way that the two following conditions are satisfied:

i) for every face F of D different from the infinity one, the polygon whose vertices are the points P_e, with e the edges in the boundary of F is a strictly convex polygon contained in F and touching its boundary only in $\{P_e\}$.

ii) if an edge e is in the boundary of the infinity face, then a straight line exists passing by P_e and not touching the diagram in any other point (a 'tangency line' on P_e).

In figure 13 we show an example of a quasiconvex diagram. This section is devoted to proof that every prime diagram has the same shape of a quasiconvex one; for non-prime diagrams the result is not true in general, but nevertheless the diagram in figure 13 is not prime.

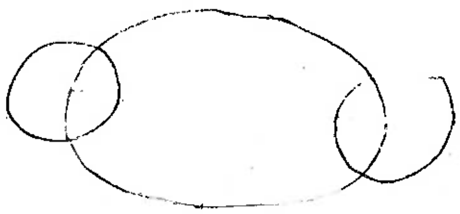

Figure 13: Quasiconvexity.

Lemma 5.2 *Let D be a quasiconvex diagram. Then every flop on D joining two different edges in a face different from the infinity one, can be made in such a way that the resulting diagram is quasiconvex.*

Proof: Let a and b the edges to make the flop, and F the face. We We make the following small peturbations in a (and b):

- if a is an edge not touching the infinity face (an interior edge), we make it to be a straight line segment in a sufficiently small neighbourhood of P_a, without altering the quasiconvexity conditions. The quasiconvexity condition remains then true if we change the point P_a by sufficiently near ones Q_a or R_a in a (figure 14-a).

- if a is one of the edges of the infinity face, we make it to be an 'angle' in P_a, without altering the quasiconvexity conditions. There exist then Q_a and R_a such that the line passing by them is parallel to the tangency line in P_a and does not cross the diagram in any other point (fig 14- b).

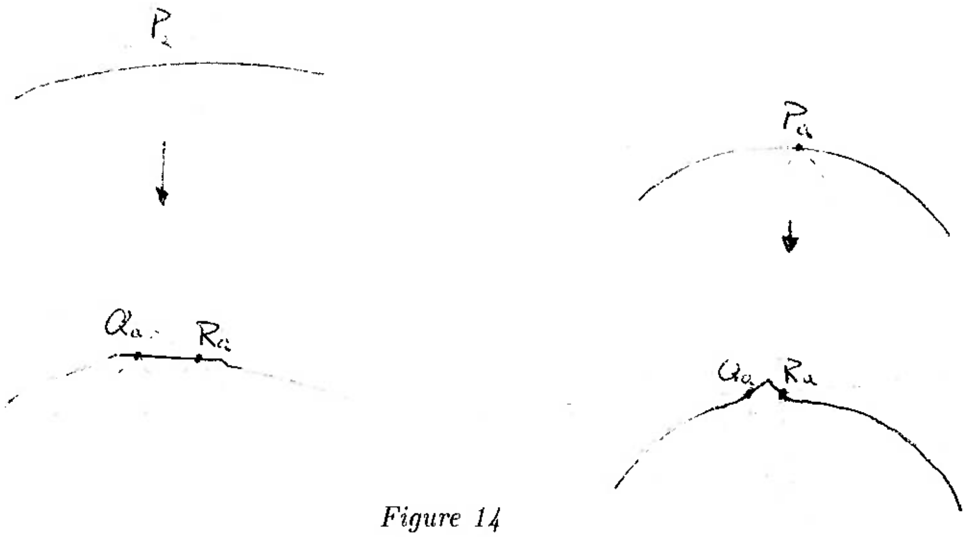

Figure 14

Now, we make the flop joining Q_a to R_b and Q_b to R_a by straight lines, and deletintg the parts of edges a and b between these points (figure 15). The diagram so obtained is quasiconvex: the quasiconvexity condition is automatically verified in the interior faces, and in the exterior one it suffices to modify a little the line passing by Q_a and R_a (or Q_b and R_b) to two lines each passing by one of them and not crossing the diagram in any other point (as in figure 15).

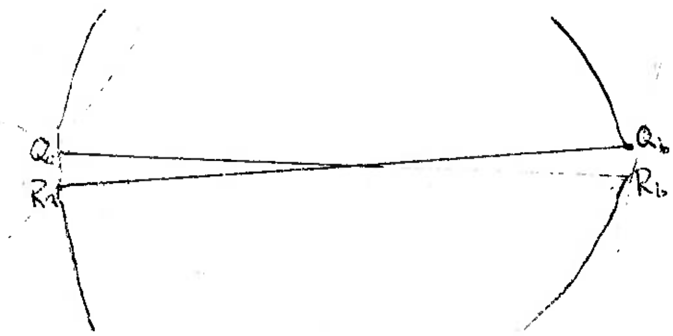

Figure 15

∎

Lemma 5.3 *Every (connected, double points) prime diagram with no interior vertices have one of the following shapes (by an interior vertex we mean a vertex not adjacent to the infinity face):*

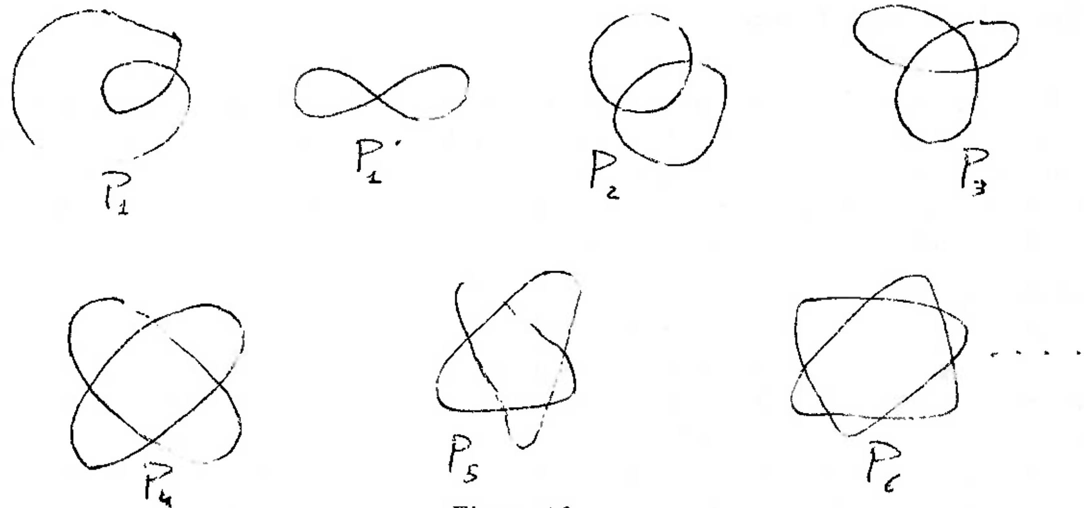

Figure 16

■

Proposition 5.4 *Every (connected, double points) prime diagram in the plane has the same topological shape than a quasiconvex one.*

Proof: Let D be our diagram, and let us prove the result by induction on the number of interior vertices. By lemma 5.3 all diagrams with no interior vertices have the shape of a quasiconvex one.

If D has $N+1$ interior vertices we choose one of them and make a prime flip on it (recall that one of the two flips in a vertex of a prime diagram gives a prime diagram). By induction hypothesis this new diagram has the shape of a quasiconvex one, and by lemma 1 the flop that recovers D from it can be made to give a quasiconvex diagram. We have used here the fact that a flop which produces a prime diagram must be made joining two different edges. ■

The lemmas and proposition 5.4 prove that if D is a prime diagram with only double points a sequence of flops on it can lead to one of the prime diagrams in lemma 5.3 and that the flops that recover the shape of D from this final diagram can be made preserving quasiconvexity. This is going to be the procedure we will use to construct an algebraic curve with the topological shape of a given prime diagram, and moreover let us see that making flops in the interior vertices we are never going to arrive to diagrams P_1 and P_1' of lemma 5.3:

Lemma 5.5 *Let D be a prime diagram (connected, with only double points) with at least one interior vertex. Then, D has at least 2 exterior vertex, i.e. a sequence of flops in its interior vertices cannot lead to the diagrams P_1 nor P_1'.*

Proof: The lemma reduces to proof that there are no prime diagrams with only one exterior vertex and at least one interior vertex.

This is true, because if there is only one exterior vertex, say V, then the cycle of edges of the exterior face has either one only edge $[V,V]$, or two edges, $[V,V][V,V]$. In this second case the diagram can only be P_1', and in the first case it is either P_1 or not prime (the other two edges of V apart from $[V,V]$ are different and disconnect the diagram).

■

6. Algebraic Construction of Curves with Given Shape. The General Case.

In this final section we are going to show how we can use the 'flip-flop' techniques on diagrams to construct an algebraic curve with a given in advance topological shape, and how we can profit of the quasiconvexity properties of prime diagrams to obtain the optimal degree $2N+2$ for the realization of any compact curve in the plane with only double points. We introduce first some well known concepts in algebrain geometry:

Definitions 6.1

-By an *algebraic plane curve* in $\mathbb{R}^2$ we mean a polynomial $f \in \mathbb{R}[X,Y]$, and also its *zero set* $V_f = \{f(X,Y)=0\} \subset \mathbb{R}^2$ when there is no ambiguity in the polynomial we consider to define V_f. It is necessary to remark this because different curves (polynomials), can have the same zero set. We say that a curve f *realizes* a diagram D if the zero set of f is isotopic to D.

-A point $P=(a,b) \in \mathbb{R}^2$ is called a *singular point* of the curve $f(X,Y)$ if $f(a,b) = f_X(a,b) = f_Y(a,b) = 0$ (where f_X and f_Y are the derivatives of f). We consider only curves with a finite number of singular points. (If a plane curve has an infinite number of singular points it means that it has a repeated factor).

-The *order* of a singular point is the least order of a derivative which is not zero in (a,b). If $P=(0,0)$, then the multiplicity of P is the least degre of the monomials of f. For P arbitrary the same thing holds if we develop f around the point P.

An important class of singular points are *nondegenerate* singular points. The general definition for complex curves is well known (see, for example, **[Wa]**), but we give here a slightly different one for the case of real curves:

Definitions 6.2 Let $f(X,Y)$ be a real curve of degree n, and let $P=(a,b)$ be a singular point of f of order m. We can then write $f(X,Y) = f_m(X-a,Y-b) + f_{m+1}(X-a,Y-b) + \ldots + f_n(X-a,Y-b)$, where f_k are homogeneous polynomials of degree k. We will say that P is *real-nondegenerate* (or real-ordinary) if f_m decomposes in m real different linear factors. (Remark: a bivariate homogeneous polynomial always decomposes totally in complex linear factors, here we demand this factors to be different and real. The usual definition of ordinary points demands them only to be different).

We are going to be specially interested in singular points of order 2. The local structre of a real algebraic curve in a neighborhood of such an order 2 point is either that of one order 2 analytic branch (this is the case of a 'cusp') or that of two nonsingular branches crossing at the point, and in this later case these branches can be either both complex or both real. We are going to call *singularities of type* A^- those order 2 singularities which consist on two real analytic branches (the name comes from the terminology used in **[AGV]** to clasify singularities). An example of these A^- singularities (in fact the only one we are going to be concerned with in the constructions) is the product of two curves both passing by a point P which is regular for both of them.

Finally we say that a curve f of degree n *has no points at infinity* if the monomial of highest degree f_n of f has no real zeroes different from the origin (i.e. if the projective curve associated to f has no points on the infinity line of the projective plane $\mathbb{R}\mathbb{P}^2$). Note that if two curves f and g have no points at infinity, then neither the product fg has.

Our construction of algebraic curves is based on perturbation techniques: a perturbation on a polynomial is a small, continuous change in its coefficients. A particular case of a perturbation of a polynomial f is the family of polynomials $f + \epsilon g$, where ϵ is supposed to be a small parameter which varies continuously and g is supposed to be of degree lower or equal to g (due to technical reasons). We call it a *perturbation of f by g* and say that a property is true for *sufficiently small* perturbations of f by g if it is true for every curve $f + \epsilon g$ with $|\epsilon|$ smaller than a certain ϵ_0.

The following result says how a small perturbation of this type affects the topological shape of the polynomial f, in a particular case that will suffice to our purposes. We give it without proof, for it is the affine version of theorem 2.7 in **[Sa2]**, and it can also be deduced from two lemmas in **[Gu]** (the 'lemma on the class of a point' and 'the lemma on isotopy').

Proposition 6.3 *Let f be a real curve with no points at infinity and of degree n, and suppose that the singular points of f are $P_1, P_2, \ldots, P_k$ and $Q_1, Q_2, \ldots, Q_l$, from which the P_i are real-nondegenerate and the Q_i are of A^- type. Let g be a curve of the degree $\deg(g) \leq \deg(f)$ which has a singular point of at least the same order in the points P_i, and which does not pass by the points Q_i.*

Then, any sufficiently small perturbation of f by g of the form $f + \epsilon g$:

- has a real-nondegenerate singular point of the same order in each of the P_i,

-has no other singular point, and no points at infinity, and

-its topological shape in $\mathbb{R}^2$ can be obtained modifying each A^- singularity of f in that of the two ways in figure 17 which is compatible with the signs of f, g and ϵ.

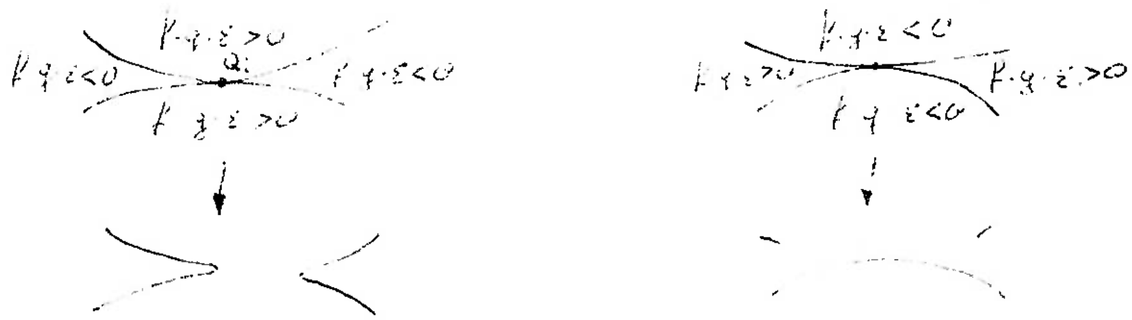

Figure 17

■

Proposition 6.3 is enough to describe the construction of algebraic 'flops', as we do in the proof of the following lemma:

Theorem 6.4 *Let D be a diagram which is realized by an algebraic curve $f(X,Y)$, f having only real-nondegenerate points and no points at infinity. Let D' be a diagram obtained by a flop on D. Then D' can also be realized by an algebraic curve f' with only real-nondegenerate singular points and no point at infinity.*

Proof: Let $P_1, P_2, \ldots, P_{k-1}$ be the singular points of f, which are all real-nondegenerate and with orders $m_1, m_2, \ldots, m_{k-1}$. We can identify D with the zero set of f.

The flop of order m in D is given by one of its faces F, an m-petals flower in the face, and m paths joining the petals with points in the boundary of F.

An m-petals flower can be constructed algebraic by the formula: $R = \cos(mt)$ if m is odd, and $R^2 = \cos(mt)$ if m is even (it is easy to check that these equations define algebraic curves of degrees $m + 1$ and $m + 2$ respectively, and that they have the shape of an m-petals flower, no points at infinity, and their only singular point is real-nondegenerate of order m).

Now we can place the m-petals flower in the face F by translations and homoteties, and call $f*$ the product of f with the polynomial defining the flower.

To make the flop we have to join each petal of the flower to the corresponding points in f, along some given paths. To do this algebraically, we first cover each path with a 'chain of circles' satisfying:

-The first circle is tangent to the point in the petal, the last one to the point in f, and each circle is tangent to the next one.

-The circles in the chains do not intersect each other nor $f*$ in other points than the mentioned tangencies (see figure 18).

(To construct the chains we first put a tangent circle in each of the two extremal points of the path, sufficiently small not to touch $f*$ in other points than the tangency one, and then cover the part of the path not covered by these two circles with a finite number of circles not touching $f*$. If we delete the superfluous circles and reduce the remaining ones to be each tangent to the next one the circles will satisfy the conditions).

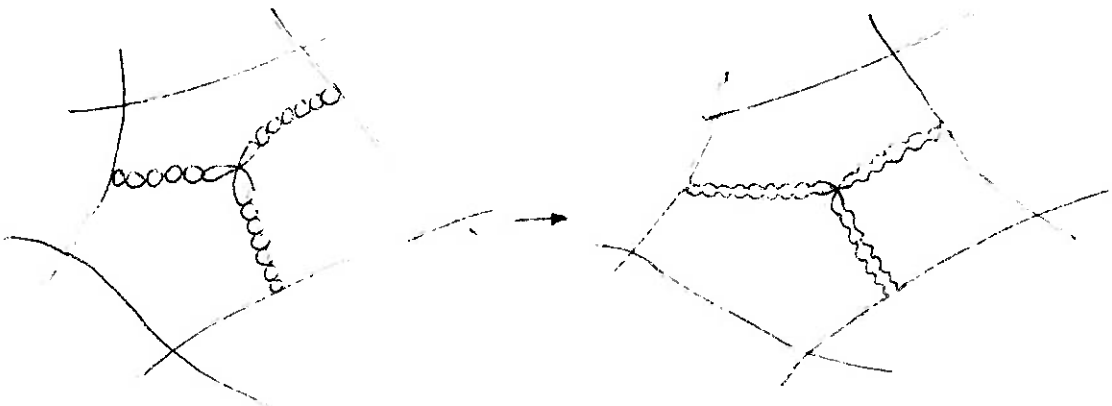

Figure 18

We still call $f*$ The product of $f*$ with all the circles in the chains.

$f*$ satisfies the hypothesis in proposition 6.3, if we call $P_k = P$ (the singular point in the flower), and $Q_1, Q_2, \ldots, Q_l$ the tangency points in the chains of circles. We can moreover suppose that $f*$ has positive sign outside the circles and the flower, and negative inside. We are going to perturb the curve $f*$ to have the same shape than the diagram obtained from the flop:

For each nondegenerate point $P_i = (a_i, b_i), i = 1, \ldots, k$ of f we consider the polynomial $g_i = ((X - a_i)^2 + (Y - b_i)^2)^{m*_i}$, where $m*_i$ is an exponent to make P_i be of order at least m_i in g_i. (It suffices $m*_i \geq m_i/2$), and call g the product of the g_i's. g is everywhere positive (except in the points P_i which are its zeroes), and we can suppose that its degree is smaller than the degree of $f*$ (if it is not we multiply $f*$ by a factor not affecting its zero set, such as $(X^2 + Y^2 + 1)^p$).

In these conditions, proposition 6.3 ensures that for a small positive ϵ the curve $f' = f * -\epsilon g$ realizes the wanted diagram D' of the flop: At each tangency point the deformation compatible with the signs of f, g and ϵ is that which joins the petals of the flower with the original curve f along the chains of circles. ∎

Corollary 6.5 *Every diagram in the plane can be realized algebraically, with only real-nondegenerate points and no points at infinity.*

Proof: By induction on the number of singular points. A diagram with no singular points is a finite collection of ovals which can always be realized by some product of circles. To realize a diagram D_N with N singular points, we make a flip to it, obtaining a diagram D_{N-1} with $N-1$ singular points, and by induction suppose this new diagram realized algebraically by a curve f_{N-1}

with only real-nondegenerate points and no points at infinity, and apply the theorem to the inverse flop to the flip made. That gives a realization of D_N ∎

7. The Case of only Double Points.

Corollary 6.5 gives a constructive proof of the characterization of the possible shapes of compact algebraic sets in the plane: every diagram is realizable as an algebraic set implies that a sufficient condition for something to have the shape of an algebraic set is to be an imbedded graph with even order in all the vertices, and as we mentioned in the introduction this is also an easy to proof necessary condition for an algebraic compact set in the plane. But more interesting that this is that this kind of construction permits us to think in controlling the degree of the curves we use to realize a diagram. In theorem 6.4 and corollary 6.5 this is not possible because we do not know a priori how to bound the number of circles needed in the chains of circles for the flop. Nevertheless if we restrict ourselves to diagrams having only double points we can refine the construction thanks the 'prime factors decomposition' of diagrams and the 'quasiconvexity' properties of prime diagrams.

For algebraic curves we need a slightly different definition of quasiconvexity than for diagrams:

Definition 7.1 We say that an algebraic curve f (connected, with only double points) is *quasiconvex* if its zero set is quasiconvex (in the sense of definition 5.1 for diagrams), and moreover the points P_e in the esterior edges of the curve are not flexes.

Note that the exterior condition of quasiconvexity, in the case of algebraic curves, implies that the tangent line to the curve at points P_e in the exterior edges does not have any other intersections with the curve. The additional assumption of the points P_e not being flexes (i.e. having finite curvature) implies that a sufficiently big circle tangent to the curve at P_e has the same property: it does not intersect the curve in any other point. This will be used in the next proposition to 'glue' the quasiconvex algebraic realizations of the prime factors of a diagram. The proposition is true for any number of prime factors, but we proof it for 2 factors, for the sake of simplicity.

Proposition 7.2 *Let D be a connected diagram with only double points which decomposes in two factors D_1 and D_2 such that D_1 and D_2 are one outside another, or D_2 inside D_1, (but not the converse). Suppose that D_1 and D_2 are realized by two algebraic curves f_1 and f_2 with only real-nondegenerate singular points (of order 2), and no points at infinity. Suppose also that $d = \deg(f_1) + \deg(f_2) \geq 2N$, where N is the number of double points in D, and that f_2 is quasiconvex. Then D can be realized by an algebraic curve f of degree d.*

Proof: In any of the two dispossitions of D_1 and D_2 (D_2 inside D_1 or one outside another), to recover D from D_1 and D_2 we need only to place a copy of D_2 in the appropiate face of D_1 (which would be the exterior face if D_1 and D_2 lie one outside another), and join them by the appropiate edges.

Let us then do that with the algebraic curves f_1 and f_2 which realize D_1 and D_2. We can put the curve f_2 in the appropiate face of f_1 by translations and homotecies, which do not affect its degree. Let e_1 and e_2 be the edges by which we must join f_2 to f_1, and let P_1 and P_2 be the points of the quasiconvexity conditions in these edges. e_2 is an exterior edge, and thus we can construct a big circle tangent to the curve f_2 at P_2 and containing the whole curve f_2. We can also construct a small circle tangent to f_1 at P_1 and contained in the appropiate face of f_1 (because P_1 is regular), and by some rotations, translations and homotecies in f_2 make the two circles coincide, and identify the points P_1 and P_2, which become a tangency point between f_1 and f_2 (see figure 19).

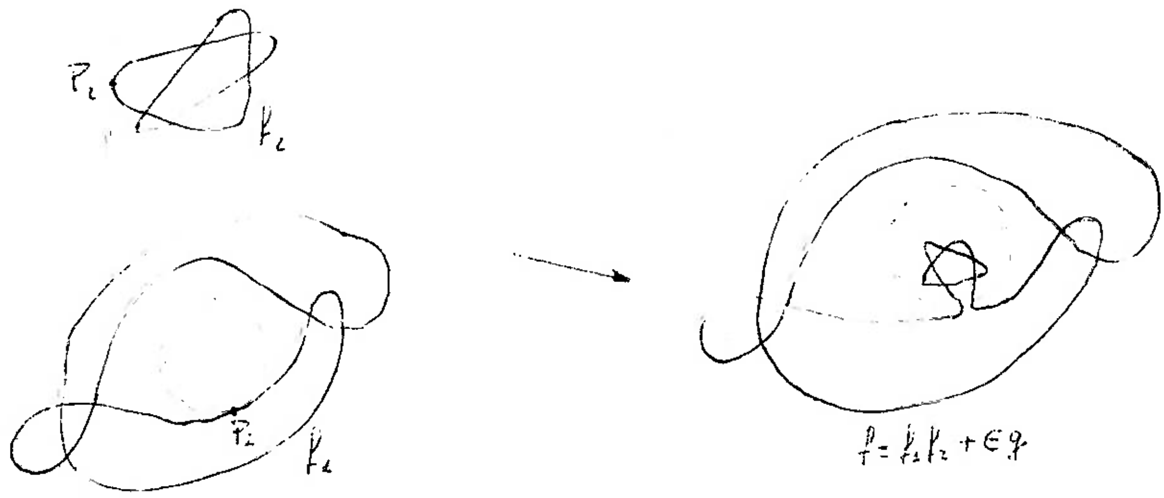

Figure 19

Consider the product f_0 of f_1 and f_2 under these conditions. It has degree $d \geq 2N$, and N order 2, real-nondegenerate singular points, which correspond to the double points of D, plus another singular point which is also of order 2 and type A^-: the tangency point. Moreover, it has the same shape of the diagram D except for the tangency point. Thus, we want to perturb it to make the tangency disappear, and it is easy to do this thanks to proposition 6.3:

f_0 satisfies the conditions of the proposition, and we can take as perturbing curve the product g of the factors $g_i = (X - a_i)^2 + (Y - a_i^2)$, (where (a_i, b_i), with $i = 1, \ldots, N$ are the coordinates of the singular points). g has degree $2N \leq d = \deg(f_0)$ and has a singular point of order 2 at each of the real-nondegenerate singular points of f_0 (which are also of order 2). Then, the perturbed curve $f = f_0 + \epsilon g$, with ϵ sufficiently small has the same shape of f_0 except for the tangency which disappears and thus, with the adequate sign for ϵ, the shape of D. ∎

Proposition 7.2 (generalized to any number of prime factors) will permit to realize any connected diagram in the plane by an algebraic curve of controlled degree if we know how to realize its prime factors by a quasiconvex curve. To realize the prime factors we are going to make use of their quasiconvexity properties, but we must make note that in the induction process we describe, we have not a proof that quasiconvexity can be preserved by the perturbations made (see conjecture 7.5). Thus the induction hypothesis is not ensured, and thus 7.6, 7.7 and 7.8 are true only if the conjecture is.

We start realizing the 'basic' prime diagrams, which are prime diagrams with no interior points:

Lemma 7.3 *Every prime diagram with only double points and no interior vertices can be realized by a quasiconvex algebraic curve of degree 2N (where N is the number of exterior vertices), with no points at infinity and only real-nondegenerate singular points, except for P_1 and P_1', which can be realized with degree 4 (and the same properties).*

Proof: : We recall lemma 5.3 which said that the only possible prime diagrams without interior vertices where the P_1, P_1' and the P_i, for $i = 2, \ldots$; we will show the algebraic construction for each of them:

P_1 is realized by the lemniscata $(X^2+Y^2)^2 = X^2-Y^2$, and P_1' can be constructed perturbing the product of two circles which intersect transversally (we consider one of the intersection points as real-nondegenerate and the other one as of type A^- to apply proposition 6.3), as shown in figure 20, and that gives degree 4. The quasiconvexity properties needed are easily verified.

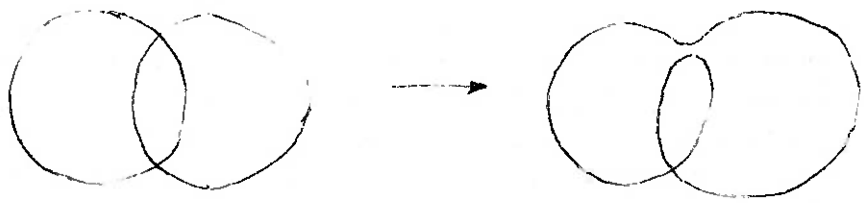

Figure 20

To realize the rest of the P_i we use the following procedure, which we describe only for P_5: we consider 5 different ratii of the unit circle from the origin, and find the 5 circles which are tangent to two consecutive ones in the points where the ratii touch the circle (two consecutive such circles are tangent to one another, as in figure 21.a). We call f the product of the 5 circles, with positive sign at the origin and at infinity. f has clearly degree 10 and no points at infinity, and we are going to perturb it by the curve $g = (X^2+Y^2-1)^2$, which is positive everywhere except in the unit circle. This perturbation is not included in proposition 6.3, because g passes by the singular points of f which are degenearte, but it is easy to describe its effect on the curve:

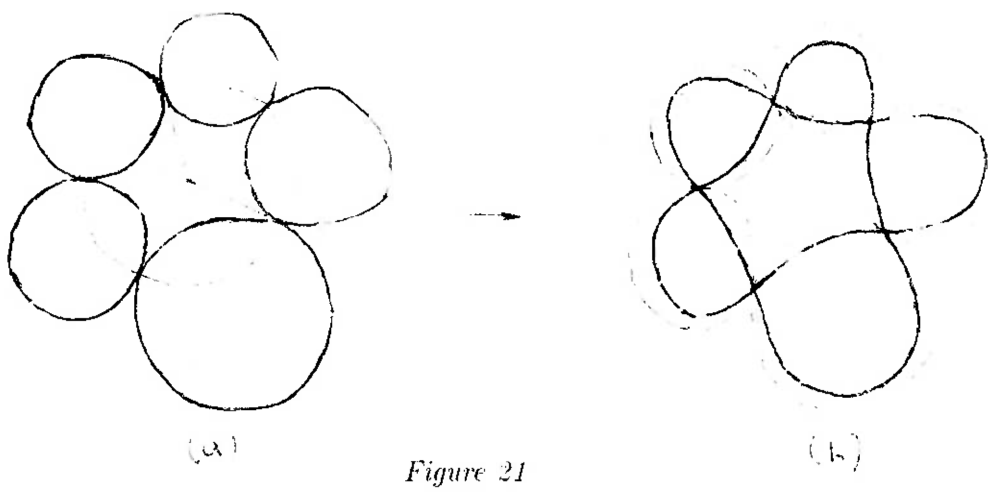

Figure 21

First, the perturbed curve $f+\epsilon g$, for sufficiently small, positive ϵ, must be included in the interior of the 5 circles (because these are the regions in which f and g have opposite sign), and is locally isotopic to f in its non singular points. It rests only to see what happens at the tangency points of the circles. If we translate one of this points to be the origin, and rotate the figure until the tangency is horizontal, then the terms of lower degree of f and g are $f = -Y^2 + \ldots$, and $g = X^2 + \ldots$, and thus $f + \epsilon g = -Y^2 + \epsilon X^2 + \ldots$, which corresponds to two real branches with tangents $Y = \sqrt(\epsilon)X$, i.e. to a real-nondegenerate order 2 point, which gives the shape of figure 21.b. The quasiconvexity properties are automatically satisfied, as shown in the figure. ∎

Now let us see how two add the interior double points to the realized diagrams:

Lemma 7.4 *Let D be a prime diagram with N vertices, which are of order 2, and with at least one interior vertex V. Let D' be a prime diagram obtained by a flip on D at V, and suppose that D' is realized by a quasiconvex algebraic curve f' with only real-nondegenerate double points and no points at infinity. Then, D can be realized by an algebraic curve of degree $2N$ with only real-nondegenerate singular points and no points at infinity.*

Proof: All we have to do is an algebraic flop on f' to recover the initial D, and we are going to do this by a similar process as made in proposition 6.4. The difference now is that we can profite the quasiconvexity properties of f'.

Let a and b be the edges of f' in which we must make a flop to recover the shape of D, and let P_a and P_b be the points of the quasiconvexity definition in the edges a and b. Then, the face for the flop (the only face which has a and b in its boundary, for f' is prime) must be an interior face of f', because the vertex of D in which we made the flip was interior. Then, by quasiconvexity, there exists a convex polygon with vertex at P_a and P_b inscribed in the face, and in particular the segment P_aP_b is contained in the face. Moreover P_a and P_b are regular points in f' (they are not vertices), and this implies that an ellipse can be constructed being tangent to f' at P_a and P_b, and sufficiently close to the segment P_aP_b to be contained in the face (see figure 22.a). What we want to do is to perturb the product of f' with this ellipse in the way shown in figure 22.b.

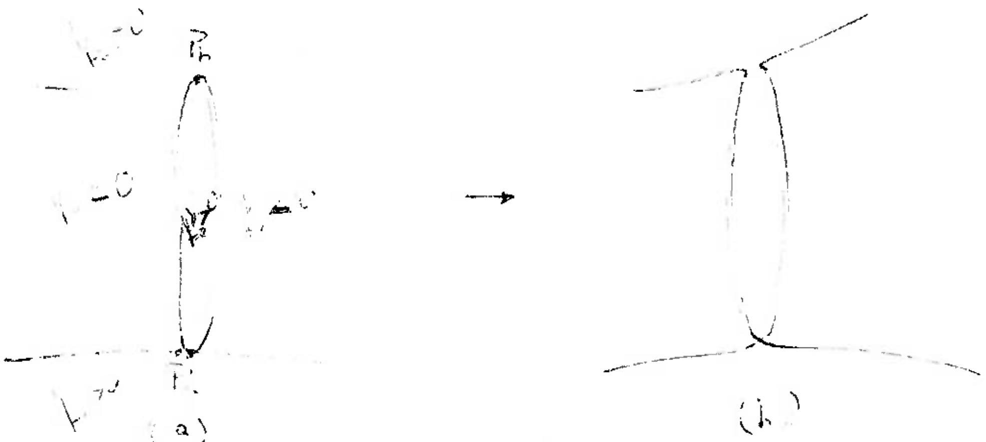

Figure 22

Call f_0 this product, $P_1, \ldots, P_{N-1}$ the singular points of f', and suppose that P_a is placed at the origin, with horizontal tagent, and with the signs dispossition for f_0 shown in figure 22.a.

For a perturbation of type $f_0 + \epsilon g$ (with ϵ small and positive) to have the shape of figure 22.b (and not to change anything elsewhere), we need a polynomial g with degree at most $2N$ (the degree of f_0), with a singular point at each of the $P_1, \ldots, P_{N-1}$ (this ensures that the real-nondegenarate singular points of f' are preserved by the perturbation), with positive sign at P_b (to break this tangency in the appropiate way) and with a singular point at P_a, such that the singularity of $f + \epsilon g$ at P_a be real-nondegenerate.

The last condition is achieved if g has no terms of degree lower than 2 and its degree 2 term is $-X^2$, and this is achieved, for example, if $g = g_1 g_2$, with g_1 positive at P_a and $g_2 = (X^2 + Y^2 + rX)(X^2 + Y^2 - rX)$, i.e. the product of two circles vertically tangent at P_a, where the radius r is chosen sufficiently small to not interfere with the rest of the figure (i.e. such that g_2 is positive at P_b and at every P_i).

Now the conditions for g_1 are only to be positive at P_a and P_b, to have a singular point at each $P_i, i = 1, \ldots, N-1$, and to be of degree at most $2N-4$ (because g_2 is of degree 4.

f' has at least two vertices (in fact, at least two exterior vertices), and we can suppose without loss of generality that P_a and P_b are not in the line passing by P_1 and P_2, because the quasiconvexity properties are also satisfied if we move a little P_a and P_b along the edges a and b of f'. Thus, we can take as g_1 the square of this line, times a factor $(X-x_i)^2+(Y-y_i)^2$ for each $i = 3, \ldots, N-1$, where (x_i, y_i) are the coordinates of the points P_i. This g_1 has degree exactly $2N-4$, and is everywhere positive except in the points P_i and in the line passing by P_1 and P_2.

That ends the construction of the algebraic flop. By a perturbation theorem similar to proposition 6.3, the curve $f = f_0 + \epsilon g$ has the same shape than the diagram D, has no points at infinity and its N singular points which correspond to the N double points of D are real-nondegenerate. ∎

We would like to use lemma 7.4 inductively to construct every prime diagram, but we do not know how to preserve the qusiconvexity conditions in the flop. Thus we state a conjecture, and give only a partial proof:

Conjecture 7.5 *In the conditions of lemma 7.4 the final curve f can be constructed quasiconvex.*

Proof: In fact, the only quasiconvexity conditions that we cannot ensure to be true are those concerning only the four new edges that appear from a and b by the flop. The rest are preserved, because the quasiconvexity conditions are 'open' in the sense that they remain true if we perturb a little the points in the edges or the edges themselves, and this perturbation is made 'smoothly' (i.e. varying continuously not only the points but also the slopes).

In the special case that the edges a and b have their curvature towards the outside at points P_a and P_b, the quasiconvexity conditions of the new four edges are also preserved, if the ellipse joining P_a and P_b is chosen sufficiently narrow: this is so because, in this case, there exists a rectangle with vertices in the edges a and b, close to P_a and P_b (as in figure 22.a), and this rectangle construction is preserved by the perturbaton, if the ellipse is contained in the rectangle and the perturbation is small (see figure 22.b).

Figure 23

If the curvatures at P_a and P_b are towards the inside this construction is not possible, but possibly with a sufficiently narrow ellipse quasiconvexity is still preserved.

∎

Corollary 7.6 *If conjecture 7.5 is true, then every prime diagram with only double points can be realized by a quasiconvex algebraic curve of degree $2N$ (where N is the number of vertices), with no points at infinity and only real-nondegenerate singular points, except for P_1 and P_1', which can be realized with degree 4 (and the same properties).*

Proof: The proof is made by induction on the number of interior vertices. Lemma 7.3 gives the proof for 0 interior vertices, and for a diagram D with at least one interior vertex V, we make to D a flip at V, obtaining a new diagram D', which can be supposed prime, by proposition 4.5(ii). Besides, lemma 5.5 ensures that D' is not P_1 nor P_1', so by induction hypothesis we can suppose D' realized by a quasiconvex algebraic curve f' of degree $2N-2$, with no points at infinity and $N-1$ real-nondegenerate order-2 singular points.

Lemma 7.4 enables us to construct the curve f with degree $2N$, only real-nondegenerate points and no point at infinity, and by conjecture 7.5 we can suppose that f is also quasiconvex. ∎

Finally we state the general theorem about the construction of real algebraic compact curves in the real plane:

Theorem 7.7 *We suppose that conjecture 7.5 is true. Let D be a connected diagram with N vertices, all of order two. If at least one of the prime factors of D is not P_1 nor P_1' then D can be realized by an algebraic curve of degree $2N$, with only real-nondegenerate singular points and no points at infinity. If not, D can be realized in the same conditions with degree $2N+2$.*

Proof: For prime diagrams the theorem is already proved (corollary 7.6), and for non prime diagrams we use the same techniques of proposition 7.2: we decompose D in its prime factors D_i, and realize each by a quasiconvex algebraic curve of degree $2N_i$, where N_i is the number of double points in D_i; this can be done by proposition 7.6, except if the factor is a P_1 or a P_1' (we will treat this case separately).

Now, proposition 7.2 gives a procedure to reglue all the prime factors one by one and gives as final degree the sum of the degrees needed to realize the factors, that is $2N$, where N is the total number of double points in D, the only thing to take care of is that to use proposition 7.2 we must first realize the most exterior prime factor of D (or one of the most exterior ones, if there are more than one), and then glue the others from the exterior to the interior.

When there is some P_1 or P_1' factor this procedure would not give degree $2N$, because these prime factors can only be realized with degree 4, and they add just one singular point. Nevertheless we can 'glue' them in another, equivalent way: insert a tangent circle in the appropiate face of the curve, and then perturb the tangency (which is a singular degenerate point) to be real-nondegenerate (this can be done in the same way we did in the proof of lemma 7.4). With this procedure each P_1 or P_1' factor increases the degree only by 2, and than the final degree $2N$ is mantained.

The only case in which this cannot be done is if all the prime factors of D are P_1 or P_1', for in these case we need degree 4 to realize the first prime factor, and thus the final degree becomes $2N+2$ instead of $2N$. ∎

Corollary 7.8 *(If conjecture 7.5 is true) every diagram in the plane with only double points can be realized with degree lower or equal to $2N+2K$, where N and K are the numbers of double points and connected components, respectively.*

Proof: Let $D_1, \ldots, D_K$ be the connectec components of D. Theorem 7.7 permits to realize each D_i with degree at most $2N_i+2$, where N_i is the number of double points in D_i. Realizing all of them and then placing them in the appropiate place from one another we will have the desired curve realizing D (the product of the curves realizing the connected components), whose degree will be at most $\sum(2N_i+2) = 2N+2K$. ∎

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Continuous sums of squares of rational functions

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0. Introduction. Suppose R is a real closed field, with the usual, order topology; let K be a subfield, with the inherited order. Let $X := \{X_1, \ldots, X_n\}$ be indeterminates. We call $f \in K[X]$ *positive semidefinite* (psd) over R if $\forall x := (x_1, \ldots, x_n) \in R^n$, $f(x) \geq 0$. Hilbert's 17th problem [15], solved by Artin [1] in 1927, was to prove a conjecture which logicians in the fifties ([13], [16], [22]) refined into the following theorem:

$$\text{if } f \in K[X] \text{ is psd over } R, \text{ then } f = \textstyle\sum p_i r_i^2, \text{ with } 0 \leq p_i \in K \text{ and } r_i \in K(X). \quad (0.0.1)$$

For general background on the 17th problem, including Kreisel's 1962 question [17] on whether various 'continuously varying' versions of (0.0.1) are possible, see, for example, [4] and [11]. The purpose of this note is to prove (0.1) that the p_i and the coefficients of the r_i may be chosen to be functions b of the coefficients $c := (c_1, \ldots)$ of f, depending only on n and $d := \deg f$, and having the following special *sup-inf-polynomially definable* (SIPD; also called 'semi-polynomial') form: $b(c) = \sup_k \inf_l h_{kl}(c)$, where the h_{kl} are finitely many polynomials in $\mathbf{Z}[c]$. As an immediate corollary, we get a proof of the following conjecture, first stated in our thesis of 1980, and repeated in [3], [5], [6], and [7]: the b can be chosen to be continuous and even 'piecewise-polynomial.' By [4], 'piecewise' cannot be dropped when $d \geq 4$ (*cf.* [8] on the impossibility of analytic variation, as well).

To state the result precisely, we introduce some definitions and notation. Now let f be the general polynomial of degree d in X with coefficients C:

$$f = \sum_{|\theta| \leq d} C_\theta X^\theta \in \mathbf{Z}[C; X],$$

where $\theta := (\theta_1, \ldots, \theta_n) \in \mathbf{N}^n$, $|\theta| = \sum_{j=1}^n \theta_j$, $C := \{\, C_\theta \mid |\theta| \leq d \,\}$ is a set of $m := \binom{n+d}{n}$ indeterminates, and $X^\theta = X_1^{\theta_1} \cdots X_n^{\theta_n}$. Write

$$P_{nd} := \{\, c \in R^{\binom{n+d}{n}} \mid f(c; X) \text{ is psd in } X \text{ over } R \,\}. \quad (0.0.2)$$

It is well known[2] that one can construct finitely many $h_{kl} \in \mathbf{Z}[C]$ such that

$$P_{nd} = \bigcup_k \bigcap_l \{\, c \in R^{\binom{n+d}{n}} \mid h_{kl}(c) \geq 0 \,\}. \quad (0.0.3)$$

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[2] See [3], [4], [5], or the following summary: (a) By Tarski's elimination of quantifiers (e.g., [20]), P_{nd} is ($\mathbf{Z}$-)semialgebraic ('s.a.'), i.e., equal to a (finite) Boolean combination of sets of the form $\{\, c \in R^m \mid h(c) > 0 \,\}$, where $h \in \mathbf{Z}[C]$. (b) P_{nd} is closed, either by noting that the limit of a sequence of psd polynomials is psd, or by noting that $R^m \setminus P_{nd}$ is obviously open, since if $f(c; x) < 0$, then for all $c' \in R^m$ close enough to c, $f(c'; x) < 0$. (c) The 'finiteness theorem' for (closed) s.a. sets (e.g., [3]) says that every closed, s.a. set can be written as in (0.0.3) above; i.e., we do not need *strict* inequalities. (The fact that the coefficients of the h_{kl} can be taken to be integers is well known, though rarely stated.)

Define functions h_k and h from $R^{\binom{n+d}{n}}$ to R by $h_k(c) = \inf_l h_{kl}(c)$ and $h(c) = \sup_k h_k(c)$; then

$$\begin{aligned} P_{nd} &= \bigcup_k \{ c \in R^{\binom{n+d}{n}} \mid h_k(c) \geq 0 \} \\ &= \{ c \in R^{\binom{n+d}{n}} \mid h(c) \geq 0 \}. \end{aligned}$$

Let B be the smallest ring of functions from $R^{\binom{n+d}{n}}$ to R containing h, the h_k, and $\mathbf{Z}[C]$ (where, in the latter, formal polynomials are identified with the functions which they determine). Since the set SIPD is actually a *ring* (i.e., closed under sums and products; *cf.* [7] or [14], for example), $B \subseteq$ SIPD. Note also that B depends on the particular choice of h_{kl} satisfying (0.0.3).

Theorem 0.1 (Main Theorem). *There is an identity*

$$f(c;X) = \sum_i p_i(c) \left(\frac{q_i(c;X)}{s(c;X)} \right)^2, \text{ where} \tag{0.1.1}$$

(a) the $p_i \in B$ are—not necessarily distinct—products of the functions 1, h, $h - h_k$, and $h_{kl} - h_k$, and hence $p_i \geq 0$ on P_{nd} (obvious meaning),

(b) $q_i, s \in A := B[X]$, and $\exists e \in \mathbf{N}$ such that $q_1 = f^{2e}$ and $s(c;x) = f(c;X)^{2e} + \sum_j t_j(c) g_j(c;X)^2$, where $g_j \in A$, and the $t_j \in B \subseteq A$ are—not necessarily distinct—products of the functions h, $h - h_k$, and $h_{kl} - h_k$, which are all ≥ 0 on P_{nd}; thus $s(c;x) = 0$ only if $f(c;x) = 0$ for $(c;x) \in P_{nd} \times R^n$; hence, by (0.1.1),

(c) each $p_i(q_i/s)^2$ extends (by 0) to a semialgebraic,[3] locally uniformly (even locally Lipschitz) continuous function $P_{nd} \times R^n \to R$.

We shall prove (0.1) in §1. In §2 we shall contrast this proof with the proof given in [9].

I thank G. Kreisel and A. Prestel for their interest and help.

1. Proof of (0.1) (à la Prestel). We shall need

Lemma 1.1 ((1.4) of [20]). *Let G be a ring (commutative, with 1, as always), and let S be a 'pre-ordering' of G, i.e., a subset of G such that $S + S \subseteq S$, $S \cdot S \subseteq S$, $G^2 := \{ g^2 \mid g \in G \} \subset S$, and $-1 \notin S$. Then S can be extended to an 'ordering' T of G, i.e., a pre-ordering such that $S \subseteq T$, $T \cup -T = G$, and the ideal* $\operatorname{supp} T := T \cap -T$ *is prime.* Q.E.D.

Identify the A in (0.1)(b) with a subring of its localization

$$A_f := \{ a/f^k \mid a \in A \ \& \ k \in \mathbf{N} \}.$$

[3] A function from one s.a. set (footnote 2(a)) to another is called *semialgebraic* if its graph is a s.a. subset of the product space.

in the usual way. Let S_f be the subsemiring of A_f generated by

$$A_f^2 \cup \{\, h,\ h - h_k,\ h_{kl} - h_k \mid \text{all } k, l \,\}.$$

Note that

$$\prod_l (h_{kl} - h_k) = \prod_k (h - h_k) = 0 \in A \subseteq A_f, \tag{1.1.1}$$

by the definition of h_k and h after (0.0.3).

Lemma 1.2. *There exist $s_1, s_2 \in S_f$ such that $(1 + s_2)f = 1 + s_1$.*

Proof of 1.2: Otherwise, $f - 1 \notin S'_f := S_f - fS_f$. Since $1 - f \in S'_f$, and S'_f is closed under multiplication, we conclude that $-1 \notin S'_f$. Thus S'_f is a pre-ordering of A_f. By (1.1) with $G = A_f$ and $S = S'_f$, S'_f extends to an ordering T_f of A_f. Write $\overline{A_f} = A_f / \operatorname{supp} T_f$, ordered by $\overline{T_f} := \{\, \overline{a} \mid a \in T_f \,\}$. $\overline{T_f}$ induces an ordering on the field L of fractions of $\overline{A_f}$. Let M be the real closure of L.

First, $\overline{f} < 0$, since $-f \in S'_f \subseteq T_f$ and $f \notin \operatorname{supp} T_f$ (prime!). Second, for all k, l,

$$\overline{h} \geq 0,\ \ \overline{h} \geq \overline{h_k},\ \text{ and } \overline{h_{kl}} \geq \overline{h_k},$$

by the choice of S_f. Third, for each k, since $\prod_l \left(\overline{h_{kl}} - \overline{h_k}\right) = \overline{0}$ (1.1.1) and $\overline{A_f}$ is an integral domain, we conclude that for some l, $\overline{h_{kl}} = \overline{h_k}$, whence

$$\inf_l \overline{h_{kl}} = \overline{h_k};$$

likewise,

$$\sup_k \overline{h_k} = \overline{h};$$

hence

$$\sup_k \inf_l \overline{h_{kl}} = \overline{h} \geq 0.$$

Since f and the h_{kl} are polynomials, $\overline{f} = f\left(\overline{C}; \overline{X}\right)$ and $\overline{h_{kl}} = h_{kl}\left(\overline{C}\right)$. Thus, re-indexing the C_θ's, we see that the statement

$$\exists b_1, \ldots, b_m, y_1, \ldots, y_n \in M\, [f(b_1, \ldots; y_1, \ldots) < 0 \ \&\ \sup_k \inf_l \{h_{kl}(b_1, \ldots)\} \geq 0],$$

which is easily seen to be elementary, is true (namely, take $b_i = \overline{C_i}$ and $y_j = \overline{X_j}$). By Tarski's transfer theorem (e.g., [20]), it remains true if we replace M by any other real closed field; we choose the field R, yielding a point $(b, y) \in P_{nd} \times R^n$ such that $f(b, y) < 0$, a contradiction. This proves (1.2). Q.E.D.

Returning to the proof of (0.1), for $u = 1, 2$, write the s_u given by (1.2) in the form a_u / f^{2n_u}, where $n_u \in \mathrm{N}$ and $a_u \in S :=$ the subsemiring of A generated by

$$A^2 \cup \{\, h,\ h - h_k,\ h_{kl} - h_k \mid \text{all } k, l \,\}.$$

Write $e = n_1 + n_2$. Multiplying the equation $(1+s_2)f = 1+s_1$ by f^{2e} gives

$$(f^{2e} + a_2 f^{2n_1})f = f^{2e} + a_1 f^{2n_2}.$$

Since $b_1 := a_2 f^{2n_1}$ and $b_2 := a_1 f^{2n_2}$ both lie in S, we get

$$f = \frac{f^{2e}+b_2}{f^{2e}+b_1} = \frac{f^{2e}+b_2}{f^{2e}+b_1} \cdot \frac{f^{2e}+b_1}{f^{2e}+b_1} = \frac{f^{4e}+b_3}{(f^{2e}+b_1)^2} = \sum_i p_i(c)\left(\frac{q_i(c;X)}{s(c;X)}\right)^2,$$

where also $b_3 \in S$, $s = f^{2e} + b_1$ (proving (0.1)(b)), and the p_i are as in (0.1)(a).

Finally, for (0.1)(c), the semialgebraicity of the extension of $p_i(q_i/s)^2$ is obvious ((0.1)(c), footnote 3). And the only points $(c;x) \in P_{nd} \times R^n$ at which the (locally Lipschitz) continuous extendibility of $p_i(q_i/s)^2$ is in question are those where $s = 0$; by (b), also $f = 0$; by (0.1.1), each $p_i(q_i/s)^2$ tends to 0 near $(c;x)$ (compare [18]); the fact that this pointwise continuity of the extension is actually locally Lipschitz follows from the corresponding property for f. Q.E.D.

2. Comparison of §1 with [9]. We proved (0.1) in 1988; that original proof will appear in [9]; the proof presented in §1 above contains some modifications on the earlier proof, due to Prestel, which we now explain.

In the seventies, several versions of the 'Nichtnegativstellensatz' for the polynomial ring $A := K[X]$ were discovered; they all give weighted sum-of-squares representations of any polynomial which is nonnegative on a 'basic s.a.' set. In the eighties, both the statement and proof of this were generalized to arbitrary rings A (*cf.*, e.g., [19]); in these abstract versions, ring elements are considered as functions on the real spectrum $\mathrm{Spec}_r A$ of A, and now a function must be nonnegative on a 'basic constructible' set in order to have a sum-of-squares representation. It is then (almost) straightforward to apply these abstract versions to the particular ring A mentioned in (0.1)(b); this was our original approach [9] to proving (0.1).

Actually, only the underlying set (as opposed to the topology) of $\mathrm{Spec}_r A$ is used in the abstract Nichtnegativstellensatz, and this set goes back to Lemma 1.4 of Prestel's 1975 book [20] (replicated here as Lemma 1.1 above); there he gave the modern definition of an 'ordering' of A (but with different vocabulary), and $\mathrm{Spec}_r A$ is just the set of all orderings of A. Furthermore, Lemma 1.2 above is based on Theorem 5.10 in [20]; the latter applied only to polynomials $\in K[X]$. So the main difference between Prestel's proof in §1 above and the proof in [9] is that the former replaces explicit use of the real spectrum and the abstract Nichtnegativstellensatz with Lemmas 1.1 and 1.2, respectively. (In addition, Prestel showed me some helpful simplifications of my original proof; these are incorporated in both §1 and [9].)

The differences in the 2 proofs of (0.1) have some bearing on the directions in which (0.1) can be extended. (1) In [2], Prestel asked whether (0.1) extends to higher even powers; cf. the abstract in [21] when $n = 1$. (2) From the proof in [9] we extract a few examples and general results on the real spectrum of partially ordered rings and (almost) f-rings; one surprise is that the (SIPD) absolute value function $c_1 \mapsto |c_1|$, while obviously psd on R, is not psd when considered as an abstract function on $\mathrm{Spec}_r A$. We also generalize (0.1) in 2 directions: first, we prove Stellensätze for arbitrary SIPD functions (generalizing the Stellensätze for polynomials), and then we obtain 'SIPD-varying' Stellensätze in all cases where continuous variation is possible (improving Scowcroft's continuous, *semialgebraically* varying Stellensätze [23]).

While mentioning different proofs of (0.1), we should mention also that in 1991 Laureano González-Vega and Henri Lombardi, together, re-discovered (0.1); *cf.* the historical note in my joint paper with them [10], and their other paper [12] on this subject. The main difference between their proof and that in [9] is that theirs reduces it to the (concrete) Nichtnegativstellensatz (for polynomials), and not the abstract version thereof.

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GROUPES FAIBLEMENT RÉTICULÉS

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Dans cet exposé, nous allons nous occuper des groupes semi-ordonnés et faiblement réticulés. Il s'agit des notions qui généralisent celles de groupes (partiellement) ordonnés (*po*-groupes) et de groupes réticulés (*l*-groupes) de cette manière que la transitivité n'est pas, en général, exiger. Donc les notions de semi-ordre et de treillis faiblement associatif sont fondamentales pour cette situation.

On appelle <u>treillis</u> <u>faiblement</u> <u>associatif</u> (<u>*wa*-treillis</u>) une algèbre $A = (A, \vee, \wedge)$ avec deux opérations binaires telles que

1. $\forall a \in A;\ a \wedge a = a\ ,\ a \vee a = a$
2. $\forall a,b \in A;\ a \wedge b = b \wedge a\ ,\ a \vee b = b \vee a$
3. $\forall a,b \in A;\ a \wedge (a \vee b) = a\ ,\ a \vee (a \wedge b) = a$
4. $\forall a,b,c \in A;\ ((a \wedge c) \vee (b \wedge c)) \vee c = c\ ,\ ((a \vee c) \wedge (b \vee c)) \wedge c = c\ .$

(Cette notion a été introduite par E.Fried [1] et H.L.Skala [5].)

On peut définir la relation binaire "≤" sur A en posant

$\forall a,b \in A;\ a \le b \Leftrightarrow_{df} a \wedge b = a \quad (a \vee b = b)$. Alors on a:

5. $\forall a \in A;\ a \le a$
6. $\forall a,b \in A;\ a \le b\ \&\ b \le a \Rightarrow a = b$
7. $\forall a,b \in A \exists d \in A;\ (a \le d\ \&\ b \le d)\ \&\ (\forall u \in A;\ (a \le u\ \&\ b \le u) \Rightarrow d \le u))$
8. $\forall a,b \in A \exists e \in A;\ (e \le a\ \&\ e \le b)\ \&\ (\forall v \in A;\ (v \le a\ \&\ v \le b) \Rightarrow v \le e))\ .$

On a aussi invérsement: Si une relation "≤" vérifie les propriétés 5-8, alors, pour $a \vee b = d$ et $a \wedge b = e$, on obtient l'algèbre $(A, \vee, \wedge)$ qui est un treillis faiblement associatif.

Une relation "≤" vérifiant les conditions 5 et 6 est dite une relation de semi-ordre sur A et $(A,\leq)$ est un ensemble semi-ordonné (*so*-ensemble). Un ensemble semi-ordonné est dit un tournoi (ou un ensemble totalement semi-ordonné) si quels que soient deux éléments $a,b\in A$ sont comparables, i.e. si pour quels que soient $a,b\in A$ on a $a\leq b$ ou $b\leq a$.

Un système $G=(G,+,\leq)$ est appelé un groupe semi-ordonné (*so*-groupe) si $(G,+)$ est un groupe, $(G,\leq)$ est un *so*-ensemble et

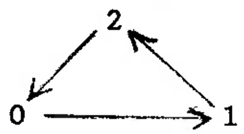
$$\forall a,b,c,d\in G;\ a\leq b \Rightarrow c+a+d\leq c+b+d\ .$$

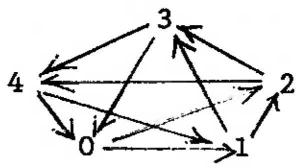
Si $(G,\leq)$ est un *wa*-treillis, alors $(G,+,\leq)$ est un groupe faiblement réticulé (*wal*-groupe). Si $(G,\leq)$ est un tournoi, $(G,+,\leq)$ est un groupe totalement semi-ordonné (*to*-groupe).

Remarque. On sait que si un groupe admet un ordre (transitif) linéaire (i.e. il est un O-groupe), alors il est sans torsion, et que, dans le cas des groupes abéliens, cette condition est aussi suffisante. E.Fried [3] a démontré que la classe de groupes admettants des semi-ordres totaux est essentiellement plus large que celle de O-groupes.

Par exemple, un groupe avec torsion admet un semi-ordre total si et seulement si il ne contient aucun élément d'ordre 2.

Exemple 1. Pour les groupes cycliques finis $\mathbb{Z}_n$, n impairs, on a les semi-ordres totaux "naturels":

$\mathbb{Z}_3$: 

, $\mathbb{Z}_5$: 

, etc.

Soit G un *so*-groupe. Notons $G^+ = \{x\in G;\ 0\leq x\}$. G^+ est appelé le cône positif de G. Il est claire qu'on a:

Proposition 1. a) Si $(G,+,\leq)$ est un *so*-groupe, alors G^+ est un sous-ensemble invariant de G tel que $a \in G^+$ et $-a \in G^+$ entraînent $a=0$, pour quel que soit $a \in G$.

b) Si $(G,+)$ est un groupe et P un sous-ensemble invariant avec 0 de G, alors $(G,+,\le)$, où $a \le b$ ssi $b-a \in P$, est un *so*-groupe et $G^+ = P$.

Nous voyons immédiatement qu'il est possible caractériser les *po*-groupes dans la classe de *so*-groupes en utilisant les propriétês des cônes positifs.

Proposition 2. Un *so*-groupe $(G,+,\le)$ est un *po*-groupe si et seulement si G^+ est un sous-demi-groupe de $(G,+)$.

Soit G un *so*-groupe et $\emptyset \ne A \subseteq G$. On dit que A est convexe si $a \le x$ et $x \le b$ entraînent $x \in A$, quels que soient $a,b \in A$ et $x \in G$. (Notons que a et b ne sont pas necéssairement comparables.) Le cône positif n'est pas, en général, convexe. (Par exemple, dans $\mathbb{Z}_3$, on a $1<2$, $2<0$, et $1,0 \in \mathbb{Z}_3^+$, mais $2 \notin \mathbb{Z}_3^+$.) Pour G avec G^+ convexe, on obtient:

Proposition 3. Si G est un *so*-groupe tel que G^+ est convexe dans G, alors G satisfait l'une ou l'autre des possibilités:

a) G est un *po*-groupe.

b) $\exists a,b \in G$; $0<a$, $a<b$, $0 \parallel b$.

Corollaire. Si G est un *to*-groupe, alors les conditions suivantes sont équivalentes:

a) G est un *o*-groupe (i.e. ordonné linéairement).

b) G^+ est convexe dans G.

c) Il n'existe pas d'éléments $a,b \in G$ tels que $0<a$, $a<b$, $b<0$.

Remarque. La définition de *wal*-groupe est essentiellement plus faible que celle de *l*-groupe, mais, tout de même, beaucoup de propriétés fondamentales des *l*-groupes restent conservées aussi pour les *wal*-groupes. Par exemple: Soient G un *so*-groupe et $a,b,c,d \in G$. Alors:

- Si $b \vee c$ existe, alors $(a+b+d) \vee (a+c+d)$ existe, et $a+(b \vee c)+d =$

$(a+b+c)\vee(a+c+d)$.

- Si $a\vee b$ existe, alors $-a\wedge -b$ existe, et $-a\wedge -b=-(a\vee b)$.
- Si $a\vee b$ existe, alors $a\wedge b$ existe, et $a\wedge b = b+(-(a\vee b))+a$.
- G est un *wal*-groupe ssi $\forall a\in G$, $a\vee 0$ existe.
- Si $a\wedge b$ existe, et c,x,y sont des éléments de G tels que

$$a=x+(a\wedge b),\ b=y+(a\wedge b),\ c=a-b,$$

alors

$$x\wedge y=0,\ x-y=c,\ x=c\vee 0,\ y=-c\vee 0.$$

- Si pour $G=(G,+,\leq)$, $(G,\leq)$ est un $\vee$-*wa*-demi-treillis (i.e. il satisfait les conditions 5-7), alors G est un *wal*-groupe ssi

$$\forall a,b,c\in G;\ a+(b\vee c)+d = (a+b+d)\vee(a+c+d)\ .$$

- Tout *wal*-groupe est engendré par son cône positif G^+, plus précisement $G=G^+-G^+$.

Nous voyons que les *wal*-groupes constituent une variété (i.e. une classe primitive) d'algèbres de type ⟨2,0,1,2⟩ avec deux opérations binaires "+" et "∨", une opération 0-aire "0", et une opération 1-aire "-(.)".

<u>Exemple 2</u>. On peut considérer le groupe $G=(\mathbb{Z},+)$ comme un *wal*-groupe en posant (de manière assez naturelle) $G^+=\{0,1,2,4,6,...\}$. Alors pour $x\in G$:

a) $x\in G^+ \Rightarrow x\vee 0=x$

b) $-x\in G^+ \Rightarrow x\vee 0=0$

c) $x\notin G^+$ et $-x\notin G^+ \Rightarrow x\vee 0 = \max\{x,0\}+1$, où $\max\{x,0\}$ est considéré dans l'ordre naturel de $\mathbb{Z}$.

Donc $(G,+,\leq)$, où '≤" est le semi-ordre défini par G^+, est un *wal*-groupe qui n'est pas ni *l*-groupe ni *to*-groupe.

<u>Remarque</u>. a) Notons que pour un *wal*-groupe G, G^+ ne doit pas être un $\vee$-*wa*-demi-treillis. Par exemple, dans le *wal*-groupe G de l'exemple 2, $1\in G^+$, $4\in G^+$, mais $1\vee 4=5\notin G^+$.

b) Tout *l*-groupe G est, comme un treillis, distributif, i.e.

$$\forall a,b,c\in G;\ a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)\ .$$

Mais cette identité n'est pas vérifiée, en général, dans tous les *wal*-groupes. Par exemple, pour $(\mathbb{Z}_3,+)$ totalement semi-ordonné par

$0<1$, $1<2$, $2<0$, on a

$$0\wedge(1\vee 2) = 0\wedge 2 = 2\ , \quad (0\wedge 1)\vee(0\wedge 20 = 0\vee 2 = 0\ .$$

Néanmoins, on a:

Proposition 3. Si G est un *wal*-groupe et $a,b,c\in G$, alors

$$(a\vee c=b\vee c\ \&\ a\wedge c=b\wedge c) \Rightarrow a=b\ .$$

(Il est bien connu que pour les treillis, cette propriété est équivalente à la distributivité.)

On peut maintenant caractériser la classe de *l*-groupes dans la classe de *wal*-groupes en termes de distributivité:

Proposition 4. Un *wal*-groupe G est un *l*-groupe si et seulement si

$$\forall a,b,c\in G;\ a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)\ .$$

Il est aussi possible de caractériser les *l*-groupes dans la classe de *wal*-groupes en utilisant la notion de l'orthogonalité.

Des éléments a et b de G^+ (G est un *wal*-groupe) sont dits othogonaux si $a\wedge b=0$. (On écrit $a\perp b$.)

Proposition 5. Pour un *wal*-groupe G, les conditions suivantes sont équivalentes.

a) G est un *l*-groupe.

b) $\forall a,b,c\in G;\ a\perp b\ \&\ c\geq 0 \Rightarrow a\wedge c=a\wedge(b+c)$.

c) $\forall a,b,c\in G;\ a\perp b\ \&\ a\perp c \Rightarrow a\perp(b+c)$.

Soit maintenant $(G,+,\leq)$ et $(G',+,\leq)$ des *so*-groupes. Une application $\varphi: G\to G'$ est dite un *so*-homomorphisme si φ est un homomorphisme des groupes et des *so*-ensembles (i.e. $a\leq b \Rightarrow \varphi(a)\leq\varphi(b)$). Un *wal*-homomorphisme d'un *wal*-groupe $(G,+,\leq)$ dans un *wal*-groupe $(G',+,\leq)$ est un *so*-homomorphisme qui est en plus un homomorphisme des *wa*-treillis.

Proposition 6. Soient $G=(G,+,\leq)$ un *so*-groupe et A un sous-groupe distingué de G. Alors A est le noyau d'un *so*-homomorphisme si et seulement si A est convexe.

Dans ce cas, on peut munir le quotient de G par A, G/A, de la relation de semi-ordre: $x+A\leq y+A \Leftrightarrow_{df} \exists a\in A;\ x+a\leq y$. (Le semi-ordre induit.)

Soient G un *wal*-groupe et A un sous-groupe de G. Alors on dit que A est un *wal*-sous-groupe de G si A est un *wa*-sous-treillis de *wa*-treillis $(G,\leq)$. Un *wal*-sous-groupe convexe distingué A de G est dit un *wal*-idéal de G s'il satisfait la propriété:

(*) $\forall a,b\in A$ et $x,y\in G$ tels que $x\leq a, y\leq b$, il existe $c\in A$ tel que $x\vee y\leq c$.

Il est claire que le noyau d'un *wal*-homomorphisme est un *wal*-sous-groupe distingué convexe. On a:

Proposition 7. Soit A un *wal*-sous-groupe distingué convexe d'un *wal*-groupe G. Alors les conditions suivantes sont équivalentes:

a) A est le noyau d'un *wal*-homomorphisme de G dans un *wal*-groupe G'.

b) A est un *wal*-idéal de G.

c) (**) $\forall a,b,c\in A, x,y\in G;\ x\leq a,\ y\leq b \Rightarrow (x\vee y)\vee c\in A$.

On appelle sous-groupe solide de G tout *wal*-sous-groupe convexe vérifiant la condition (**).

On va noter $\mathcal{L}(G)$ l'ensemble des *wal*-idéaux et $\mathcal{C}(G)$ l'ensemble des sous-groupes solides d'un *wal*-groupe G. Il est évident que $\mathcal{L}(G)$ et $\mathcal{C}(G)$, ordonnés par inclusion, forment des treillis complets, avec l'élément minimum $\{0\}$ et avec le maximum G, et que les infima sont égaux aux intersections dans les deux treillis.

Soient G un *wal*-groupe et $H\in\mathcal{C}(G)$. Considérons les conditions suivantes:

(1) Si $x,y\in G$ et $0\leq x\wedge y\in H$, alors $x\in H$ ou $y\in H$.

(2) Si $x,y\in G$ et $x\wedge y=0$, alors $x\in H$ ou $y\in H$.

(3) $G/_1H$ (l'ensemble des classes de gauche de G par H) est un

tournoi.

(4) $\{A\in\mathcal{C}(G);\ H\subseteq A\}$ est un ensemble linéairement ordonné.

(5) Si $A,B\in\mathcal{C}(G)$ et $A\cap B=H$, alors $A=H$ ou $B=H$.

Théorème 8. Si H est un sous-groupe solide d'un *wal*-groupe G, alors

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Rightarrow (5) .$$

Un sous-groupe solide H d'un *wal*-groupe G vérifiant les conditions (1), (2) et (3) est dit sous-groupe redressant. Un sous-groupe solide H de G est dit sous-groupe premier si H vérifie la condition (5).

Remarque. a) Si $H\in\mathcal{L}(G)$, alors H est redressant si et seulement si G/H est un groupe totalement semi-ordonné.

b) Il est bien connu que pour les sous-groupes solides d'un *l*-groupe, toutes les conditions (1) - (5) sont équivalentes. Mais pour les *wal*-groupes, en général, cela n'a pas lieu, parce qu'il existe des sous-groupes premiers qui ne sont pas redressants.

Exemple 3. Soit G le produit direct $\mathbb{Z}\times\mathbb{Z}$, où $(\mathbb{Z},+)$ est semi-ordonné par le même semi-ordre que dans l'exemple 2, i.e. $G^+=\{0,1,2,4,6,\dots\}$. G est un produit direct des *wal*-grouoes, donc il est un *wal*-groupe lui-même. Notons $H=\{(x,0);\ x\in\mathbb{Z}\}$. H est un *wal*-idéal de G. Mais H n'est pas un sous-groupe redressant parce que, par exemple, $(1,4)\wedge(4,1) = (0,0)$, mais $(1,4)\notin H$ et $(4,1)\notin H$.

H est un sous-groupe premier. En effet, soit $A\in\mathcal{C}(G)$ tel que H est un sous-groupe propre de A. Soit $(a_1,a_2)\in A\setminus H$. Alors $a_2\neq 0$ et $(0,a_2) = (a_1,a_2)-(a_1,0)\in A$. Parce que le sous-groupe convexe de $\mathbb{Z}$ engendré par a_2 est égale à $\mathbb{Z}$, on a $(x_1,x_2)=(x_1,0)+(0,x_2)\in A$ pour tout élément $(x_1,x_2)\in G$, donc $A=G$, et A est un sous-groupe premier.

Un sous-groupe H de G est dit régulier si $H=\cap(A_i;\ i\in I)$, $A_i\in\mathcal{C}(G)$, entraînent l'existence d'un $i_0\in I$ tel que $H=A_{i_0}$.

Proposition 9. a) Tout sous-groupe solide d'un *wal*-groupe est l'intersection des sous-groupes réguliers.

b) Tout sous-groupe redressant est l'intersection d'un système linéairement ordonné de sous-groupes réguliers.

On dit qu'un *wal*-groupe G est représentable s'il est isomorphe à un produit sous-direct de *to*-groupes.

Théorème 10. Un *wal*-groupe est représentable si et seulement si l'intersection de tous ses idéaux redressants est égale à $\{0\}$.

Corollaire. Si un *wal*-groupe G est représentable, alors G contient un système des idéaux premiers tel que l'intersection de ce système est égale à $\{0\}$.
(L'implication inverse n'est pas valable. Voir par exemple G de l'exercice 3, où H et $H'=\{(0,y);\ y\in\mathbb{Z}\}$ sont premiers et $H\cap H'=\{0\}$.

Remarque. Il est bien connu que tout *l*-groupe abélien est représentable, donc la classe de *l*-groupes abéliens est strictement contenue dans la classe de *l*-groupes représentables. Mais cette inclusion n'est pas valable pour les *wal*-groupes.

Par exemple, le *wal*-groupe abélien $G=(\mathbb{Z},+,\leq)$, où $G^+=\{0,1,2,4,6,\ldots\}$ n'est pas représentable. (G n'a pas de sous-groupes redressants distincts de G.)

Donc la classe de *wal*-groupes abéliens et celle de *wal*-groupes représentables sont incomparables. En plus, les *l*-groupes représentables constituent une variété de *l*-groupes. Par exemple, ils sont caractérisés par l'identité

(1) $$(x\wedge(-y-x+y))\vee 0 = 0 \quad ,$$

et aussi par l'identité

(2) $$2(x\wedge y) = 2x\wedge 2y \ .$$

Mais les *wal*-groupes représentables ne doivent pas vérifier ces identités. Par exemple: Pour $G=(\mathbb{Z}_3,+,\leq)$, où $G^+=\{0,1\}$, on a

$$(2\wedge(-2))\vee 0 = (2\wedge 1)\vee 0 = 1 \neq 0 \ ,$$

et aussi

$$2(1\wedge 2) = 2 \ , \quad 2\times 1 \wedge 2\times 2 = 1 \ , \quad 2\neq 1 \ .$$

G est un to-groupe, donc représentable.

Il est claire que, plus général, la première identité n'est pas vraie dans aucun wal-groupe abélien qui contient un élément $a \in G^+$ tel que $0<a$, $a<-a$, $-a<0$.

Pour la deuxième identité: Il existe aussi des to-groupes abéliens infinis qui ne vérifient pas cette identité. Par exemple, soit $G=(\mathbb{Z},+)$ totalement semi-ordonné par $G^+=\{0,1,-2,3,4,-5,6,7,-8,9,10,-11,\ldots\}$. Alors

$$2(1\wedge 5) = 2 \quad \text{et} \quad 2\times 1 \wedge 2\times 5 = 10 \quad .$$

Il y a maintenant une question assez naturelle: Les l-groupes représentables, sont-elles caractérisés dans la classe de wal-groupes représentables comme ceux qui vérifient les conditions (1) et (2)? Mais cette conjecture est fausse. Par exemple, considérons le wal-groupe $G=(\mathbb{Z},+,\leq)$, où $G^+=\{0,1,2,-3,4,-5,-6,-7,8,-9,-10,\ldots,-15,16,-17,\ldots\}=\{2^k;\ k\geq 0\} \cup -\{\mathbb{Z}^+\setminus\{2^k;\ k\geq 0\})$. G est un to-groupe, donc représentable. (Il n'est pas un o-groupe.)

On a, pour $k\geq 0$: $2^k - (-2^k) = 2^{k+1} \in G^+ \Rightarrow -2^k \leq 2^k$, et on a $-2^k \leq 0$.

Pour $k\geq 1$: $-(2k+1) - (2k+1) = -2(2k+1) \in G^+ \Rightarrow 2k+1 < -(2k+1)$, et on a $2k+1<0$.

Pour $k\geq 3$, $k\neq 2^l$, $\forall l\geq 0$: $-2k-2k = 2(-2k) \neq 2^m$, $\forall m\geq 0 \Rightarrow -4k \in G^+ \Rightarrow 2k<-2k$, et $2k<0$.

Donc $\forall x\in G$; $(x\wedge -x)\vee 0 = 0$, c'est-à-dire l'identité (1) est satisfaite.

Pour la condition (2): G est totalement semi-ordonné, donc tous deux éléments x,y de G sont comparables. Soit, par exemple, $x\leq y$. Alors $2(x\wedge y) = 2x$.

Supposons que $y-x = 2^k$, $k\geq 0$. Alors $2(x+2^k)-2x = 2^{k+1} \in G^+ \Rightarrow 2x\leq 2y \Rightarrow 2x\wedge 2y = 2x$.

Soit $y-x = -(2k+1)$, $k\geq 1$. Alors $2(x-(2k+1))-2x = -2(2k+1) \in G^+ \Rightarrow 2x\wedge 2y=2x$.

Finalement, soit $y-x = -2k$, $k\geq 3$, $k\neq 2^l$, $\forall l\geq 0$. Alors $2(x-2k)-2x = -2(2k) \neq 2^m$, $\forall m\geq 0 \Rightarrow 2x\wedge 2y = 2x$.

Donc G vérifie aussi l'identité (2).

C'est-á-dire, la classe de wal-groupes vérifiants les conditions (1) et (2) est plus large que celle de l-groupes représentables.

Il reste une <u>question</u> <u>ouverte</u> si les *wal*-groupes représentables constituent ou non une variété de *wal*-groupes.

Soient maintenant T un tournoi et $AutT$ l'ensemble de tous les automorphismes de T. $AutT$ muni de la loi de composition des applications est un groupe. Si on pose, pour $f,g \in AutT$,

$$f \le g \Leftrightarrow_{df} \forall t \in T;\ f(t) \le g(t)\ ,$$

alors $AutT$ avec "≤" constituent un *wal*-groupe. Soit G un *wal*-sous-groupe de $AutT$. Si t est un élément de T, alors l'ensemble $G_t = \{g \in G;\ g(t) = t\}$ est dit le <u>stabilisateur de t</u>.

<u>Proposition 11</u>. G_t est un sous-groupe redressant de G, quel que soit $t \in G$.

On dit qu'un *wal*-groupe G est <u>transitif</u> s'il existe un tournoi T et un *wal*-homomorphisme injectif $u: G \to AutT$ tels que $u(G)$ opère transitivement dans T.

<u>Théorème 12</u>. Un *wal*-groupe est transitif si et seulement si il contient un sous-groupe redressant A tel que l'intersection des conjugués de A est égale à $\{0\}$.

<u>Corollaire</u>. Un *wal*-groupe abélien est transitif si et seulement si il est un *to*-groupe.

<u>Théorème 13</u>. Si un *wal*-groupe G contient un système de sous-groupes redressants $(G_i;\ i \in I)$ tel que $\cap(G_i;\ i \in I) = \{0\}$, alors G est isomorphe à un produit sous-direct des *wal*-groupes transitifs.

<u>Théorème 14</u>. Si un *wal*-groupe abélien contient un système de sous-groupes redressants avec l'intersection nulle, alors G est un produit sous-direct de *to*-groupes.

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Sur le Groupe de Galois de l'Extension abélienne maximale d'un Corps

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Dans cet exposé on va donner un aperçu des resultats obtenus en collaboration avec M. Prestel ([JP1],[JP2]).

D'abord nous faisons mention des questions, qui originellement faisaient le point de départ de nos études.

Pour simplifier nous ne considérons que des corps de caractéristique zéro.

Soient G un groupe fini ou profini et K un corps. Par une G-extension de K on entend une extension normale de K, dont le groupe de Galois est isomorphe à G.

$\nu(G, K)$ soit le nombre de K-non-isomorphes G-extensions de K. A l'aide des résultats dans [FJ] (Theorem 16.13 et Lemma 15.1) on voit sans peine qu'il existe pour chaque groupe fini G un corps K (dépendant sur G) tel que $0 < \nu(G, K) < \infty$. On appelle la multiplicité de réalisation de G, notée $\nu(G)$, le minimum $\nu(G, K)$ où K parcourt les corps pour lesquels $\nu(G, K) > 0$. D'après le résultat ci-dessus on a $\nu(G) < \infty$.

Il se trouve que $\nu(G) = 1$ pour "beaucoup de groupes", par exemple $\nu(G) = 1$ si G est simple ou un groupe symmétrique. Le plus petit groupe G où $\nu(G) > 1$ est le groupe quasidihédrale d'ordre 16 pour lequel $\nu(G) = 2$. De plus $\nu(G) = 3$ si G est le produit direct du groupe cyclique d'ordre 2 et du groupe dihédral d'ordre 8.

Il est une question ouverte de savoir si $\nu(G) = 1$ pour tout groupe G abélien fini.

Cette question mène de façon naturelle à l'étude de l'extension abélienne maximale K_{ab} d'un corps K. Rappelons que K_{ab} est la réunion filtrante de toutes les extensions finies normales, dont le groupe de Galois est abélien. Le groupe (profini) de Galois $\mathrm{Gal}(K_{ab}/K)$ est le quotient $G(K)/[G(K), G(K)]$ du groupe absolu de Galois $G(K)$ par rapport à son sous-groupe des commutateurs $[G(K), G(K)]$.

Nous nous proposons de caractériser les groupes profinis abéliens de type fini, qui peuvent être réalisé comme le groupe de Galois de l'extension abélienne maximale d'un corps de caractéristique zéro.

La structure d'un groupe profini abélien est bien connue. Mais, auparavant, nous fixons des notations.

Si n est un nombre naturel, Z_n désigne le groupe cyclique d'ordre n. Si p est un nombre premier, $\hat{\mathbf{Z}}_p$ désigne le groupe additif des nombres p-adiques entiers.

Un groupe profini abélien G est produit direct de ses p-groupes de Sylow G_p, et un pro-p-groupe abélien de type fini est un module de type fini sur l'anneau des nombres p-adiques entiers. Puisque cet anneau est principal, chaque pro-p-groupe abélien de type fini est un produit direct

$$G_p = \prod_{i=1}^{\infty} \mathbf{Z}_{p^i}^{\mu_i} \times \hat{\mathbf{Z}}_p^{\hat{\mu}} \qquad (*)$$

où $\mu_i = 0$ pour presque tout i.

Si G est le groupe de Galois de l'extension abélienne maximale d'un corps K, alors $G(p, K)$, le p-groupe de Sylow de G, est le groupe de Galois de la p-extension abélienne maximale $K_{ab}(p)$ de K (c.à.d. $K_{ab}(p)$ est la réunion de toutes les p-extensions abéliennes finies de K).

Nous commençons à donner des conditions suffisantes pour qu'un pro-p-groupe abélien de type fini G_p soit isomorphe à $G(p, K)$ pour un corps K convenable.

THEOREME 1. *G_p soit un pro-p-groupe abélien de type fini écrit comme produit direct des groupes procycliques $(*)$. Si G_p satisfait à la condition suivante*

$$\#\{i \mid \mu_i = 0\} \leqq \hat{\mu} \qquad \text{(Ap)}$$

alors il existe un corps K de caractéristique zéro tel que G_p est isomorphe à $G(p, K)$.

Si $p = 2$ la condition plus faible

$$\#\{i \mid i > 1 \ \text{ et } \ \mu_i = 0\} \leqq \hat{\mu} \qquad \text{(A2)}$$

est suffisante pour l'existence d'un corps K de caractéristique zéro tel que G_2 est isomorphe à $G(2, K)$.

Si l'on se restreint à considérer des corps K qui sont algébriques sur Q (c.à.d. des sous-corps de $\overline{Q}$, le corps des nombres algébriques), nous avons un résultat plus précis:

THEOREME 2. *Soient p un nombre premier > 2 et G_p un pro-p-groupe abélien. Alors il existe un sous-corps K de $\overline{Q}$ tel que $G_p = G(p, K)$ si et seulement si les invariants μ_i, $\hat{\mu}$ de G_p dans $(*)$ satisfont à la condition*

$$\sum_{i=1}^{\infty} \mu_1 \leqq \hat{\mu} \qquad \text{(Bp)}$$

Soient $p = 2$ et G_2 un pro-2-groupe abélien, alors il existe un sous-corps K de $\overline{Q}$ tel que $G_2 = G(2, K)$ si et seulement si les invariants μ_i, $\hat{\mu}$ de G_2 dans $(*)$ satisfont à la condition

$$\sum_{i=2}^{\infty} \mu_i \leqq \hat{\mu} \qquad \text{(B2)}$$

Le rôle exceptionel du nombre premier $p = 2$ est dû au fait que les seuls sous-groupes finis non-triviaux du groupe absolu de Galois $G(K)$ d'un corps K sont d'ordre 2; le corps des invariants correspondant à un tel sous-groupe est un corps ordonné maximal, qui définit un ordre sur K. Si l'on prend en considération le nombre des ordres sur les corps, on a le résultat suivant

THEOREME 3. *Soit K un sous-corps de $\overline{Q}$ tel que $G(2, K)$ est de type fini. Alors les invariants de $G(2, K)$ dans $(*)$ satisfont à la condition*

$$\sum_{i=1}^{\infty} \mu_i \leqq m + \hat{\mu} \leqq \mu_1 + \hat{\mu} \qquad (*)$$

où m est le nombre des ordres sur K.

Réciproquement si m, μ_i et $\hat{\mu}$ sont des nombres satisfaisant à la condition $(*)$, alors il existe un sous-corps K de $\overline{Q}$ ayant exactement m ordres et tel que les invariants de $G(2, K)$ coïncident avec les nombres μ_i et $\hat{\mu}$.

On déduit maintenant des critères pour qu'un p-groupe abélien fini peut être réalisé comme le groupe de Galois d'exactement une extension normale.

THEOREME 4. *Soit G un p-groupe abélien fini, écrit comme*

$$G = \prod_{i=1}^{n} \mathbf{Z}_{p^i}^{\mu_i}$$

où $\mu_n > 0$.

Alors il existe un corps K pour lequel il y a exactement une G-extension de K, si

$$\#\{i \mid i < n \text{ et } \mu_i > 0\} \leqq \mu_n \,. \tag{Cp}$$

Si $p = 2$ une condition plus faible suffit:

Il existe un corps K pour lequel il y a exactement une G-extension de K si

$$\#\{i \mid 1 < i < n \text{ et } \mu_i > 0\} \leqq \mu_n \,. \tag{C2}$$

De plus, si l'on se restreint aux sous-corps K de $\overline{Q}$ on aura un résultat plus précis.

THEOREME 5. *Soit*

$$G = \prod_{i=1}^{n} \mathbf{Z}_{p^i}^{\mu_i}$$

où $\mu_n > 0$.

Soit $p > 2$. Alors il existe un sous-corps K de $\overline{Q}$ tel qu'il y a exactement une G-extension de K si et seulement si

$$\sum_{i=1}^{n-1} \mu_i \leqq \mu_n \,. \tag{Dp}$$

Dans le cas $p = 2$ il existe un sous-corps K de $\overline{Q}$ pour lequel il y a exactement une G-extension de K si et seulement si

$$\sum_{i=2}^{n-1} \mu_i \leqq \mu_n \,. \tag{D2}$$

REMARQUE. Il est une question ouverte de savoir si les conditions suffisantes dans les théorèmes 1 et 3 sont nécessaires. Par example, on ne sait pas s'il existe un corps K tel que $G(2, K) = \mathbf{Z}_4 \times \mathbf{Z}_8 \times \hat{\mathbf{Z}}_2$. C'est une question équivalente de savoir s'il existe un corps K tel qu'il y a exactement une $(\mathbf{Z}_4 \times \mathbf{Z}_8 \times \mathbf{Z}_{16})$-extension de K. Si p est un nombre premier impair on ne sait pas s'il existe un corps K tel que $G(p, K) = \mathbf{Z}_p \times \mathbf{Z}_{p^2} \times \hat{\mathbf{Z}}_p$ ou s'il existe un corps K pour lequel il y a exactement une $(\mathbf{Z}_p \times \mathbf{Z}_{p^2} \times \mathbf{Z}_{p^3})$-extension de K.

Nous regardons maintenant le cas général d'un groupe profini abélien de type fini. On a le résultat suivant.

THEOREME 6. *Soit G un groupe profini abélien de type fini. Il existe un corps K tel que* $\mathrm{Gal}(K_{ab}/K) = G$ *si les invariants du p-groupe de Sylow de G satisfont aux conditions* (Ap) *du théorème 1 pour chaque nombre premier impair p et les invariants du 2-groupe de Sylow de G satisfont aux conditions* (A2) *du théorème 1.*

En ce concerne les sous-corps de $\overline{Q}$ le THEOREME 2 se généralise comme suit.

THEOREME 7. *Soit G un groupe profini abélien de type fini et de torsion finie. (Cette dernière condition signifie que la p-torsion de G s'annulle pour presque tout nombre premier p.) Alors il existe un sous-corps K de $\overline{Q}$ tel que $G =$* $\mathrm{Gal}(K_{ab}/K)$ *si et seulement si les invariants du p-groupe de Sylow de G satisfont aux conditions* (Bp) *du théorème 2 pour chaque nombre premier impair p et les invariants du 2-groupe de Sylow de G aux conditions* (B2) *du théorème 2.*

Comme dans le cas des p-groupes finis on déduit des critères pour qu'un groupe abélien fini peut être réalisé comme le groupe de Galois pour exactement une extension normale.

THEOREME 8. *Soit G un groupe abélien fini. Alors il existe un corps K pour lequel il y a exactement une G-extension de K si le p-groupe de Sylow de G satisfait à la condition* (Cp) *du théorème 4 pour chaque diviseur premier impair p de l'ordre de G et le 2-Sylow groupe de G satisfait à la condition* (C2) *du théorème 4.*

De même on a:

THEOREME 9. *Soit G un groupe abélien fini. Alors il existe un sous-corps K de $\overline{Q}$ pour lequel il y a exactement une G-extension de K si et seulement si le p-groupe de Sylow de G satisfait à la condition* (Dp) *du théorème 5 pour chaque diviseur premier p impair de l'ordre de G et le 2-Sylow groupe de G satisfait à la condition* (D2) *du théorème 5.*

Pour terminer nous esquissons des exemples qui illustrent le rapport entre la structure d'un corps K et le groupe $G(2, K)$. Pour simplifier nous supposons que $G(2, K)$ est de type fini, c.à.d. $[K^* : K^{*2}]$ est fini et donc $[K^* : K^{*2}] = 2^n$ pour un entier n. Pour eviter des cas triviaux on suppose $n > 0$.

Ceci étant le groupe $G(2, K)$ peut s'écrire

$$G(2, K) = \prod_{i=1}^{\infty} \mathbf{Z}_{2^i}^{\mu_i} \times \hat{\mathbf{Z}}_2^{\hat{\mu}}$$

où $\mu_i = 0$ pour presque tout i et $\sum_{i=1}^{\infty} \mu_i + \hat{\mu} = n$.

Un résultat célèbre de Whaples [W] implique

(E1): $\hat{\mu} = 0 \iff \mu_1 = n \iff$ K est pythagoricien et formellement réel.

Rappelons que le niveau $s(K)$ d'un corps K est ∞ si K est formellement réel et dans le cas non-réel $s(K)$ est le plus petit entier s tel que -1 est une somme de s carrés. L'assertion suivante est évidente:

(E2): $\mu_1 = 0 \Rightarrow s(K) = 1$ ou 2.

De plus on a

(E3): $n = 2\,,\ \mu_1 = \hat{\mu} = 1 \Rightarrow s(K) = 2$ ou ∞.

En général on a (en utilisant "k" p.191 dans [R])

(E4): $\mu_1 > 0 \Rightarrow s(K) = \infty$ ou $s(K) = 2^e\,,\ 1 \leqq e \leqq \dfrac{3 + \mu_1}{2}$

Un résultat (Theorem 3) dans [K] implique

(E5): $\mu_2 > 0 \Rightarrow 2 \notin K^2$ et $-2 \notin K^2$.

On peut en déduire

(E6): $$\mu_1 > 0\,,\ \mu_2 > 0\,,\ \mu_i = 0 \text{ pour } i > 1\,,\ \hat{\mu} = 1 \Rightarrow$$
$$s(K) = \infty \text{ ou } s(K) = 2^e\,,\ 2 \leqq e \leqq \frac{3+\mu_1}{2}$$

et

(E7): $$\mu_1 > 0\,,\ \mu_2 = \hat{\mu} = 1\,,\ \mu_i = 0 \text{ pour } i > 1 \Rightarrow$$
$$s(K) = \infty \text{ ou } s(K) = 2^e\,,\ 3 \leqq e \leqq \frac{3+\mu_1}{2}\,.$$

En particulier,

(E8): $G(2,K) = \mathbf{Z}_2^{\mu_1} \times \mathbf{Z}_4 \times \hat{\mathbf{Z}}_2\,,\ \mu_1 \leq 2 \not\Rightarrow K$ est formellement réel.

Il restent plusieurs questions ouvertes. Par exemple on ignore si un corps K est forcément formellement réel, si $\mu_1 > 0$, $\hat{\mu} = 1$ et $\mu_i > 0$ pour un $i > 1$.

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ORDERINGS AND VALUATIONS IN COMMUTATIVE RINGS

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INTRODUCTION

Valuation theory is one of the main tools for studying the reduced theory of quadratic forms and higher level orders and the reduced theory of forms over fields, see, for example [L], [BR]. There is a general theory of valuations in commutative rings (see [LM], [M], and [G]), which in [Ma] is used to study orderings and the reduced theory of quadratic forms over commutative rings. In this paper we list some further results obtained, namely on the real holomorphy ring of a commutative ring and on orderings of higher level in commutative rings. Results on the holomorphy ring of a commmutative ring are from joint work with E. Becker. We do not give proofs here, since the work will appear elsewhere.

§1. PRELIMINARIES

Let R be a commutative ring with 1 and R^* the units of R. For any subset $S \subseteq R$, S^* denotes $S \cap R^*$. For a prime ideal $\mathfrak{p} \subseteq R$, let $R(\mathfrak{p})$ denote the quotient field of $R/\mathfrak{p}$ and $\alpha_\mathfrak{p}$ the canonical map $R \to R/\mathfrak{p} \hookrightarrow R(\mathfrak{p})$.

Valuations in commutative rings Let Γ be an ordered abelian group, written additively, and set $\Gamma_\infty = \Gamma \cup \{\infty\}$, where $\alpha + \infty = \infty + \alpha = \infty$ and $\alpha < \infty$ for all $\alpha \in \Gamma$. A mapping $v : R \to \Gamma_\infty$ is a *valuation on* R if $v(0) = \infty$, $v(1) = 0$, and for all $x, y \in R$, $v(x + y) \geq \min\{v(x), v(y)\}$ and $v(xy) = v(x) + v(y)$. We always assume that Γ is the group generated by $\{v(r) \mid r \in R\}$. (If not we replace Γ by this group.) Γ is called the *value group* of v. If v is surjective, we say v is a *Manis valuation.*

Suppose $v : R \to \Gamma_\infty$ is a valuation. Then it is easy to check that $v^{-1}(\infty)$ is a prime ideal in R, called the *support of* v and denoted $\operatorname{supp}(v)$. Let $\mathfrak{q} := \operatorname{supp}(v)$, then there exists a unique valuation $\hat{v} : R(\mathfrak{q}) \to \Gamma_\infty$ with $v = \hat{v} \circ \alpha_\mathfrak{q}$. Conversely, if $\mathfrak{q}$ is a prime ideal in R and $\hat{v} : R(\mathfrak{q}) \to \Gamma_\infty$ is a valuation, then $v := \hat{v} \circ \alpha_\mathfrak{q}$ is a valuation on R. Since $\hat{v}(x) = \infty$ iff $x = 0$, it follows that $\mathfrak{q} = \operatorname{supp}(v)$. Two valuations v and w are equivalent if $\operatorname{supp}(v) = \operatorname{supp}(w)$ and $\hat{v} = \hat{w}$. Note that if v and w are equivalent and v is Manis, then w is Manis. We identify equivalent valuations, thus there is a 1-1 correspondence between valuations v and pairs $(\mathfrak{q}, \hat{A})$, where $\mathfrak{q}$ is a prime ideal in R and $\hat{A}$ is a valuation ring in $R(\mathfrak{q})$. We write $v = (\mathfrak{q}, \hat{A})$, where $\mathfrak{q} = \operatorname{supp}(v)$ and $\hat{A}$ is the valuation ring of $\hat{v}$.

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Suppose A is a subring of R and I is a prime ideal in A. Then (A, I) is called a ***Manis valuation pair*** if given any $r \in R \setminus I$ there exists some $x \in I$ such that $xr \in A \setminus I$. The connection between Manis valuations and valuation pairs is given by the following:

Proposition 1.1. *Given $v = (\mathfrak{p}, \hat{A})$ a valuation in R. Set $A := \alpha_{\mathfrak{p}}{}^{-1}(\hat{A})$ and $I := \alpha_{\mathfrak{p}}{}^{-1}(\hat{I})$, where $\hat{I}$ denotes the maximal ideal of $\hat{A}$. Then v is a Manis valuation iff (A, I) is a Manis valuation pair. Conversely, given a Manis valuation pair (A, I) then there exists a Manis valuation $v = (\mathfrak{p}, \hat{A})$ such that $A = \alpha_{\mathfrak{p}}{}^{-1}(\hat{A})$ and $I = \alpha_{\mathfrak{p}}{}^{-1}(\hat{I})$.*

Definition. Given a valuation $v = (\mathfrak{p}, \hat{A})$, let $A = \alpha_{\mathfrak{p}}{}^{-1}(\hat{A})$ and $I = \alpha_{\mathfrak{p}}{}^{-1}(\hat{I})$. Then A is the called the *valuation ring of v* and I the *prime ideal of A*.

Remark. Given a valuation v with valuation ring A and prime ideal I. Then $A = \{r \in R \mid v(r) \geq 0\}$ and $I = \{r \in R \mid v(r) > 0\}$. This follows easily from the definitions. Also note that if v is a Manis valuation, then A determines v, since in this case $I = \{r \in R \mid xr \in A \text{ for some } x \in R \setminus A\}$, see [G].

Proposition 1.2. *Suppose there exists $k \in \mathbb{N}$ such that $1 + x \in R^*$ for all $x \in R^k$. Then every valuation in R is a Manis valuation.*

Definition. Suppose A is a subring of R and $\mathfrak{p}$ a prime ideal in A. Define $A_{\mathfrak{p}} = \{r \in R \mid xr \in A \text{ for some } x \in A \setminus \mathfrak{p}\}$ and $\mathfrak{p}^{\#} = \{r \in R \mid xr \in \mathfrak{p} \text{ for some } x \in A \setminus \mathfrak{p}\}$. Then $A_{\mathfrak{p}}$ is a subring of R and $\mathfrak{p}^{\#}$ is a prime ideal in $A_{\mathfrak{p}}$.

We say A is a *Prüfer ring in R* if $(A_{\mathfrak{p}}, \mathfrak{p}^{\#})$ is a Manis valuation pair for all prime ideals $\mathfrak{p}$ in A.

Theorem 1.3. *Suppose A is a subring of R such that $\frac{1}{1+x} \in A$ for each $x \in \Sigma R^{2n}$. Then A is a Prüfer ring in R.*

Remark. When R is a field and $n = 1$, Theorem 1.8 is a result of Dress [D]. Becker proved Theorem 1.8 for R a field and general n [B3].

Higher Level Preorders and Orders For details on higher level orders and preorders in commutative rings, see [MW, §1].

A subset $T \subseteq R$ is a *preorder of level n* if $T + T \subseteq T$, $T \cdot T \subseteq T$, $-1 \notin T$, and $R^{2n} \subseteq T$. If F is a field, then a preorder P of level n in F is an order of level n if F^*/P^* is cyclic. In general, a preorder P of level n in R is an *order of level n* if there exists a prime ideal $\mathfrak{p}$ in R and an order $\bar{P}$ on $R(\mathfrak{p})$ such that $P = \alpha_{\mathfrak{p}}{}^{-1}(\bar{P})$. In this case we will write $P = (\mathfrak{p}, \bar{P})$. Note $\mathfrak{p} = P \cap -P$. In this paper, "order" will always mean an order of some level n. For a preorder T in R, let O_T denotes the set of orders P such that $T \subseteq P$. (We reserve X_T for the T-signatures of R, see §4.)

A prime ideal $\mathfrak{p}$ in R is a *real prime* if $R(\mathfrak{p})$ has an order, iff there exists an order P in R with $P \cap -P = \mathfrak{p}$. Given a preorder T in R of level n and a prime ideal $\mathfrak{p}$, let $T(\mathfrak{p}) = \{\alpha_{\mathfrak{p}}(t)\alpha_{\mathfrak{p}}(s)^{-2n} \mid t \in T \text{ and } s \in R \setminus \mathfrak{p}\}$. We say $\mathfrak{p}$ is *T-compatible* if $T(\mathfrak{p})$ is a preorder in $R(\mathfrak{p})$. It is easy to see that $\mathfrak{p}$ is T-compatible iff $-1 \notin T(\mathfrak{p})$.

Given an order $P = (\mathfrak{p}, \bar{P}) \in O_T$, then by [B1, 3.4] $A(\bar{P}) = \{x \in R(\mathfrak{p}) \mid k \pm x \in \bar{P}$ for some $k \in \mathbb{N}\}$ is a valuation ring in $R(\mathfrak{p})$ with maximal ideal $I(\bar{P}) = \{x \in R(\mathfrak{p}) \mid k \pm x \in \bar{P}$ for all $k \in \mathbb{N}\}$. Thus we have a valuation $(\mathfrak{p}, A(\bar{P}))$, with valuation ring $A(P) := \{r \in R \mid k \pm r \in P$ for some $k \in \mathbb{N}\}$ and prime ideal $I(P) := \{r \in R \mid k \pm r \in P$ for all $k \in \mathbb{N}\}$. We denote this valuation by v_P.

§2. The Real Holomorphy Ring of a Commutative Ring

The results in this section are from joint work with E. Becker.

The real holomorphy ring of a field is the intersection of all valuation rings with a formally real residue field. Marshall [Ma] has defined the real holomorphy ring for commutative rings in which $1 + \Sigma R^2 \subseteq R^*$. In this section we define the real holomorphy ring of a general commutative ring and extend many of Marshall's results to our setting.

Definition. The *real holomorphy ring of R* is

$$H(R) := \bigcap_{P \in X(R)} A(P)$$

Proposition 2.1. $H(R) = \{r \in R \mid$ *there exists* $k \in \mathbb{N}$ *with* $k \pm r \in P$ *for all* $P \in X(R)\}$.

Corollary 2.2.

(i) $H(R) = \{r \in R \mid$ *there exists* $k \in \mathbb{N}$ *with* $k \pm r \in P^+$ *for all* $P \in X(R)\}$.
(ii) $H(R) = \{r \in R \mid$ *there exists* $k \in \mathbb{N}$ *with* $k^2 - r^2 \in P^+$ *for all* $P \in X(R)\}$.

We use frequently the following theorem of R. Berr:

Theorem 2.3. ([Be, Theorem 6]) *Given* $n \in \mathbb{N}$ *and* $r \in R$. *Then* $r \in P^+$ *for all* $P \in X_n(R)$ *iff there exist* $t, t' \in \Sigma R^{2n}$ *with* $rt = 1 + t'$.

Theorem 2.4.

(i) $H(R) = \{r \in R \mid$ *there exists* $k \in \mathbb{N}$ *and* $t, t' \in \Sigma R^2$ *with* $(k^2 - r^2)t = 1 + t'\}$.
(ii) $H(R) = \bigcap \alpha_{\mathfrak{p}}{}^{-1}(H(R(\mathfrak{p})))$, *the intersection over all real prime ideals* $\mathfrak{p}$ *in* R.

Theorem 2.5. *Fix* $n \in \mathbb{N}$. *Then* $H(R) = \{r \in R \mid$ *there exists* $k \in \mathbb{N}$ *and* $t_1, t_1', t_2, t_2' \in \Sigma R^{2n}$ *with* $(k + r)t_1 = 1 + t_1'$ *and* $(k - r)t_2 = 1 + t_2'\}$.

Since $X(R) \neq \emptyset$,we have $-1 \notin \Sigma R^2$. Let $R_\Sigma := (1 + \Sigma R^2)^{-1} R$, then $1 + \Sigma R^{*2} \subseteq R_\Sigma{}^*$ and there is a canonical isomorphism between $X(R)$ and $X(R_\Sigma)$: $(\mathfrak{p}, \bar{P}) \leftrightarrow (\mathfrak{p}_\Sigma, \bar{P})$, where $\mathfrak{p}_\Sigma$ denotes the image of $\mathfrak{p}$ in R_Σ.

Proposition 2.6. *Given $n \in \mathbb{N}$. Then there is a canonical isomorphism between R_Σ and $(1+\Sigma R^{2n})^{-1}R$.*

For the rest of this section we replace R by R_Σ.

proclaimTheorem 2.7 (cf. [B3, 2.16, 3.3].)

(i) $H(R)$ is a Prüfer ring in R.
(ii) Fix $n \in \mathbb{N}$. Then $H(R) = \{r \in R \mid k \pm r \in \Sigma R^{2n}$ for some $k \in \mathbb{N}\}$.
(iii) Fix $n \in \mathbb{N}$. Then $H(R)$ is generated as a subring by the elements $\frac{1}{1+x}$ with $x \in \Sigma R^{2n}$.

Theorem 2.8. *(cf. [B2, 3.3]) Let $E(R)$ denote the group of units in $H(R)$, then*

$$E(R) \cap \Sigma R^2 \subseteq \bigcap_n R^{2n}$$

Remark. By [B3, 1.7] $\frac{1+x^2}{2+x^2} \in \Sigma\mathbb{R}(x)^{2n}$ for each $n \in \mathbb{N}$. We can improve this result: Given $S = \mathbb{R}[x]$, let $R = (1+\Sigma S^2)^{-1}S$. Since $1 \pm \frac{1+x^2}{2+x^2} \in \Sigma R^2$ and $1 \pm \frac{2+x^2}{1+x^2} \in \Sigma R^2$, $\frac{1+x^2}{2+x^2} \in E(R) \cap \Sigma R^2$ by Corollary 2.2. Hence $\frac{1+x^2}{2+x^2} \in \Sigma R^{2n}$ for each $n \in \mathbb{N}$ by Theorem 2.8.

§3. Higher Level Orders in Commutative Rings

Marshall and Walter [MW] have generalized results on higher level orders and reduced forms from fields to rings with many units (a generalization of semilocal rings). They do not use valuation theory and it appears that their techniques will not extent to general commutative rings. In this section we use valuations to generalize ideas and results on higher level orders to commutative rings.

We fix a preorder T of level n. Let $S = 1+T$, a mulitplicative set in R, then $S^{-1}R$ a nonzero ring. It is easy to check that $S^{-1}T$ is a preorder in $S^{-1}R$ and there is a 1-1 correspondence between O_T and $O_{S^{-1}T}$ given by $P \mapsto \{xs^{-2n} \mid x \in P$ and $s \in S\}$. Under this bijection we have $(\mathfrak{p}, \bar{P}) \leftrightarrow (\mathfrak{p}', \bar{P})$ where $\mathfrak{p}'$ denotes the image of $\mathfrak{p}$ in $S^{-1}R$. *For the rest of this paper we replace R by $S^{-1}R$ and T by $S^{-1}T$, i.e., we assume throughout that $1+T \subseteq R^*$.*

Lemma 3.1.
(i) Given $r \in R$ such that $r \notin P \cap -P$ for all $P \in O_T$. Then $r \in R^*$.
(ii) $T^* = \bigcap_{P \in O_T} P^*$.
(iii) $R = T^* - T^*$.

Corollary 3.2. *All valuations in R are Manis valuations.*

Definition. Let $A(T) = \{r \in R \mid k \pm r \in T$ for some $k \in \mathbb{N}\}$.

By Theorem 1.3, we have:

Proposition 3.3. *$A(T)$ is a Prüfer ring in R.*

Compatible valuations One of the key notions in studying higher level orders and forms in fields is that of compatibility between orders and valuations. For a field F, a valuation ring A with maximal ideal I, and an order P on F, we say A is compatible with P if $1 + I \subseteq P$. In this case the "pushdown of P along A", the image of $P \cap A$ in the field A/I, is an order. For details, see [BR, §2].

In our case the situation is a bit more complicated since in general a given order and a given valuation will come from different residue fields of R. In order to define compatibility we first need the following:

Proposition 3.4. *Suppose $v = (\mathfrak{q}, \hat{A})$ is a valuation with valuation ring A and prime ideal I. Given $P = (\mathfrak{p}, \bar{P}) \in O_T$, then the following are equivalent:*
(i) $\mathfrak{p} \subseteq \mathfrak{q}$ and $P \cap (A \setminus I) + I \subseteq P$.
(ii) $P(\mathfrak{q})$ is an order in $R(\mathfrak{q})$, $\hat{A}$ is compatible with $P(\mathfrak{q})$, and ${\alpha_{\mathfrak{q}}}^{-1}(P(\mathfrak{q})) = P \cup \mathfrak{q}$.

Definition. Suppose v is a valuation and $P \in O_T$. If the equivalent conditions of 2.1 hold, then we say v is *compatible with* P, written $v \sim P$. We say v is *compatible with* T if v is compatible with some $P \in O_T$, written $v \sim T$. If v is compatible with all $P \in O_T$ then we say v is *fully compatible with* T, written $v \sim_f T$.

Remark. If R is a field then $P \cap (A \setminus I) + I \subseteq P$ iff $1 + I \subseteq P$. Hence our definitions agree with the usual definitions for fields, cf. [BR, §2].

Lemma 3.5. *For all $P \in O_T$, $v_P \sim P$.*

Proposition 3.6. *A valuation $v = (\mathfrak{q}, \hat{A})$ is compatible with T iff $\mathfrak{q}$ is a T-compatible prime ideal and $\hat{A} \sim T(\mathfrak{q})$ in $R(\mathfrak{q})$.*

We want to define the pushdown of an order along a valuation. Again, the situation is complicated by the fact that the order and the valuation may come from different fields.

Definition. Given a valuation v with valuation ring A and prime ideal I, let D_v denote the domain A/I and K_v the quotient field of D_v.

Proposition 3.7. *Suppose v is a valuation with valuation ring A and prime ideal I and v is compatible with $P \in O_T$. Let $\tilde{P}$ denote the image of $A \cap P$ in D_v and define $f : K_v \to \hat{A}/\hat{I}$ by $f\big((a + I)/(b + I)\big) = \big(\alpha_{\mathfrak{q}}(a)/\alpha_{\mathfrak{q}}(b)\big) + \hat{I}$. Then f is an isomorphism such that $f\big(\tilde{P}(\{0\})\big) = \overline{P(\mathfrak{q})}$, where $\overline{P(\mathfrak{q})}$ denotes the pushdown of $P(\mathfrak{q})$ along $\hat{A}$. (Here $\{0\}$ denotes the zero prime ideal in D_v.)*

Definition. Given v and P as above, $\tilde{P}$ is the *pushdown of P along* v. Note that since $\hat{A} \sim P(\mathfrak{p})$, we have that $\overline{P(\mathfrak{q})}$, the pushdown of $P(\mathfrak{q})$ along $\hat{A}$, is an order in $\hat{A}/\hat{I}$. Hence, by 2.4, $\tilde{P}$ is an order in D_v, namely $\tilde{P} = (\{0\}, \overline{P(\mathfrak{q})})$.

We define the *pushdown of T along* v to be the image of $T \cap A$ in D_v, denoted $\tilde{T}$.

Lemma 3.8. *Suppose v is a valuation in R which is compatible with T. Then $\tilde{T}$ is a preorder in D_v.*

Proposition 3.9. *Suppose v is a valuation fully compatible with T. Then the map $\theta : R^*/T^* \to R(\mathfrak{q})^*/T(\mathfrak{q})^*$ given by $\theta(rT^*) = rT(\mathfrak{q})^*$ is an isomorphism, where $\mathfrak{q} = supp(v)$.*

Theorem 3.10. *Suppose $v = (\mathfrak{q}, \hat{A}, A, I)$ is a valuation in R with value group Γ which is fully compatible with T. Then the sequence*

$$1 \to \tilde{A}^*/\tilde{T}^* \xrightarrow{\alpha} R^*/T^* \xrightarrow{\beta} \Gamma/v(T^*) \to 1$$

where $\alpha((a+I)\tilde{T}^*) = aT^*$ and $\beta(rT^*) = v(r)v(T^*)$ is exact.

Remark. Theorem 3.10 is proven for the field case in [BR, §2].

As in the field case (see [BR, §5]), we can define an equivalence relation on O_T using the valuations v_P. This allows us to "break up" T in pieces which are fully compatible with a valuation.

Definition.

(i) Suppose v_1 and v_2 are nontrivial valuations in R. For $i = 1, 2$, let Γ_i denote the value group, A_i the valuation ring, and I_i the prime ideal of v_i. Following [G], we say v_2 is *coarser* than v_1, denoted $v_2 \leq v_1$, if there is an order homorphism $f : \Gamma_1 \to \Gamma_2$ such that $v_2 = f \circ v_1$, iff (by [G, Proposition 4]) $A_1 \subseteq A_2$ and $I_2 \subseteq I_1$.

(ii) Nontrivial valuations v_1 and v_2 are *dependent valuations* if there exists a nontrivial valuation coarser than both. Otherwise, they are *independent*.

(iii) We define the relation of *dependency*, denoted $\sim$, on O_T as follows: Given $P, Q \in O_T$. If P is archimedean, then $P \sim Q$ iff $I(Q) = I(P)$. If P is nonarchimedean, then $P \sim Q$ if Q is nonarchimedean and v_P and v_Q are dependent valuations.

Proposition 3.11. *The relation of dependency is an equivalence relation on O_T.*

Definition. For $P \in O_T$, let $[P]$ denote the equivalence class of P, called the *dependency class of P.*

Proposition 3.12. *Suppose there are only finitely many valuations among $\{v_P \mid P \in O_T\}$. Then O_T has only finitely many dependency classes. Let $[P_1], \ldots [P_k]$ be the dependency classes of the nonarchimedean elements of O_T and set $T_i := \bigcap_{P \in [P_i]} P$. Then for each i*

(i) $O_{T_i} = [P_i]$.

(ii) There exists a valuation v_i such that $v_i \leq v_P$ for each $P \in [P_i]$ and if $i \neq j$, then v_i and v_j are independent valuations.

Theorem 3.13. *Suppose there are only finitely many valuations among $\{v_P \mid P \in O_T\}$ and O_T contains no archimedean orders. Let $[P_1], \ldots, [P_k]$ and $T_1, \ldots T_k$ be as in 3.6. Then the map*

$$\theta : R^*/T^* \to R^*/T_1{}^* \times \cdots \times R^*/T_k{}^*,$$

where $\theta(rT^*) = (rT_1{}^*, \ldots, rT_k{}^*)$, *is an isomorphism.*

T-Forms and the Reduced Witt Ring We define signatures, T-forms and the reduced Witt ring of T as in [MW].

For any abelian group G of finite exponent, let $G^\vee$ denote $\mathrm{Hom}(G, \mu)$, where μ denotes the complex roots of unity.

If F is a field and Q a preorder in F then a Q-signature is any $\chi \in (F^*)^\vee$ such that $Q^* \subseteq \ker\chi$ and $\ker\chi$ is additively closed. Note that if χ is a Q-signature then $\ker\chi \cup \{0\} \in O_Q$. A *$T$-signature in R* is a character $\sigma \in (R^*)^\vee$ such that there exists a T-compatible prime ideal $\mathfrak{p}$ and a $T(\mathfrak{p})$-signature χ with $\sigma = \chi\circ\alpha_\mathfrak{p}|_{R^*}$, where $|_{R^*}$ denotes restriction to R^*. In this case we have $P = \alpha_\mathfrak{p}{}^{-1}(\ker\chi \cup \{0\}) \in O_T$ and $P^* = \ker\sigma$. Conversely, given $P = (\mathfrak{p}, \bar{P}) \in O_T$ then there is a $T(\mathfrak{p})$-signature χ with $\bar{P}^* = \ker\chi$. Hence there is a T-signature σ, defined by $\sigma = \chi\circ\alpha_\mathfrak{p}|_{R^*}$, such that $\ker\sigma = P^*$. We write X_T to denote the set of T-signatures.

An r-dimensional form over T is an r-tuple $\rho = \langle a_1, \ldots, a_r\rangle$, where $a_i \in R^*$. The sum and product of forms are defined in the usual way: For ρ as above and $\tau = \langle b_1, \ldots, b_k\rangle$,

$$\rho \oplus \tau = \langle a_1, \ldots, a_r, b_1, \ldots, b_k\rangle$$

and

$$\rho \otimes \tau = \langle a_1b_1, \ldots, a_1b_k, \ldots, a_rb_1, \ldots, a_rb_k\rangle.$$

If $\rho = \langle a_1, \ldots a_r\rangle$ and σ is a T-signature, we define $\sigma(\rho) = \Sigma_{i=1}^r\sigma(a_i)$. Two forms ρ and τ are *T-equivalent*, denoted $\rho \sim \tau$, if $\sigma(\rho) = \sigma(\tau)$ for all T-signatures σ. If in addition ρ and τ have the same dimension, they are *T-isometric*, denoted $\rho \cong \tau$. The *Witt ring of T*, denoted $W_T(R)$, consists of T-equivalence classes of forms with operations induced by $\oplus$ and $\otimes$.

Definition.

(i) We say a form $\rho = \langle a_1, \ldots a_r\rangle$ is *isotropic* if there exist $t_1, \ldots t_r \in T^* \cup \{0\}$, not all 0, such that $a_1t_1 + \cdots + a_rt_r = 0$. Otherwise, ρ is *anisotropic.*

(ii) The *represented set of ρ*, denoted $D_T(\rho)$, is $Ta_1 + \cdots + Ta_r$.

Proposition 3.14. *Suppose $\rho = \langle a_1, \ldots, a_r\rangle$ is a form, and $b \in R^*$. Then $b \in D_T(\rho)^*$ iff $\alpha_\mathfrak{p}(b) \in D_{T(\mathfrak{p})}(\alpha_\mathfrak{p}(\rho))$ for all T-compatible primes $\mathfrak{p}$, where $\alpha_\mathfrak{p}(\rho) = \Sigma\alpha_\mathfrak{p}(a_i)$.*

Theorem 3.15. *Suppose ρ and τ are T-forms such that $\rho \sim \tau$ and $\dim\rho < \dim\tau$. Then τ is isotropic.*

Corollary 3.16. *$\rho \cong \tau$ implies $D_T(\rho) = D_T(\tau)$.*

Remark. Theorem 3.15 and Corollary 3.16 are proven for rings with many units in [MW, 3.5].

Spaces of Signatures Spaces of signatures (hereafter SOS) provide an abstract setting for studying the reduced theory of higher level forms over fields. For details and terminology see [Mu] and [MM]. The advantage of this abstract approach is that once we prove we have a SOS then much of the theory for fields generalizes immediately to our setting. In [MW] it is shown that a preordered ring with many units gives rise to a SOS. We cannot prove this in general in our setting, but we can prove it for preorders T which satisfy the conditions of 3.12.

We generalize some ideas from the theory of SOS's:

Definition. A ***signature pair*** is a pair (X, G) where G is an abelian group of finite even exponent and X is a subset of $G^\vee$. Two signature pairs (X_1, G_1) and (X_2, G_2) are *equivalent* if there is an isomorphism $\alpha : G_1 \to G_2$ such that $\alpha^\vee(X_2) = X_1$, where $\alpha^\vee$ is the dual isomorphism.

Given signature pairs $\{(X_i, G_i)\}_{i=1}^k$, we set $G = G_1 \times \cdots \times G_k$ and $X = X_1 \dot\cup \ldots \dot\cup X_k$, where X_i is identified with its image in $G^\vee$, and $\dot\cup$ denotes disjoint union. Then (X, G) is a signature pair, called the *direct sum* of the (X_i, G_i)'s. We write $(X, G) = \bigoplus_{i=1}^k (X_i, G_i)$.

Remark.

(i) A SOS is a signature pair which satisfies certain axioms, see [Mu], [MM].

(ii) Given $\sigma \in X_T$, we identify σ with its image in $(R^*/T^*)^\vee$ and thus $(X_T, R^*/T^*)$ is a signature pair. If R is a field then $(X_T, R^*/T^*)$ is a SOS by [Mu, 1.10].

(iii) If a signature pair is equivalent to a SOS, then it is also a SOS.

(iv) The direct sum of finitely many SOS's is a SOS. This follows from [Mu, 2.6].

Theorem 3.17. *Suppose T satisfies the conditions of 3.12, i.e., the set $\{v_P \mid P \in O_T\}$ is finite and O_T contains no archimedean orders. Then $(X_T, R^*/T^*)$ is a SOS.*

Corollary 3.18. *Suppose T satisfies the conditions of 3.12. Then there exists a field K and a preorder $Q \subseteq K$ such that $(X_T, R^*/T^*)$ and $(X_Q, K^*/Q^*)$ are equivalent SOS's. In particular, $W_T(R)$ is isomorphic to $W_Q(K)$.*

Remark. The assumption that O_T contains no archimedean orders is strange and perhaps unnecessary. The problem is that the approximation theorem for independent valuations only applies to nontrivial valuations. In the field case, one uses the approximation theorem for V-topologies, see [PZ]. However it is not clear (to the author, at least) how one could generalize this theorem to our situtaion.

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