N° 32

Séminaire de Structures Algébriques Ordonnées

1990–1991

F. DELON, M.A. DICKMANN, D. GONDARD

Janvier 1992

Séminaire de Structures Algébriques Ordonnées.

Delon – Dickmann – Gondard

1990 – 1991

Liste des exposés.

20.11.90 – **C. Roche** (Dijon)
Géométrie des variétés Pfaffiennes.

23.11.90 – **J. Koenigsmann** (Konstanz, Allemagne).
Half–ordered fields.

27.11.90 –**D. Gluschankof** (Angers).
Groupes réticulés hyper–réguliers I.

4.12.90 – **R. Cignoli** (Buenos–Aires, Argentine).
L'algèbre de la logique de Lukasiewicz.

11.12.90 – **D. Gluschankof** (Angers).
Groupes réticulés hyper–réguliers II.

15.01.91 – **J.P. Françoise** (Paris VI)
Décidabilité effective des ensembles algébriques réels.

29.01.91 – **F. Cucker** (Université Catalogne, Barcelone, Espagne).
Problèmes P–complets pour le modèle de Blum, Shub et Smale.

06.03.91 – **F. Pop** (Heidelberg).
Local–Global principle in the field of totally p–adic numbers.

12.03.91 – **A. Guergueb** (Rennes).
Exemples de démonstration automatique en géométrie réelle.

14.05.91 – **D. Gluschankof** (Angers).
Groupes cycliquement ordonnés et MV–algèbres.

21.05.91 – **M.F. Coste–Roy** (Rennes).
Complexité d'algorithme pour la description des composantes connexes d'un ensemble semi–algébrique.

1.07.91 – **M. Marshall** (Univ. de Saskatchewan –Canada).
Spaces of orderings and their Witt rings.

2.07.91 – **R. Berr** (Dortmund, Allemagne).
Orderings of higher level and sums of mixed powers in fields.

Sèminaire de Structures Algébriques Ordonnées.

Delon – Dickmann – Gondard

1990 – 1991

Liste des contributions.

F. Pop (Heidelberg, Allemagne).
A universal Local–Global principle for the field of totally Σ–adic numbers.

D. Gluschankof (Angers).
Cyclic ordered groups and MV–algebras.

M. Marshall (Saskatoon, Canada).
Spaces of orderings of fields, generalizations and applications.

R. Berr (Dortmund, Allemagne).
On the existence of orderings of prescribed level.

A universal Local–Global Principle for the field of totally Σ–adic numbers

by *Florian Pop* at Heidelberg

— Abstract —

Let K be a field. We say that a place $\mathfrak{p}$ of K is of local type if the completion $K_\mathfrak{p}$ of K at $\mathfrak{p}$ is a local field. Equivalently, the valuation $\mathfrak{v}_\mathfrak{p}$ defined by $\mathfrak{p}$ is either archimedean, or non-archimedean discrete and with finite residue field. In the first case $K_\mathfrak{p}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and in the second case, if $\mathbb{F}$ is the residue field of $\mathfrak{v}_\mathfrak{p}$ and $p = \operatorname{char}\mathbb{F}$ is the residual characteristic, then $K_\mathfrak{p}$ is finite over the p–adic field $\mathbb{Q}_p$ or isomorphic to the power series field $\mathbb{F}((t))$.

As usual, if $\mathfrak{p}$ is a place of local type of K we say that an algebraic element a over K is totally $\mathfrak{p}$–adic if for all K–embeddings

$$\imath : \tilde{K} \longrightarrow \tilde{K}_\mathfrak{p}$$

$\imath(a)$ lies in $K_\mathfrak{p}$. Here $\tilde{K}$ denotes the algebraic closure of K and $\tilde{K}_\mathfrak{p}$ denotes the algebraic closure of $K_\mathfrak{p}$.

Let Σ be a finite family of places of local type of K. We say that an algebraic element a over K is *totally* Σ*–adic* if a is totally $\mathfrak{p}$–adic for all $\mathfrak{p} \in \Sigma$. By general valuation theory it follows that set of all totally Σ–adic elements of $\tilde{K}$ is a normal extension K^Σ of K and moreover, if $K_\mathfrak{p}|K$ is a separable field extension for all $\mathfrak{p} \in \Sigma$ then $K^\Sigma|K$ is a Galois extension. One has the following:

Let $K_\mathfrak{p}^{\mathrm{alg}}$ be the relative algebraic closure of K in $K_\mathfrak{p}$. Then the field K^Σ is the intersection of all conjugates in $\tilde{K}$ of the fields $K_\mathfrak{p}^{\mathrm{alg}}$ $(\mathfrak{p} \in \Sigma)$.

In particular, the separable part of K^Σ is the maximal separable extension of K in which all $\mathfrak{p} \in \Sigma$ split totally.

Let now $\mathcal{K}^\Sigma$ denote the prolongation of Σ to K^Σ. Then for any $\mathfrak{q} \in \mathcal{K}^\Sigma$ and the corresponding $\mathfrak{p} = \mathfrak{q}|_K$ we have by the observation above: $K_\mathfrak{p} = K^\Sigma_\mathfrak{q}$.

The main result we proved is the following:

Main Theorem. *The field K^Σ satisfies the following universal* Local–Global Principle *for the existence of rational points on varieties:*

Let V be a geometrically normal and irreducible projective variety defined over K^Σ. Then V has K^Σ–rational points, provided V has simple $K^\Sigma_\mathfrak{q}$–rational points for all $\mathfrak{q} \in \mathcal{K}^\Sigma$.

The above Main Theorem provides us with the first interesting examples of PAC, PRC, PpC and more generally, pseudo classically closed fields:

Corollary. *Let K be a number field and Σ a finite set of places of K. Then K^{Σ} is pseudo classically closed, ie it satisfies the following universal Lokal–Global Principle:*

Let V be an absolutely irreducible variety define over K^{Σ}. Then V has simple K^{Σ}–rational points, provided V has $K^{\Sigma}_{\mathfrak{q}}$–rational points for all $\mathfrak{q} \in K^{\Sigma}$.

In particular:

(1) *The field of all algebraic totally real numbers is pseudo real closed.*

(2) *The field of all algebraic totally p–adic numbers is pseudo p–adically closed.*

As a further corollary of the main result above we can construct "visible" pseudo algebraically closed fields of algebraic numbers by the following method: Let K be a number field and Σ a finite set of places of K. Further consider $L|K^{\Sigma}$ an arbitrary algebraic extension having the property: The compositums $LK^{\Sigma}_{\mathfrak{q}}$ are algebraically closed for all $\mathfrak{q} \in \Sigma$. Then L *is pseudo algebraically closed.*

This construction has interesting consequences for some special choices of L, namely: Let K be a number field and $L = K^{\Sigma,\mathrm{nil}}$ denote the compositum of K^{Σ} and $\mathbb{Q}^{\mathrm{nil}}$. Here as usual, $\mathbb{Q}^{\mathrm{nil}}$ denotes the maximal nilpotent extension of $\mathbb{Q}^{\mathrm{ab}}$. Then $K^{\Sigma,\mathrm{nil}}$ is a Hilbertian PAC field. Applying a recent result of Fried–Völklein, proved independently and by other methods also by Matzat, on the absolute Galois group of a Hilbertian PAC field one gets the following interesting fact:

Theorem. *The absolute Galois group of $K^{\Sigma,\mathrm{nil}}$ is ω–free.*

This makes the conjecture that the absolute Galois group of $\mathbb{Q}^{\mathrm{nil}}$ is ω–free very plausible, which in turn is further evidence for the Shafarevich conjecture. We recall that the Shafarevich conjecture asserts that the absolute Galois group of $\mathbb{Q}^{\mathrm{ab}}$ is ω–free.

The most important ingredient in the proof of the main theorem is an existence theorem of Rumely type. The classical existence theorem of Rumely [RU], Theorem 1.3.1 is one of the fundamental facts used in the proof of the decidability of the ring $\tilde{\mathbb{Z}}$ of all algebraic integers, see Cantor–Roquette [C–R], van den Dries [vdD] and others. We followed the approach of Roquette from [R], where a very simple proof of the Rumely existence theorem is given. By making some adjustments the idea from [R] can be used to get the existence theorem with *rationality conditions*. Using this one first proves the main theorem for 1–dimensional varieties, ie for curves, by using the continuity of the roots of algebraic functions of one variable. From the 1–dimensional situation one gets the general case by a "Bertini type argument" using a result from [G–J].

We should remark that similar results for global fields are contained in Moret-Bailly's paper [M-B], where the same idea of proof as in [R] is used for the Rumely

existence theorem. Nevertheless, our proof is rather elementary compared with the one in [M-B].
The results will appear in the DMV-Jahresbericht series.

References

[C–R] D. CANTOR, P. ROQUETTE, *On diophantine equations over the ring of all algebraic integers*, J. Number Theory **18** (1984), 1—26

[vdD] L. VAN DEN DRIES, *Elimination theory for the ring of algebraic integers*, J. reine angew. Math. **388** (1988), 189—205.

[G–J] W.-D. GEYER, M. JARDEN, *On stable fields in positive characteristic*, Geom. Dedic. **29** (1989), 335–376.

[M-B] L. MORET-BAILLY, *Groupes de Picard et problèmes de Skolem I*, Ann. Sci. Ec. Norm. Super. **22** (1989), 161—179.

[R] P. ROQUETTE, *Solving diophantine equations over the ring of all algebraic integers*, Atas de 8^e Escola de Algebra, Vol 2, IMPA 1984.

P. ROQUETTE, *Reciprocity in valued function fields*, J. reine angew. Math. **375/376** (1987), 238—258.

[RU] R. RUMELY, *Capacity theory on algebraic curves*, Springer LNM **1378**, Berlin Heidelberg New-York 1989.

Mathematisches Institut, Im Neuenheimer Feld 288, D–6900 Heidelberg

Cyclic ordered groups and MV-algebras

Daniel Gluschankof

In the forties and fifties two -at the moment- unrelated concepts derived from that of an ordered group appeared. The notion of cyclic-ordered group (*c*-group) (see [9], [10], [13] and [14]) and that of *MV*-algebra (see [4] and [5]). The first one appeared as a way of generalizing the notion of totally ordered groups. That notion was further extended to that of partially cyclically ordered groups. The notion of *MV*-algebras resulted from a succesfull attempt of giving an algebraic structure to the infinite-valued Łukasievicz propositional logics. In the last decade, that theory was fruitfully linked with that of a class of $\mathbb{C}^*$-algebras (see [8]). The objective of this work is to show that suitable subclasses of that notions can be linked by the way of a covariant functor.

1. Definitions and first facts. A *cyclically ordered group* (*c*-group) is a system $\langle G,+,-,0,T\rangle$ where $\langle G,+,-,0\rangle$ is a group (not necessarily commutative) and T is a ternary relation verifying the following properties:

C1. $\forall abc$ (if $a \neq b \neq c \neq a$ then exactly one of $T(a,b,c)$ and $T(a,c,b)$ holds);

C2. $\forall abc$ $(T(a,b,c) \Longrightarrow a \neq b \neq c \neq a)$;

C3. $\forall abc$ $(T(a,b,c) \Longrightarrow T(c,a,b))$;

C4. $\forall abcd$ $(T(b,c,a)$ & $T(c,d,a) \Longrightarrow T(b,d,a))$;

C5. $\forall abcd$ $(T(a,b,c) \Longrightarrow T(d+a,d+b,d+c)$ & $T(a+d,b+d,c+d))$.

A fundamental result of Rieger (see [9]) says that any such a group is isomorphic to a quotient of a totally ordered group (*o*-group) by the subgroup generated by a strong unit (a cofinal element in its centre). In that case, if $G = \langle G,+,-,0,u,\leq\rangle$ is an o-group with strong unit u, the quotient group $G_u = G_{/\langle u\rangle}$ can be endowed with a cyclic order by defining $T(\mathbf{a},\mathbf{b},\mathbf{c})$ if and only if, for the only representatives a,b,c such that $0 \leq a,b,c < u$, either $a < b < c$ or $b < c < a$ or $c < a < b$ holds.

The notion of *c*-group generalizes that of totally ordered groups (o-groups) in the sense that for a c-group with the property: for all $a \in G$, $T(-a,0,a)$ implies, for all $n \in \mathbb{N}$, $T(-na,0,na)$ a total order (compatible with the group operation) can be defined by $0 < a$ if and only if $T(-a,0,a)$. Conversely, an o-group can be endowed with a c-group structure by defining $T(a,b,c)$ if and only if $a < b < c$ or $b < c < a$ or $c < a < b$.

A *partially cyclically ordered group* (pco-group) is a system $\langle G,+,-,0,T\rangle$ where the axioms C2, C3, C4, C5 and

C1p. $\forall abc\ (T(a,b,c) \Longrightarrow \neg T(a,c,b))$;

C6. $\forall abc\ (T(a,b,c) \Longrightarrow T(-c,-b,-a))$ hold.

This last axiom is consequence of axioms C1..C5.

Observe that, Rieger's theorem also holds in this case by replacing the o-group by a partially ordered group (po-group) (see [13] or [14]).

An MV-*algebra* (see [4], [5] and [8]) is a system $\langle A,\oplus,*,\neg,0,1\rangle$ which satisfies the following universal identities:

m_1 $\quad x\oplus(y\oplus z) = (x\oplus y)\oplus z$

m_2 $\quad x\oplus 0 = x$

m_3 $\quad x\oplus y = y\oplus x$

m_4 $\quad x\oplus 1 = 1$

m_5 $\quad \neg\neg x = x$

m_6 $\quad \neg 0 = 1$

m_7 $\quad x\oplus\neg x = 1$

m_8 $\quad \neg(\neg x\oplus y)\oplus y = \neg(x\oplus\neg y)\oplus x$

m_9 $\quad x*y = \neg(\neg x\oplus\neg y)$

By defining $x\vee y := (x*\neg y)\oplus y$ and, by duality, $x\wedge y := \neg(\neg x\vee\neg y)$ we have that $\langle A,\vee,\wedge,0,1\rangle$ is a bounded distributive lattice.

Another approach for this structures is that of *Wajsberg algebras* (W-algebras) (see [6] and [11]). Such an algebra is a system $\langle A,\rightarrow,\neg,0,1\rangle$ satisfying the following universal identities:

W1. $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;

W2. $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$;

W3. $(\neg x \rightarrow \neg y) \rightarrow (y \rightarrow x) = 1$;

W4. $1 \rightarrow x = x$;

W5. $x \rightarrow 0 = \neg x$;

Cyclic ordered groups and MV-algebras

Daniel Gluschankof

In the forties and fifties two -at the moment- unrelated concepts derived from that of an ordered group appeared. The notion of cyclic-ordered group (c-group) (see [9], [10], [13] and [14]) and that of *MV*-algebra (see [4] and [5]). The first one appeared as a way of generalizing the notion of totally ordered groups. That notion was further extended to that of partially cyclically ordered groups. The notion of *MV*-algebras resulted from a succesfull attempt of giving an algebraic structure to the infinite-valued Łukasievicz propositional logics. In the last decade, that theory was fruitfully linked with that of a class of $\mathbb{C}^*$-algebras (see [8]). The objective of this work is to show that suitable subclasses of that notions can be linked by the way of a covariant functor.

1. Definitions and first facts. A *cyclically ordered group* (*c*-group) is a system $\langle G,+,-,0,T\rangle$ where $\langle G,+,-,0\rangle$ is a group (not necessarily commutative) and T is a ternary relation verifying the following properties:

C1. $\forall abc$ (if $a \neq b \neq c \neq a$ then exactly one of $T(a,b,c)$ and $T(a,c,b)$ holds);

C2. $\forall abc\ (T(a,b,c) \Longrightarrow a \neq b \neq c \neq a)$;

C3. $\forall abc\ (T(a,b,c) \Longrightarrow T(c,a,b))$;

C4. $\forall abcd\ (T(b,c,a)\ \&\ T(c,d,a) \Longrightarrow T(b,d,a))$;

C5. $\forall abcd\ (T(a,b,c) \Longrightarrow T(d+a,d+b,d+c)\ \&\ T(a+d,b+d,c+d))$.

A fundamental result of Rieger (see [9]) says that any such a group is isomorphic to a quotient of a totally ordered group (*o*-group) by the subgroup generated by a strong unit (a cofinal element in its centre). In that case, if $G = \langle G,+,-,0,u,\leq\rangle$ is an o-group with strong unit u, the quotient group $G_u = G_{/\langle u\rangle}$ can be endowed with a cyclic order by defining $T(\mathbf{a},\mathbf{b},\mathbf{c})$ if and only if, for the only representatives a,b,c such that $0 \leq a,b,c < u$, either $a < b < c$ or $b < c < a$ or $c < a < b$ holds.

The notion of *c*-group generalizes that of totally ordered groups (o-groups) in the sense that for a c-group with the property: for all $a \in G$, $T(-a,0,a)$ implies, for all $n \in \mathbb{N}$, $T(-na,0,na)$ a total order (compatible with the group operation) can be defined by $0 < a$ if and only if $T(-a,0,a)$. Conversely, an o-group can be endowed with a c-group structure by defining $T(a,b,c)$ if and only if $a < b < c$ or $b < c < a$ or $c < a < b$.

A *partially cyclically ordered group* (pco-group) is a system $\langle G,+,-,0,T\rangle$ where the axioms C2, C3, C4, C5 and

C1p. $\forall abc\ (T(a,b,c) \Longrightarrow \neg T(a,c,b))$;

C6. $\forall abc\ (T(a,b,c) \Longrightarrow T(-c,-b,-a))$ hold.

This last axiom is consequence of axioms C1..C5.

Observe that, Rieger's theorem also holds in this case by replacing the o-group by a partially ordered group (po-group) (see [13] or [14]).

An MV-*algebra* (see [4], [5] and [8]) is a system $\langle A,\oplus,*,\neg,0,1\rangle$ which satisfies the following universal identities:

m_1 $\quad x\oplus(y\oplus z) = (x\oplus y)\oplus z$

m_2 $\quad x\oplus 0 = x$

m_3 $\quad x\oplus y = y\oplus x$

m_4 $\quad x\oplus 1 = 1$

m_5 $\quad \neg\neg x = x$

m_6 $\quad \neg 0 = 1$

m_7 $\quad x\oplus\neg x = 1$

m_8 $\quad \neg(\neg x\oplus y)\oplus y = \neg(x\oplus\neg y)\oplus x$

m_9 $\quad x*y = \neg(\neg x\oplus\neg y)$

By defining $x\vee y := (x*\neg y)\oplus y$ and, by duality, $x\wedge y := \neg(\neg x\vee\neg y)$ we have that $\langle A,\vee,\wedge,0,1\rangle$ is a bounded distributive lattice.

Another approach for this structures is that of *Wajsberg algebras* (W-algebras) (see [6] and [11]). Such an algebra is a system $\langle A,\longrightarrow,\neg,0,1\rangle$ satisfying the following universal identities:

W1. $(x \longrightarrow y) \longrightarrow ((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) = 1$;

W2. $(x \longrightarrow y) \longrightarrow y = (y \longrightarrow x) \longrightarrow x$;

W3. $(\neg x \longrightarrow \neg y) \longrightarrow (y \longrightarrow x) = 1$;

W4. $1 \longrightarrow x = x$;

W5. $x \longrightarrow 0 = \neg x$;

W6. $\neg 1 = 0$;

W7. $\neg 0 = 1$.

By defining $x \vee y := (x \longrightarrow y) \longrightarrow y$ and $x \wedge y := \neg(\neg x \vee \neg y)$ $\langle A, \vee, \wedge, 0, 1\rangle$ results also a bounded distributive lattice.

In [6] it is proved that a *W*-algebra can be thought of as an *MV*-algebra (and viceversa) by identifying the respective 0,1 and $\neg$ and defining:

$$a \longrightarrow b := \neg a \oplus b \qquad \text{and} \qquad a \oplus b := \neg a \longrightarrow b;$$

(recall that the operation $*$ of the *MV*-algebra can be defined in terms of $\oplus$ and $\neg$).

In [4] it is proved that any *MV*-algebra *A* can be obtained from an abelian lattice-ordered group (ℓ-group) with strong unit u $G = \langle G, \vee, \wedge, +, -, 0, u\rangle$ by defining:

$A = [0,u] = \{a \,/\, 0 \le a \le u\}$; $a \oplus b = (a + b) \wedge u$; $\neg a = u - a$ and $1 = u$.

Since any *MV*-algebra derives from an abelian ℓ-group, in the sequel group will stand for *abelian* group, *homomorphism* and *subgroup* for homomorphism and subgroup for the respective structures (*o*-groups, *c*-groups, *pco*-groups, ℓ-groups, *MV*-algebras).

2. Lattice pco-groups.

For any *pco*-group *G*, a partial order can be defined by

$$a \le b \text{ if and only if } a = b \text{ or } T(0,a,b) \text{ or } a = 0. \qquad (*)$$

This order makes every element "positive". Observe that, in general, $\le$ is not compatible with the group operation, for example, by setting $G = \mathbb{Z}_{/3\mathbb{Z}}$ with its natural cyclical order, the total order $(*)$ induced is given by the set of pairs $\{(0,0),(1,1),(2,2),(0,1),(0,2),(1,2)\}$ which is obviously non-compatible, since $1 \le 2$ holds but $2 = 1 + 1 \le 2 + 1 = 0$ does not hold.

We say that a group homomorphism $f: G \longrightarrow H$ between *pco*-groups is a *pco*-homomorphism if, for $a,b,c \in G$ such that $T(a,b,c)$, if $f(a) \ne f(b) \ne f(c) \ne f(a)$ then $T(f(a),f(b),f(c))$.

Observe that a *pco*-homomorphism is also a homomorphism for the order given in $(*)$.

<u>Definition 2.1:</u> A pco-group G will be called a *lattice-cyclical-group* (and denoted ℓc-group), if, for the order defined in (*) the structure $\langle G,0,\leq\rangle$ admits a distributive lattice structure with first element.

<u>Lemma 2.2:</u> Let G be an ℓc-group, $a,b \in G$. If $a \leq a + b$ ($b \leq a + b$) then $b \leq a + b$ ($a \leq a + b$), implying $a \vee b \leq a + b$.

Proof: Suppose $0 < a < a + b$ (the other cases are immediate). Then we have $T(0,a,a + b)$, which, adding $-(a + b)$ to each term, implies $T(-(a + b),-b,0)$ which, by axiom C6, is equivalent to $T(0,b,a + b)$, proving our claim. ■

<u>Definition 2.3:</u> Let G be an ℓc-group and H a subgroup.

i) It is called an ℓc-ideal if it is convex for the order $\leq$ (that is, for all $x \in H$, $z \in G$, $z \leq x$ implies $z \in H$), and is an ℓ-subgroup (that is, for $x,y \in H$, $x \vee y \in H$).

ii) It is called a pc-subgroup if it is convex for the relation T (that is, for $x,y \in H$ and $z \in G$, $T(x,z,y)$ implies $z \in G$).

Observe that the ℓc-ideals (pc-subgroups) are the kernels of ℓc (pc)-homomorphisms. Moreover, the ℓc-ideals are lattice-ideals for the structure $\langle G,0,\vee,\wedge\rangle$. Observe also that for cyclically ordered groups, the T-convex subgroups are always trivial.

<u>Lemma 2.4:</u> Let G be an ℓc-group and H a subgroup. H is T-convex if and only if it is $\leq$-convex. So, any pc-subgroup preserving the lattice operations is also an ℓc-ideal.

Proof: Let H be T-convex, $a \in H$, $b \in G$ such that $0 \leq b \leq a$. If $b = 0$ or $b = a$, it is immediate that $b \in H$. So we can write $T(0,b,a)$, implying, by T-convexity, that $b \in H$.

For the converse, if H is $\leq$-convex, $a,c \in H$, $b \in G$ such that $T(a,b,c)$. By axiom C5 we have $T(0,b-a,c-a)$. Since H is $\leq$-convex, we conclude that $b-a \in H$ and then $b \in H$. ■

So, without abuse of notation, we can speak about convex subgroups.

<u>Lemma 2.5:</u> Let G be an ℓc-group, $H \subsetneq G$ an ℓc-ideal. H is prime if and only if the quotient $G_{/H}$ is cyclically ordered.

Proof: By a result on distributive lattices (see [1, III.3]) we have that the lattice $\langle G/_H, 0, \vee, \wedge\rangle \cong \langle G, 0, \vee, \wedge\rangle/_H$ is totally ordered if and only if H is prime as a lattice ideal. Since the notion of primeness is a set theoretic one, H is prime as lattice ideal if and only if it is so as ℓc-ideal. It is immediate to verify that the induced order $\preceq$ on a *pco*-group is total if and only if the group is cyclically ordered. ∎

As in the case of ℓ-groups, we can define the notions of orthogonality, projectability and weak unit:

Definitions 2.6: Let G be an ℓc-group, $g, h \in G$, A, B subsets of G.

i) g and h are *orthogonal*, $g \perp h$, if $g \wedge h = 0$.

ii) The *polar* of A, $A^{\perp} = \{x \ / \ \forall a(a \in A \Longrightarrow x \perp a)\}$. B is called a *polar* if $B = A^{\perp}$ for some A. If $A = \{g\}$ we shall write $g^{\perp}$ in place of $\{g\}^{\perp}$.

iii) The *double polar* of A, $A^{\perp\perp} = \{x \ / \ \forall y(y \in A^{\perp} \Longrightarrow x \perp y)\}$. Observe that B is a double polar if and only if it is a polar.

iv) G is called *projectable* if one can define a binary operation pr on G, compatible for the left argument with the group operations, such that, $h' = pr(g, h)$ implies $h' \in h^{\perp}$ and $g - h' \in h^{\perp\perp}$.

v) $u \in G$ is called a *weak unit* if, for all $g \in G$, $g \perp u$ implies $g = 0$.

Lemma 2.7: Let G be a projectable ℓc-group. Its polars are ℓc-ideals.

Proof: Let $g, h \in G$, A a subset of G. Consider a generic $a \in A$. By distributivity, it is immediate that $(g \vee h) \wedge a = (g \wedge a) \vee (h \wedge a)$. Since $g \le h$ implies $g \wedge a \le h \wedge a$, we have that $h \in a^{\perp}$ implies $g \in a^{\perp}$. Since $A^{\perp} = \bigcup\{a^{\perp} \ / \ a \in A\}$, we conclude that $A^{\perp}$ is a lattice-ideal. Suppose $g \perp a$ and $h \perp a$. By projectability, observe that $g = pr(g, a)$ and $h = pr(h, a)$. Since pr is compatible at left with the sum and the inverse, we have that $pr(g + h, a) = g + h$ and $pr(-g, a) = -g$, implying $(g + h) \perp a$ and $-g \perp a$. So we can conclude that $A^{\perp}$ is an ℓc-ideal. ∎

Lemma 2.8: Let G be a projectable ℓc-group, $g, h_1, h_2, h_3, h_4 \in G$ such that $h_1, h_3 \in h^{\perp}$; $h_2, h_4 \in h^{\perp\perp}$ and $g = h_1 + h_2 = h_3 + h_4$ then $h_1 = h_3$ and $h_2 = h_4$.

Proof: We have $h_1+h_2 = h_3+h_4$ implies $h_1-h_3 = h_2-h_4$. Since the polars are ℓc-ideals, we have that the first member belongs to $h^{\perp}$ and the second to $h^{\perp\perp}$, implying that both equal zero. ∎

From the above proved lemma, we conclude that the decomposition in terms of $h^{\perp}$ and $h^{\perp\perp}$ given by $pr(\ ,h)$ is the only one possible and, since $pr(pr(g,h),h) = pr(g,h)$ it can be well considered a projection.

We recall (see [3; § 8.1]) that given a language $\mathcal{L}$, an $\mathcal{L}$-structure G and a family $(L_i)_{i\in I}$ of $\mathcal{L}$-structures, G is a *Boolean product* of the family $(L_i)_{i\in I}$ (denoted by $G \in \Gamma(I,(L_i)_{i\in I})$) if and only if:

i) G is a subdirect product of the family $(L_i)_{i\in I}$ and

ii) I can be endowed with a Boolean space topology such that:

α) For any atomic $\mathcal{L}$-formula $\varphi(x_1,..,x_n)$ and $g_1,..,g_n \in G$, the set $\{i \ / \ L_i \models \varphi[g_1(i),..,g_n(i)]\}$ (denoted by $[\![\varphi[g_1,..,g_n]]\!]$) is clopen;

β) For $g,h \in G$ and J a clopen set of I, there exists the element of G given by $g_{|J} \cup h_{|I\setminus J}$ (patchwork property).

Let $(C_i)_{i\in I}$ be a family of c-groups and G a subgroup of ΠC_i. G will be endowed with a pca structure by considering the product ternary relation $T = \Pi T_i$. That is (a,b,c) if and only for all $i \in I$ $T(a_i,b_i,c_i)$ holds.

The following proposition is analogous to a result of Weispfenning on ℓ-groups (see [12]):

<u>Proposition 2.9:</u> An ℓc-group G is isomorphic to a Boolean product (in the language $\langle +,-,0,T,\vee,\wedge\rangle$) of (non-trivial) c-groups if and only if it is projectable and has a weak unit.

Proof: Let $G \in \Gamma(I,(C_i)_{i\in I})$ where $(C_i)_{i\in I}$ is a family of non-trivial c-groups. For each $i \in I$ there exists $h_i \in C_i$ such that $h_i \neq 0$. Since G is a subdirect product, there exist a family $(h'_i)_{i\in I} \subseteq G$ such that, for each $i \in I$, $h'_i(i) = h_i$. By property ii-α) above, for each $i \in I$, the set $[\![h'_i \neq 0]\!]$ is clopen. By compacity of I, a finite subset J of I can be found such that the family $\{[\![h'_i \neq 0]\!] \ / \ i \in J\}$ covers I. Now, by property iii), an element $h \in G$ can be found such

that $[\![h \neq 0]\!] = 0$. (This line of argumentation on Boolean products is standard and will not be repeated in the following proofs). We shall see that h is, indeed, a weak unit. For, suppose $g \in G$ and $g \wedge h = 0$. Since G is a subdirect product and $x \wedge y = 0$ is an atomic formula, for each $i \in I$, $g(i) \wedge h(i) = 0$ holds. But, for each c-group C_i, $h(i)$ is different from 0, implying that $g(i) = 0$ for all i and then $g = 0$.
For the projectability, let $g, h \in G$. Consider the clopen subset of I $J = [\![h \neq 0]\!]$. By property iii) call h'' the restriction of g to J and h' its restriction to $I \backslash J$. It is immediate to verify (since G is a subdirect product) that $g = h' + h''$ and $h' = pr(g,h)$.

For the converse. Let G be a projectable ℓc-group with weak unit u. We consider the Boolean algebra $B(G,u)$ with underlying set $\{pr(u,g) \ / \ g \in G\}$ and operations $pr(u,g) \vee pr(u,h) = pr(u, g \wedge h)$; $\neg pr(u,g) = u - pr(u,g) = pr(u, pr(u,g))$; $0_B = pr(u,u) = 0$ and $1 = pr(u,0) = u$. It is easy to verify that, if u, u' are weak units, we have the isomorphism $B(G,u) \cong B(G,u')$. So we can forget the weak unit and write $B(G)$ for the Boolean algebra of the group. Observe that polars of G and ideals of $B(G)$ are in a bijective correspondence: If A is a polar of G, $A \cap B(G)$ is an ideal of $B(G)$. If J is an ideal of $B(G)$, $J^G = \{g \in G \ / \ u - pr(u,g) \in J\}$ is a polar of G. Both constructions are each other inverses.
Let $I = \mathcal{S}p(B(G))$ the space of prime ideals of $B(G)$. By the above remark and lemma 2.7, we can identify it as a subspace of the space of prime ℓc-ideals of G. That set of ℓc-ideals distinguishes points: In particular, if $g \in G$, $g \neq 0$, there exists a prime ideal P of $B(G)$ such that $u - pr(u,g) \notin P$. Then $g_{/P^G} \neq 0$. So G can be represented as a subdirect product of the family $(C_i)_{i \in I}$ of ℓc-grups given by the quotients by the elements of I. Since, each of those ℓc-ideals is prime, by lemma 2.5, each C_i results cyclically ordered for the quotient of the relation T.
Finally we show that G (considered as a subdirect product) has properties ii-α) and ii-β) of the Boolean product definition. Any atomic formula $\varphi(\bar{x})$ is of the form or $T(t_1(\bar{x}), t_2(\bar{x}), t_3(\bar{x}))$ or $t_1(\bar{x}) = t_2(\bar{x})$ for t_1, t_2, t_3 terms in the group language.
For the sake of simplicity, we can suppose that the terms are just variables. We have, for a c-group $T(x_1, x_2, x_3) \Leftrightarrow T(0, x_2 - x_1, x_3 - x_1) \Leftrightarrow$

$\Leftrightarrow$ $0 < x_2-x_1 < x_3-x_1$ $\Leftrightarrow$ $(x_2-x_1)\vee(x_3-x_1) = x_3-x_1$ & $x_2-x_1 \neq 0$ & $x_3-x_2 \neq 0$. Let be now $g_1, g_2, g_3 \in G$, call $b = \neg pr(u, g_2-g_1)$, $a = pr(u, g_3-g_1-((g_2-g_1)\vee(g_3-g_1)))$ and $c = \neg pr(u, g_3-g_2)$. Now, by the above considerations about the definition of T on a subgroup of a product of c-groups, the element $a\wedge b\wedge c$ of the Boolean algebra $B(G)$ corresponds to $[\![T(g_1, g_2, g_3)]\!]$. And since the elements of $B(G)$ are in correspondence with the clopen sets of $\mathcal{S}p(B(G))$, we are done. For the formula $x_1 = x_2$, and $g_1, g_2 \in G$, it suffices to take $a = pr(u, g_1-g_2)$, proving property ii-α).

Property ii-β) results from projectability. Let $g, h \in G$ and J a clopen set of I, there exists then $c_J \in G$ such that $c_J = pr(u, u-c_J) = \neg pr(u, c_J)$ and that element "corresponds" to J. So, we have the identity $g_{|J} \cup h_{|I\setminus J} = pr(g, u-c_J) + pr(h, c_J)$. ∎

3. The standard construction. We recall the result of V. Weispfenning (see [12]), which states that an ℓ-group is isomorphic to a Boolean product of totally ordered groups if and only if it is projectable and has a weak unit.

Let G be a projectable ℓ-group and $u \in G$ a strong unit. Define the ℓ-subgroup $H(u)$ generated by all the elements of the form $u_{|g^+}$ (with g ranging by all the elements of G). Consider the quotient group $G_u = G_{/H(u)}$.

<u>Proposition 3.1:</u> The group G_u admits a natural ℓc-structure.

Proof: By the above stated observation, we shall consider $G \in \Gamma(I, (L_i)_{i\in I})$ for some family $(L_i)_{i\in I}$ of totally ordered groups. First, observe that, for any $g_u \in G_u$ there exists only one $a \in [0,u) = \{h \in G \;/\; 0 \leq h < u\}$ such that $a_u = g_u$: Let be $g \in G$. Since u is a strong unit, we have that there exists $n \in \mathbb{N}$ such that $nu > |g|$. For $m \in \mathbb{Z}$ such that $-n \leq m < n$, call I_m the clopen subset of I given by $[\![mu \leq g < (m+1)u]\!]$. Calling g_m the restriction of g to I_m, we have that it has a representative in the interval $[0, u_m)$. Now, by the patchwork property, we can patch all those representatives and obtain an element $a \in [0,u)$ such that $a_u = g_u$. It is immediate that any two of the elements in the interval are not congruent modulo $H(u)$.

Now, for $a_u, b_u, c_u \in G_u$, consider the representatives $a, b, c \in [0, u)$. We shall define $T(a_u, b_u, c_u)$ if and only if

$$I = [\![a < b < c \text{ or } b < c < a \text{ or } c < a < b]\!].$$

The proof that this defines a partial cyclic order is analogous to that for the cyclic order case (see [10]).

Call $\leq_u$ the order induced by T. It is immediate to verify that $a_u \leq_u b_u$ if and only if $a \leq b$ for a, b representatives in $[0, u)$. Since for this order that interval is a distributive lattice with first element, we can conclude that its lattice structure is copied, isomorphically on G_u. ∎

The Boolean product characterization allows us to prove the converse.

Proposition 3.2: Let G be a projectable ℓc-group with weak unit. There exists an ℓ-group G' with a strong unit u such that $G \cong G'_u$ in the above sense.

Proof: We can suppose $G \in \Gamma(I, (C_i)_{i\in I})$ for some family $(C_i)_{i\in I}$ of c-groups. By Rieger's theorem, there exists a family $(L_i, u_i)_{i\in I}$ of o-groups with strong units such that for each $i \in I$, $C_i \cong L_{i/<u_i>}$. Consider now the direct product ΠL_i and identify the elements of G with the elements in the product of intervals $\Pi[0, u_i)$. Now call G' the ℓ-group spanned by G and $(u_i)_{i\in I}$ in ΠL_i. By construction, it results that $G' \in \Gamma(I, (L_i)_{i\in I})$ and it is immediate to prove that, setting $u = (u_i)_{i\in I}$, $G \cong G'_u$. ∎

4. The functorial equivalence. In the sequel we shall restrict ourselves to projectable MV-algebras, which can be defined analogously to the case of ℓc (ℓ) -groups. In particular, it holds that a projectable MV-algebra is isomorphic to an element of $\Gamma(I, (L_i))_{i\in I}$ for a family $(L_i)_{i\in I}$ of totally ordered MV-algebras. (This result is analogous of that of Weispfenning on ℓ-groups and can be found -implicitly- in [11]).

In an MV-algebra, an element a is called *boolean* if $a \perp \neg a$.

Let $A = \langle A, \oplus, *, \neg, 0, 1\rangle$ be an MV-algebra and consider the equivalence relation $\sim$ given by:

$a \sim b$ if and only if there exist boolean elements a' and b' such that

$a\oplus a' = b\oplus b'$, $a\perp a'$, $b\perp b'$ and $a'\perp b'$. By considering A as a boolean product over a space I, this corresponds to the identity $I = [\![a = b]\!]\cup[\![a = 0 \,\&\, b = 1]\!]\cup[\![b = 0 \,\&\, a = 1]\!]$. We show that $\sim$ is, indeed, an equivalence relation:

- By taking $a' = 0$, we prove that $a\sim a$.
- The reflexivity results from the definition.
- Let be $a\sim b\sim c$. We shall use the boolean product characterization of the relation $\sim$:

$I_1 = [\![a = c]\!] = ([\![a = b]\!]\cap[\![b = c]\!])\cup[\![a = 0 \,\&\, c = 0]\!]\cup[\![a = 1 \,\&\, c = 1]\!]$;

$I_2 = [\![a = 0 \,\&\, c = 1]\!] =$

$= ([\![a = b]\!]\cap[\![b = 0 \,\&\, c = 1]\!])\cup([\![c = b]\!]\cap[\![b = 1 \,\&\, a = 0]\!])$;

$I_3 = [\![a = 1 \,\&\, c = 0]\!] =$

$= ([\![a = b]\!]\cap[\![b = 1 \,\&\, c = 0]\!])\cup([\![c = b]\!]\cap[\![b = 0 \,\&\, a = 1]\!])$.

A simple set-theoretic manipulation proves that $I = I_1\cup I_2\cup I_3$ and then $a\sim c$.

We define the group operations in $G = A_{/\sim}$ by

$-(a_{/\sim}) := \neg a_{/\sim}$.

Given $a_{/\sim}, b_{/\sim} \in G$, consider the clopen set $J = [\![a\oplus b < 1]\!]$ and define

$(a_{/\sim}) + (b_{/\sim}) = ((a\oplus b)_{|J}\cup(a*b)_{|I\setminus J})_{/\sim}$.

To verify that those operations are well-defined, since we are dealing with subdirect products, it suffices to consider the totally ordered case:

For that case we have $a\sim b$ if and only if $a = b$ or ($a = 0$ and $b = 1$) or ($a = 1$ and $b = 0$). For the difference: $\neg 0_{/\sim} = 1_{/\sim} = 0_{/\sim} = \neg 1_{/\sim}$.

For the sum, it suffices to consider the case $a_{/\sim} = 0_{/\sim}$ and $0 < b < 1$. So we have $0_{/\sim} + b_{/\sim} = (0\oplus b)_{/\sim} = b_{/\sim} = (1*b)_{/\sim} = 1_{/\sim} + b_{/\sim}$.

We show that $\langle G,+,-,0\rangle$ is an abelian group:

Recall the theorem 16 in [6] which implies that the variety of *MV*-algebras is generated by the *MV*-algebra $\mathbb{Q}[0,1]$ with underlying set $\{x \in \mathbb{Q} \;/\; 0 \le x \le 1\}$ and operations $x\oplus y = 1\wedge(x + y)$ and $\neg x = 1-x$. So any equation is true in the variety if and only if it holds in $\mathbb{Q}[0,1]$. We shall consider then $A = \mathbb{Q}[0,1]$.

- The commutativity results from that of $\oplus$ and $*$;
- $a_{/\sim} + 0_{/\sim} = (a\oplus 0)_{/\sim} = a_{/\sim}$;
- $a_{/\sim} + (-(a_{/\sim})) = a_{/\sim} + \neg a_{/\sim} = (a*\neg a)_{/\sim} = 0_{/\sim}$ because $a\oplus\neg a = 1$;
- For the associativity, let $a_{/\sim}, b_{/\sim}, c_{/\sim} \in G$:

Case $(a\oplus b)\oplus c < 1$: Results from the associativity of $\oplus$;

Case $a\oplus b = 1$ and $(a*b)\oplus c = 1$: Since $a*b \le b$, we have $b\oplus c = 1$ and then $(a_{/\sim} + b_{/\sim}) + c_{/\sim} = (a*b)*c$ (1).

$a\oplus(b*c) = 1\wedge(a + (b*c)) = 1\wedge(a + \neg(\neg b\oplus\neg c)) =$

$= 1\wedge(a + (1-(1\wedge(1-b + (1-c))))) = 1\wedge(a + (1-(1\wedge(2-(b + c))))) =$

$= 1\wedge(a + (1-(2-(b + c)))) = 1\wedge(a + b + c - 1) = (a*b)\oplus c$ because $a*b = a + b - 1$. And, by hipothesis, $(a*b)\oplus c = 1$. So we have $a_{/\sim} + (b_{/\sim} + c_{/\sim}) = (a*b)*c$ which coincides with (1).

Case $a\oplus b = 1$, $(a*b)\oplus c < 1$ and $b\oplus c < 1$:

$(a_{/\sim} + b_{/\sim}) + c_{/\sim} = (a*b)\oplus c = 1\wedge(a*b + c) = 1\wedge(\neg(\neg a\oplus\neg b) + c) =$

$= 1\wedge(1-(1\wedge(1-a + (1-b))) + c) = 1\wedge(1-(1\wedge(2-(a + b))) + c) =$

$= 1\wedge(1-(2-(a + b)) + c) = 1\wedge(a + b + c - 1)$ (2).

Since $a\oplus(b\oplus c) \ge a\oplus b = 1$, we have $a_{/\sim} + (b_{/\sim} + c_{/\sim}) = a*(b\oplus c)$. An analogous treatment yields $a*(b\oplus c) = (2)$.

The rest of the cases are treated in a similar way, proving the associativy.

Now, for the relation T, given $a_{/\sim}, b_{/\sim}, c_{/\sim} \in G$, define the following clopen sets:

$I_1 = [\![(a < b < c)\ \&\ (a \ne 0 \text{ or } c \ne 1)]\!]$,

$I_2 = [\![(b < c < a)\ \&\ (b \ne 0 \text{ or } a \ne 1)]\!]$,

$I_3 = [\![(c < a < b)\ \&\ (c \ne 0 \text{ or } b \ne 1)]\!]$.

Define a pc-order by $T(a_{/\sim}, b_{/\sim}, c_{/\sim})$ if and only if $I = \bigcup_{j=1}^{3} I_j$. It is immediate that T satisfies properties C1p, C2, C3, C4, C5 and C6. The good definition results from the second condition in each I_j. Since the order $\le_c$ defined on G by $g \le_c h$ if and only if $T(0,g,h)$ or $g = 0$ or $g = h$ coincides with the order $\le$ of A (modulo $\sim$), we have that it induces a lattice structure.

For the compatibility of + and T it also suffices to consider the totally ordered case: Let be $a,b,c,d \in A$ such that $a < b < c < 1$ and $d < 1$.

- If $c\oplus d < 1$ we have $a\oplus d < b\oplus d < c\oplus d < 1$;
- If $a\oplus d = b\oplus d = c\oplus d = 1$, we have $a*d < b*d < c*d$;
- If $a\oplus d, b\oplus d < 1$ and $c\oplus d = 1$ we have $c*d < d \le a\oplus d < b\oplus d$;
- The case $a\oplus d < 1$ and $b\oplus d, c\oplus d = 1$ is analogous.

If $f: A \longrightarrow B$ is an MV-homomorphism, it is immediate to verify that $f_{/\sim}$ is well-defined and then, an ℓc-group homomoprhism.

Reciprocally, let $G = \langle G,+,-,0,u,T\rangle$ be a projectable ℓc-group with weak unit. We can identify G with an element of $\Gamma(I,(L_i)_{i\in I})$ for some family $(L_i)_{i\in I}$ of c-groups, where the Boolean space I is the one constructed in the second part of the proof of proposition 2.9. The Boolean algebra $B(I)$ of clopen sets of I (considered as a set algebra) can be also identified with the algebra of supports of elements of G.

Define $A = \{(g,\alpha) \in G\times B(I) \ / \ \mathrm{supp}(g)\cap\alpha = \emptyset\}$.

We define on A the *MV* operations:

The 0 of the *MV*-algebra will be the element $(0,\emptyset)$ and the 1 the element $(0,I)$.

Let $(g,\alpha) \in A$, call $\beta = I\backslash\mathrm{supp}(g)$. Define $\neg(g,\alpha) = (-g,(I\backslash\alpha)\cap\beta)$.

Given $(g,\alpha),(h,\beta) \in A$, consider the clopen set $\gamma = I\backslash(\alpha\cup\beta)$ and the elements of G $g' = g_{|\gamma}$ and $h' = h_{|\gamma}$. Call δ the clopen set $\gamma\cap([\![T(0,g',g'+h')]\!]\cup[\![g'=0]\!]\cup[\![h'=0]\!])$ which coincides with $\gamma\cap[\![g' \le g'+h']\!]$. (Observe that lemma 2.2 implies $T(0,g',g'+h')$ if and only if $T(0,h',g'+h')$). And finaly $\eta = [\![\neg T(0,g',g'+h')]\!]$. Now define:

$$(g,\alpha)\oplus(h,\beta) = ((g'+h')_{|\delta},\alpha\cup\beta\cup\eta).$$

The operation $*$ is defined in terms of $\oplus$ and $\neg$.

We shall proof that $A = \langle A,\oplus,*,\neg,0,1\rangle$ is in effect an *MV*-algebra.

m_1: Let $(g,\alpha),(h,\beta),(k,\gamma) \in A$.

By setting $\delta = I\backslash\alpha\cup\beta\cup\gamma$, $g' = g_{|\delta}$, $h' = h_{|\delta}$, $k' = k_{|\delta}$, $\varepsilon = [\![g' \le g'+h' \le g'+h'+k']\!]$, $\eta = \varepsilon\cap\delta$ and $\kappa = \neg[\![g' \le g'+h' \le g'+h'+k']\!]$, we have that

$((g,\alpha)\oplus(h,\beta))\oplus(k,\gamma) = (g,\alpha)\oplus(h,\beta)\oplus(k,\gamma) =$

$= ((g'+h'+k')_{|\eta},\alpha\cup\beta\cup\gamma\cup\kappa)$, implying the associativity.

m_5: Let $(g,\alpha) \in A$, $\beta = I\backslash\mathrm{supp}(g)$, then $\neg(g,\alpha) = (-g,(I\backslash\alpha)\cap\beta)$. Since $\mathrm{supp}(-g) = \mathrm{supp}(g)$, we have $\neg\neg(g,\alpha) = (g,I\backslash((I\backslash\alpha)\cap\beta)\cap\beta) = (g,\alpha)$ because $\alpha \subseteq \beta$.

m_8: We shall prove that $\neg(\neg x\oplus y)\oplus y = x\vee y$, proving then the equation $\neg(\neg x\oplus y)\oplus y = \neg(x\oplus\neg y)\oplus x$. Let $(g,\alpha),(h,\beta) \in A$. Using the Boolean product characterization, we have $\neg(\neg x\oplus y)\oplus y = x\vee y$ if and only if, for each $i \in I$,

$$(\neg(\neg x \oplus y)\oplus y)(i) = \begin{cases} x(i) \text{ if } y(i) \le x(i); \\ y(i) \text{ if } x(i) \le y(i)).' \end{cases}$$

which translated to the elements of A results:

$(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta))(i) =$

$(g,\alpha)(i)$ if $T(0,h(i),g(i))$ or $(g(i) \neq 0$ and $h(i) = g(i))$ or $(g(i) = 0$ and $\alpha(i) = 1)$ or $h(i) = \beta(i) = 0$;

$(h,\beta)(i)$ if $T(0,g(i),h(i))$ or $(h(i) \neq 0$ and $g(i) = h(i))$ or $(h(i) = 0$ and $\beta(i) = 1)$ or $g(i) = \alpha(i) = 0$;

Case $g(i) = \alpha(i) = 0$:
$\neg(g,\alpha)(i) = (0,1)$ and then $(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta))(i) =$
$= (\neg((0,1)\oplus(h,\beta))\oplus(h,\beta))(i) = ((0,0)\oplus(h,\beta))(i) = (h,\beta)(i)$.

Case $g(i) = 0$, $\alpha(i) = 1$:
$\neg(g,\alpha)(i) = (0,0)$ and then $(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta))(i) =$
$= (\neg((0,0)\oplus(h,\beta))\oplus(h,\beta))(i) = (\neg(h,\beta)\oplus(h,\beta))(i) = (0,1) = (g,\alpha)(i)$.

Case $h(i) = \beta(i) = 0$:
$(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta))(i) = (\neg(\neg(g,\alpha)\oplus(0,0))\oplus(0,0))(i) =$
$= \neg\neg(g,\alpha)(i) = (g,\alpha)(i)$.

Case $h(i) = 0$, $\beta(i) = 1$:
$(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta))(i) = (\neg(\neg(g,\alpha)\oplus(0,1))\oplus(0,1))(i) = (0,1) =$
$= (h,\beta)(i)$;

Case $T(0,g(i),h(i))$, that is $0 < g(i) < h(i)$ and $\alpha(i) = \beta(i) = 0$: that implies $\neg(g,\alpha)(i) = (-g,0)(i) > (-h,0)(i) = \neg(h,\beta)(i)$, and then $\neg(g,\alpha)(i)\oplus(h,\beta)(i) = (0,1)$, concluding that
$(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta))(i) = \neg(0,1)\oplus(h,\beta)(i) = (h,\beta)(i)$.

Case $T(0,h(i),g(i))$, that is $0 < h(i) < g(i)$ and $\alpha(i) = \beta(i) = 0$: Since $\neg(g,\alpha)(i) < \neg(h,\beta)(i)$, we have $\neg(g,\alpha)(i)\oplus(h,\beta)(i) < (0,1)$, implying $\neg(g,\alpha)(i)\oplus(h,\beta)(i) = (-g + h,0)(i)$. Then
$(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta))(i) = (\neg(-g + h,0))\oplus(h,0))(i) =$
$((g - h,0)(i))\oplus(h,0)(i)$ which is equal to $(g,0)(i)$ because we have $T(0,g(i)-h(i),g(i))$.

Case $g(i) = h(i) \neq 0 = \alpha(i) = \beta(i)$: We have $\neg(g,\alpha)(i) = \neg(h,\beta)(i)$.

So $(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta))(i) = (\neg(0,1)\oplus(h,\beta))(i) = ((0,0)\oplus(h,\beta))(i)$ which equals to $(h,\beta)(i)$.

m_2, m_3, m_4, m_6 and m_7 are immediate and m_9 can be considered a definition.

If $f\colon G \longrightarrow H$ is an ℓc-homomorphism, observe that f induces a Boolean algebra homomorphism $B(f) = B(G) \longrightarrow B(H)$, where $B(G)$ and $B(H)$ are the respective Boolean algebras of supports: Define $B(f)(\mathrm{supp}(g)) = \mathrm{supp}(f(g))$. The good definition results from the fact that f maps weak units on weak units ans preserves the lattice operations: So, let $g, g' \in G$ such that $\mathrm{supp}(g) = \mathrm{supp}(g')$. Let u be a weak unit in G. The element $g'' = pr(u, g')$ is orthogonal to both g and g', and both $g + g''$ and $g' + g''$ are weak units. So since $\mathrm{supp}(f(g) + f(g'')) = \mathrm{supp}(f(g') + f(g'')) = I'$ (where I' is the Boolean space of H) and $f(g') \perp f(g'')$ we have that $\mathrm{supp}(f(g')) \subseteq \mathrm{supp}(f(g))$. The proof of the other inclusion is analogous.

Now, if A and B are the respective MV-algebras constructed from G and H respectively, as above, define $\bar{f}\colon A \longrightarrow B$ by $\bar{f}((g,\alpha)) = (f(g), B(f)(\alpha))$. We shall proof that it is an MV-homomorphism: Let $(g,\alpha),(h,\beta) \in A$, call $\alpha' = I\backslash\mathrm{supp}(g)$ (where I is the Boolean space of G). Then $\bar{f}(\neg(g,\alpha)) = \bar{f}(-g, (I\backslash\alpha)\cap\alpha') = (f(-g), B(f)((I\backslash\alpha)\cap\alpha')) = (-f(g), (B(f)(I)\backslash B(f)(\alpha))\cap B(f)(\alpha'))) = (-f(g), (I'\backslash B(f)(\alpha))\cap B(f)(\alpha'))$. By calling $\alpha'' = I'\backslash\mathrm{supp}(f(g))$, we have also $\neg\bar{f}((g,\alpha)) = (-f(g), (I'\backslash B(f)(\alpha))\cap\alpha'')$. Since $\alpha'' = B(f)(\alpha')$ we have that $\bar{f}$ preserves the operation $\neg$.

For $\oplus$, call $\gamma = I\backslash(\alpha\cup\beta)$, $g' = g_{|\gamma}$, $h' = h_{|\gamma}$, $\delta = \gamma\cap[\![g' \leq g' + h']\!]$ and $\eta = \neg[\![g' \leq g' + h']\!]$. We have $(g,\alpha)\oplus(h,\beta) = ((g' + h')_{|\delta}, \alpha\cup\beta\cup\eta)$.
$\bar{f}((g,\alpha)\oplus(h,\beta)) = (f((g' + h')_{|\delta}), B(f)(\alpha\cup\beta\cup\eta)) =$
$= (f(g'_{|\delta}) + f(h'_{|\delta}), B(f)(\alpha\cup\beta\cup\eta))$. By the other side, calling
$\mu = B(f)(\alpha)$, $\nu = B(f)(\beta)$, $\sigma = I'\backslash(\mu\cup\nu) = B(f)(\gamma)$,
$\upsilon = \neg[\![f(g') \leq f(g') + f(h')]\!]$ (because f preserves the relation T), $g'' = f(g)_{|\sigma}$, $h'' = f(h)_{|\sigma}$, and $\tau = \sigma\cap[\![g'' \leq g'' + h'']\!]$, we have $\bar{f}((g,\alpha))\oplus\bar{f}((h,\beta)) = (f(g),\mu)\oplus(f(h),\nu) =$

$= ((f(g) + f(h))_{|\tau}, \mu\cup\nu\cup\upsilon)$. Since, for each $i \in I$,

$g'(i) \leq g'(i) + h'(i)$ if and only if $f(g')(i) \leq f(g')(i) + f(h')(i)$

because of axiom C1 and the fact that f is an ℓc-homomorphism, we have that $\upsilon = B(f)(\eta)$, proving $\tilde{f}((g,\alpha)\oplus(h,\beta)) = \tilde{f}((g,\alpha))\oplus\tilde{f}((h,\beta))$.

Finally we show that the compositions of both functors are the identity:

Call $\mathcal{LC}$ and $\mathcal{MV}$, the categories of projectable ℓc-groups with weak unit and projectable MV-algebras, respectively, $\Psi: \mathcal{MV} \longrightarrow \mathcal{LC}$ and $\Phi: \mathcal{LC} \longrightarrow \mathcal{MV}$ the above constructed functors.

Let $G \in \mathcal{LC}$, $\Phi(G) = \{(g,\alpha) \in G\times B(I) \ / \ \mathrm{supp}(g)\cap\alpha = \emptyset\}$ (as a set) and $\Psi(\Phi(G)) = \Phi(G)_{/\sim}$ (as a set). Observe that $a = (g,\alpha) \sim (h,\beta) = b$ if and only if $g = h$: by taking $a' = (0,\beta\setminus\alpha)$ and $b' = (0,\alpha\setminus\beta)$, we have $a\oplus a' = b\oplus b'$, $a'\perp b'$, $a\perp a'$ and $b\perp b'$, implying $(g,\alpha) \sim (g,\beta)$. Suppose now $g \neq h$, then the set $[\![a = b]\!]\cup[\![a = 0 \ \& \ b = 1]\!]\cup[\![a = 1 \ \& \ b = 0]\!]$ is strictly contained in I, implying that (g,α) is not equivalent to (h,β). Now, for the operations, it is immediate for 0 and $-$. Let $g,h \in G$, we can choice, for their images in $\Phi(G)$, the elements $(g,\emptyset)$ and $(h,\emptyset)$ respectively. By calling $J = [\![(g,\emptyset)\oplus(h,\emptyset) < 1]\!]$, we have, in $\Psi(\Phi(G))$, $g + h = (((g,\emptyset)\oplus(h,\emptyset))_{|J}\cup((g,\emptyset)*(h,\emptyset))_{|I\setminus J})_{/\sim}$. Observe that $J = [\![g \leq g + h]\!]$ and then $(g,\emptyset)\oplus(h,\emptyset) = ((g + h)_{|J}, I\setminus J]\!]$. So, it holds $g + h = (g + h)_{|J}\cup(((g,\emptyset)*(h,\emptyset))_{|I\setminus J})_{/\sim} =$

$= (g + h)_{|J}\cup(\neg(\neg(g,\emptyset)\oplus\neg(h,\emptyset))_{|I\setminus J})_{/\sim} =$

$= (g + h)_{|J}\cup(\neg((-g,\emptyset)\oplus(-h,\emptyset))_{|I\setminus J})_{/\sim} = (g + h)_{|J}\cup(\neg((-g-h,\emptyset))_{|I\setminus J})_{/\sim}$

because $[\![-g \leq -g-h]\!] = I\setminus J$. So, we can conclude that (in $\Psi(\Phi(G))$), $g + h = (g + h)_{|J}\cup(-(-g-h))_{|I\setminus J} = g + h$ (in G). We have, proved, then, that $\Psi\circ\Phi = \mathrm{Id}_G$.

For the converse, let $A \in \mathcal{MV}$. In $\Psi(A)$ the elements of A which coincide modulo a Boolean element are identified. Let $a \in A$. By setting $\alpha = [\![a = 1]\!]$, we have that, in $\Phi\circ\Psi(A)$ the element $(a_{/\sim},\alpha)$

corresponds to a (in A). So, it is immediate to verify that the application $a \longmapsto (a_{/\sim},\alpha)$ gives a bijection between A and $\Phi\circ\Psi(A)$ preserving the 0 and 1. For the negation, $[\neg a = 1] = [a = 0] = I\setminus(\alpha\cup\mathrm{supp}(a_{/\sim}))$ and call $\beta = I\setminus\mathrm{supp}(a_{/\sim})$. We have then $\neg(a_{/\sim},\alpha) = (-(a_{/\sim}),(I\setminus\alpha)\cap\beta)) = (\neg a_{/\sim}, I\setminus(\alpha\cup\mathrm{supp}(a_{/\sim})))$ proving that the above defined map preserves also the negation.

Finally, for the *MV* sum, let $a,b \in A$, $\alpha = [a = 1]$ and $\beta = [b = 1]$. Define $\gamma = I\setminus(\alpha\cup\beta)$, $(a_{/\sim})' = (a_{/\sim})_{|\gamma} = (a_{|\gamma})_{/\sim}$, $(b_{/\sim})' = (b_{/\sim})_{|\gamma} = (b_{|\gamma})_{/\sim}$, $\delta = \gamma\cap[\![(a_{/\sim})' \le (a_{/\sim})' + (b_{/\sim})']\!]$ and $\eta = \neg[\![(a_{/\sim})' \le (a_{/\sim})' + (b_{/\sim})']\!]$. So, we can write

$(a_{/\sim},\alpha)\oplus(b_{/\sim},\beta) = (((a_{/\sim})' + (b_{/\sim})')_{|\delta}, \alpha\cup\beta\cup\eta)$.

Call now $J = [a_{|\gamma}\oplus b_{|\gamma} < 1]$. We have then

$(a_{/\sim})' + (b_{/\sim})' = (a\oplus b)_{|J\cap\gamma}\cup(a*b)_{|(I\setminus J)\cap\gamma}$, which implies

$(a_{/\sim},\alpha)\oplus(b_{/\sim},\beta) = (a\oplus b)_{|J\cap\delta}\cup(a*b)_{|(I\setminus J)\cap\delta}\cup\alpha\cup\beta\cup\eta$. It is easy to verify that $J = \delta$, implying $(a_{/\sim},\alpha)\oplus(b_{/\sim},\beta) = (a\oplus b)_{|\delta}\cup\alpha\cup\beta\cup\eta = a\oplus b$ because $\alpha\cup\beta\cup\eta = [\![a\oplus b]\!] = 1$.

So we can state the

<u>Theorem 4.1:</u> The categories $\mathscr{LC}$ and *MV* are equivalent.

References

[1] Balbes,R. and Dwinger,P., *Distributive lattices*, University of Missouri Press, 1974.

[2] Bigard,A.,Keimel,K. and Wolfenstein,S., *Groupes et anneaux réticulés*, Springer LNM 608, 1978.08, 1978.

[3] Burris,S. and Sankappanavar,H.P., *A course in universal algebra*, Springer-Verlag, New York-Heidelberg-Berlin, 1981.

[4] Chang,C.C., *Algebraic analysis of many-valued logics*, Trans.Am.Math.Soc. 88 (1958) 467-490.

[5] Chang,C.C., *A new proof of the completeness of the Łukasiewicz axioms*. Trans.Am.Math.Soc. 93 (1959) 74-80.

[6] Font,J.,Rodriguez,A. and Torrens,A., *Wajsberg algebras*, Stochastica, VIII (1984), 5-31.

[7] Lacava, F., *Alcune proprietà delle l-algebre e delle t-algebre essistenzialmente chiuse*, Boll. Unione Mat. Italiana, (5) 16A (1979), 360-366.

[8] Mundici, D., *Interpretation of AF* $\mathbb{C}^*$*-Algebras in Łukasiewicz sentential calculus*. J.Func.An. 65 (1986), 15-63.

[9] Rieger, L., *On ordered and cyclically ordered groups* I, II, III, Vestnik kral. české spol. nauk 1946, 1-31; 1947, 1-33; 1948, 1-26. (In Czech).

[10] Swierczkowski, S., *On cyclically ordered groups*, Fund. Math. 47, 1959, 161-166.

[11] Torrens, A., *W-algebras which are Boolean products of members of SR[1] and CW-algebras*, Studia Logica, XXLVI, 33 (1987), 263-272.

[12] Weispfenning, V., *Elimination of quantifiers for certain ordered and lattice-ordered abelian groups*, Bull. Soc.Math.Belg., XXXIII (1981) Fasc.I, serie B, 131-155.

[13] Zabarina, A.I. and Pestov, G.G., *Swierczkowski's theorem*, Sibirski Math. J., 25 (1984), 46-53.

[14] Zheleva, S.D., *Cyclically ordered groups*, Sibirski Math. J., 17 (1976), 1046-1051.

Département de Mathématiques
U.F.R. Structures et Matériaux
Université d'Angers
2, Bd. Lavoisier
49000 Angers - France

SPACES OF ORDERINGS OF FIELDS, GENERALIZATIONS AND APPLICATIONS

by

Murray Marshall

Department of Mathematics, University of Saskatchewan

Saskatoon, Saskatchewan, Canada, S7N 0W0

The object of this paper is to survey the theory of spaces of orderings, its origins in field theory, and various generalizations and applications of the theory that have emerged in the past 15 years. A special feature of the paper is a large list of references. Hopefully this will be of some value to a novice to the area. See [96,97,142] for additional references.

Spaces of orderings were originally introduced [90,107-111,113] as an abstract device for studying orderings and the reduced theory of quadratic forms over fields [11,12,19-21,28,30,39-40,133,139-141,147]. In the field case, spaces of orderings arise as follows: Take a field F and a preordering $T \subseteq F$. (For example, take $T = \Sigma F^2$.) Take $X = X_T$:= the set of all orderings of F lying over T and $G = G_T := F^*/T^*$. Then the pair $(X,G) = (X_T,G_T)$ is a space of orderings. Elements of X are viewed as characters on G. The theory has since been generalized in two different directions:

(1) To abstract quadratic form schemes satisfying the "linkage axiom" [32,94,112,121,148]; equivalently, abstract Witt rings which are "strongly representational" [83]. These were introduced initially to study the (non-reduced) theory of quadratic forms over fields. Linked quadratic form schemes satisfying $D\langle 1,1\rangle = \{1\}$ correspond exactly to strongly representational Witt rings which are reduced (nilradical = 0) and these, in turn, correspond exactly to spaces of orderings.

(2) To Becker's reduced theory of diagonal forms of higher degree [7-10,13]. Here, the abstract objects being studied are called "spaces of signatures" [116,119,130-132,134,135,137], a level 1 space of signatures being just a space of orderings. Here again, the initial examples come from field theory: If $T \subseteq F$ is a preordering of level n (for example, $T = \Sigma F^{2n}$) then one has an associated space of signatures (X_T,G_T) where $G_T := F^*/T^*$ and X_T := all signatures of higher level lying over T.

Additional examples of spaces of orderings (and also of the generalizations (1) and (2) above) are now known. In fact, (1) and (2) both apply not just to fields, but to semi-local rings [83,84] and, more generally, to rings with many units [120,158]. The theory of spaces of orderings applies to skew fields [44,50,157], and even to planar ternary rings [68,72,73]. (2) also applies to skew fields [136,138] and possibly also to planar ternary rings, but there seems to be some problem in interpreting (1) even in the case of skew fields; see [149].

Interestingly enough, in the abstract case, there is a natural common generalization of (1) and (2) (see the Remark in [116]) and this is probably worth investigating further. But first one needs to show, in the field case, that the type of "higher level scheme" described in [116] actually occurs, and this is not at all clear.

For a space of orderings (X,G), we give X the topology induced by the embedding $X \subseteq \mathrm{Hom}(G,\{\pm1\})$. A "fan" in X is a closed set $V \subseteq X$ satisfying $\alpha,\beta,\gamma \in V \Rightarrow \alpha\beta\gamma \in V$. For any finite fan $V \subseteq X$, $|V| = 2^{k-1}$ where k is the $\mathbf{Z}/2\mathbf{Z}$-dimension of $G/V^{\perp}$. If $\alpha,\beta \in X$ (possibly $\alpha = \beta$) then $V = \{\alpha,\beta\}$ is a fan (called a "trivial" fan). In the field case, non-trivial fans all arise in a natural way from valuations on F [20,30].

The Witt ring $W = W(X,G)$ of a space of orderings (X,G) is the subring of the function ring $\mathrm{Cont}(X,\mathbf{Z})$ generated by the functions $\sigma \to \sigma(a)$, $a \in G$. The cokernel of the embedding $W \subseteq \mathrm{Cont}(X,\mathbf{Z})$ is 2-primary torsion [85]. (X,G) can be recovered from W (G = the group of units of W, X = all homomorphisms $\sigma : W \to \mathbf{Z}$). Thus, the study of spaces of orderings is equivalent to the study of their Witt rings. Elements of W are represented by anisotropic quadratic forms $\rho = \langle a_1,\ldots,a_n\rangle$, $a_1,\ldots,a_n \in G$, just as in the classical case.

One can also form the graded Witt ring $GW = GW(X,G) = \oplus_{k\geq 0} I^k/I^{k+1}$ where $I \subseteq W$ denotes the unique ideal of index 2, i.e., the ideal of even dimensional forms. The following question arises naturally in studying GW: Is it true that $\rho \in W$, $\rho(\sigma) \equiv 0 \bmod 2^k$ $\forall\, \sigma \in X \Rightarrow \rho \in I^k$? This is Lam's "Open Problem B" [95]. This is true, for example, if the chain length is finite [29,88,106,107], but little progress has been made on this problem.

For a space of signatures (X,G) of higher level, the Witt ring is defined similarly. But now $\mathbf{Z}$ gets replaced by the ring of algebraic integers $\mathbf{Z}[\omega]$, ω a primitive $2n$-th root of 1, and $\{\pm1\}$ gets replaced by the cyclic group $(\omega) = \{1,\omega,\dots,\omega^{2n-1}\}$, where n is the level.

There are several main results in the theory of spaces of orderings:

(a) Classification of finite spaces of orderings (more generally, spaces of orderings of finite chain length). Spaces of this sort are built up from the singleton space by two operations called "direct sum" and "group extension" The construction is essentially unique.

(b) A local-global principle for isotropy: A quadratic form ρ is anisotropic over (X,G) iff it is anisotropic over some finite subspace of (X,G).

(c) A representation theorem for the Witt ring: A continuous function $\rho : X \to \mathbb{Z}$ is in $W = W(X,G)$ iff $\rho|_V$ is in $W(V,G/V^{\perp})$ for all finite fans $V \subseteq X$ iff $\sum_{\sigma\in V} \rho(\sigma) \equiv 0 \bmod |V|$ for all finite fans $V \subseteq X$.

These results were proved first in the field case using valuation theory [11,19,21,39,140,141] and, in the field case, valuation theory is still the quickest way of proving these results. See [30] for a proof of (b),(c) using the theory of $\mathbb{R}$-places. For the proof in the general case, see [108,110,111]. Also, see [113] for a generalization of (b). Actually, there are two proofs of (c) in the general case: One is given in [110]. The other involves using (b) to reduce to the case where (X,G) is finite, and then applying (a) as in [11].

If n is given then, using (a), one can "count" the number of spaces of orderings (X,G) with $|G| = 2^n$. One can also compute the possible values for $|X|$ for given n [21,41,122,126]. (One knows $n \le |X| \le 2^{n-1}$, but one can say which values in the interval $[n,2^{n-1}]$ are actually achieved.)

An interesting consequence of (c) is that G is determined by the topological space X together with its fans: The natural embedding $G \hookrightarrow \mathrm{Cont}(X,\{\pm1\})$ identifies G with the group of all continuous functions $\rho : X \to \{\pm1\}$ satisfying $\sum_{\sigma\in V} \rho(\sigma) \equiv 0 \pmod 4$ for all 4-element fans $V \subseteq X$ [110].

In the field case (a),(b),(c) generalize to the higher level situation [13,135]. (b),(c) also generalize to the higher level situation in the case of a ring with many units, using results in [120] to first reduce to the field case. For abstract spaces of signatures, (a) is generalized in [137] (but only in the finite 2-power level case) and (c) is generalized in [119] and [132]. (b) is still open for abstract spaces of signatures.

The non-reduced analogue of (a) would be to classify all finite linked quadratic form schemes. This has not been done, even in the field case. The "elementary type conjecture" [112] asserts that all finite linked quadratic form schemes are built up from quadratic form schemes of finite fields and local fields by the non-reduced analogues of the direct sum and group extension operations. See [15,16,18,32,34,35,52-57,76,93,94,112,115,148,153,154, 165,167] for work related to this conjecture. If the elementary type conjecture were proved true, them there would be interesting applications to Galois cohomology [4,6,63,64], But unfortunately, the non-reduced theory is not very well developed (as compared to the reduced theory). In particular, there is no indication that the elementary type conjecture will be settled in the near future.

For a field F, Char $F \neq 2$, let GWF denote the (non-reduced) graded Witt ring of F and let H_q^*F denote the graded cohomology ring $H^*(\mathrm{Gal}(F_q/F), \mathbf{Z}/2\mathbf{Z})$ where F_q = the quadratic closure of F. There is a natural relation $e_F^* : \mathrm{GWF} \to H_q^*F$ which is conjectured to be a well-defined isomorphism. In case the (non-reduced) Witt ring WF is of elementary type, this conjecture is true [4,6]. Also, in this same case, $\mathrm{Gal}(F_q/F)$ is describable recursively in terms of WF and the action of $\mathrm{Gal}(F_q/F)$ on the roots of unity [63,64]. However, the question of which Demuškin groups can occur (corresponding to the dyadic local factors) is still open. An interesting consequence of the results in [63,64] is the following: WF is of elementary type $\Rightarrow$ $WF[\sqrt{a}\,]$ is of elementary type. This lends some credence to the elementary type conjecture, at least in the field case.

In the category of (abstract) spaces of orderings, there are several constructions for producing new spaces of orderings from old. Direct sum and group extension have already been mentioned. In addition there is an inverse limit construction [109], a direct limit construction [92] and a sheaf construction [112]. Direct limits of finite spaces of orderings are classified in [92].

Given an abstract space of orderings (X,G), one would like to be able to find F and a preordering $T \subseteq F$ such that $(X,G) \cong (X_T,G_T)$ (as spaces of orderings). This is referred to as the "realization problem". One would prefer that F be a field although some other structure (e.g., skew-field, semi-local ring) might be admitted. The "best" situation would be if we could choose F to be a Pythagorian field and $T = \Sigma F^2$. If (X,G) has finite chain length then there does indeed exist a Pythagorean field F such that $(X,G) \cong (X_T,G_T)$ where $T = \Sigma F^2$. The proof involves the classification theorem (a) above, and a fair amount of valuation theory [21,39]. Another case when a solution is known is when (X,G) has stability index ≤ 1. In this case also, by a result of Craven [37,140], (X,G) is realized as (X_T,G_T) with F Pythagorean, $T = \Sigma F^2$.

The realization theorem for finite chain length spaces of orderings extends: (1) To quadratic form schemes: Any linked quadratic form scheme of elementary type is isomorphic to the quadratic form scheme of a field [89]. (2) To spaces of signatures of higher level which are built up from the singleton space using the two standard constructions [135].

For any space of orderings (X,G), $\sigma,\tau \in X$ are said to be "connected" if either $\sigma = \tau$ or there exists a non-trivial fan $V \subseteq X$ with $\sigma,\tau \in V$. This is an equivalence relation on X. The equivalence classes are called the "connected components" of (X,G) [108,111,113]. The proof of the classification theorem (a) for finite chain length spaces involves a careful analysis of connected components. Generally speaking, connected components are useful in that they provide, in the abstract situation, a partial substitute for the valuation theory which is available in the field case.

Suppose (X,G) has finite chain length and F is a Pythagorian field whose space of orderings realizes (X,G). In [62], Jacob constructs valuations on F associated to the non-trivial connected components of (X,G); also see [122]. In this way, each non-trivial connected component of (X,G) is realized as the space of orderings of a certain 2-Henselian extension of F in a natural functorial way. This construction has since been generalized to the non-reduced case [5,15,16,160], culminating in [63,64].

Suppose $(X,G) = (X_T,G_T)$ for some preordering $T \subseteq F$, F a field. Then there exists a natural surjection $\lambda : X_T \to M_T$ where M_T is the space of $\mathbb{R}$-places on F compatible with T [11,28,30]. This satisfies:

(i) Each fiber $\lambda^{-1}\{\alpha\}$, $\alpha \in M_T$ is a fan.

(ii) Each fan in X_T lies in $\lambda^{-1}\{\alpha\} \cup \lambda^{-1}\{\beta\}$ for some $\alpha,\beta \in M_T$. (Possibly $\alpha = \beta$.)

(iii) M_T is finite iff (X_T,G_T) has finite chain length.

This set-up is generalized in [111] and [113] to the idea of a "P-structure" on a space of orderings. P-structures always exist (even in the abstract case), although they are not unique. One cannot expect uniqueness since, even in the field case, $(X_S,G_S) \cong (X_T,G_T) \not\Rightarrow M_S \cong M_T$ [122].

In the case of a planar ternary ring F (in particular, a skew-field) one has, as in the field case, a natural P-structure corresponding to the space of T-compatible $\mathbb{R}$-places on F, provided $\mathbb{Q} \subseteq F$ [71]. The same is true for any semi-local ring or ring with many units, if one takes the definition of $\mathbb{R}$-place given in [117].

The "stability index" or "s-invariant" of a space of orderings (X,G) is the smallest integer $s \geq 0$ such that $2^s\rho \in W(X,G)$ $\forall\ \rho \in \mathrm{Cont}(X,\mathbb{Z})$ (or ∞ is no such finite s exists). This is also characterized as the smallest integer s such that each basic clopen set $S \subseteq X$, $S \neq \emptyset$, has the form $S = \{\sigma \in X\colon \sigma(a_i) = 1,\ i = 1,\dots,s\}$ for some $a_1,\dots,a_s \in G$. Applying the representation theorem (c), we also have a characterization of stability in terms of fans: It is the largest integer s such that there exists a fan $V \subseteq X$ with $|V| = 2^s$ [11,110]. The reader can refer to [113,147] for the definition of the "local stability index" and the relationship between this and the (global) stability index defined here. As might be expected, the case where the stability index is finite is somewhat better understood but, at the same time, it is also more important, at least from the viewpoint of application to real algebraic geometry [22,23,25,26, 105,118,144].

Recently, Bröcker introduced another invariant of spaces of orderings called the "t-invariant" [23,25,26]. This is defined to be the least integer $t \geq 1$ such that each clopen set $S \subseteq X$ is expressible as a union of t basic clopen sets. Using the isotropy criterion (b), and the classification of finite spaces of orderings (a), Bröcker gives a bound for t in terms of the stability index s [25]. Unfortunately, it appears unlikely that this bound is best possible, and moreover, computation of the best bound appears to be difficult. Since the t-invariant also has application to real algebraic geometry [25,26], it would be nice to understand better the relationship between s and t.

See [17,87,98] for an introduction to real algebra and real algebraic geometry. Here, the theory of spaces of orderings (mainly of fields, but also of semi-local rings) has application to the problem of minimal generation of semi-algebraic sets in an algebraic set $V \subseteq R^n$, R a real closed field [22,23,25,105,144] and also to the more general problem of minimal generation of constructible sets in the real spectrum X(A) of an arbitrary commutative ring A [26,118]. Specifically, one uses results on the s-invariant and t-invariant and (in the case of algebraic sets) results on the behaviour of the s-invariant under field extension [19,105].

At the same time (although, a priori, this has nothing to do with spaces of orderings), one knows that, for any commutative ring A with Witt ring WA, the cokernel of the total signature $\hat{}$: WA $\to$ Cont(X(A),$\mathbb{Z}$) is 2-primary torsion [31,102]. Applying this where A is the coordinate ring of the algebraic set $V \subseteq R^n$ implies, for example, that the semi-algebraic components of V can be separated by quadratic forms. Also, when A is the coordinate ring of an algebraic set V, there is some smallest integer s such that 2^sCont(X(A),$\mathbb{Z}$) is in the image of $\hat{}$. In [103], Mahé determines bounds for s in terms of the dimension of V.

References

1. B. Alpers, Round quadratic forms, J. Alg. 137(1) (1991) 44-55.

2. C. Andradas, L. Bröcker, and J.M Ruiz, Minimal generation of basic open semi-analytic sets, Invent. Math. 92 (1988) 409-430.

3. J. Arason, R. Elman, and W. Jacob, The graded Witt ring and Galois cohomology I, in "Quadratic forms and Hermitian forms", CMS Conf. Proc. 4 (1984) 17-50.

4. J. Arason, R. Elman, and W. Jacob, Graded Witt rings of elementary type, Math. Ann. 272 (1985) 267-280.

5. J. Arason, R. Elman, and W. Jacob, Rigid elements, valuations, and realization of Witt rings, J. Alg. 110 (1987) 449-467.

6. J. Arason, R. Elman, and W. Jacob, The graded Witt ring and Galois cohomology II, Trans. Amer. Math. Soc. 314(2) (1989) 745-780.

7. E. Becker, Summen n-ter Potenzen in Körpern, J. reine angew. Math. 307/308 (1979) 8-30.

8. E. Becker, Partial orderings on a field and valuation rings, Comm. Alg. 7(18) (1979) 1933-1976.

9. E. Becker, Local global theorems for diagonal forms of higher degree, J. reine angew. Math. 318 (1980) 36-50.

10. E. Becker, The real holomorphy ring and sums of 2n-th powers, in "Géométrie algébrique réelle et formes quadratiques", Springer Lecture Notes 959 (1982) 139-181.

11. E. Becker and L. Bröcker, On the description of the reduced Witt ring, J. Alg. 52(2) (1978) 328-346.

12. E. Becker and E. Köpping, Reduzierte quadratische Formen und semi-ordnungen reeler Körpern, Abh. Math. Sem. Univ. Hamburg 46 (1977) 143-177.

13. E. Becker and A. Rosenberg, Reduced forms and reduced Witt rings of higher level, J. of Alg. 92 (1985) 477-503.

14. E. Becker and N. Schwartz, Reduced Witt rings of infinite level, to appear.

15. L. Berman, "The Kaplansky radical and values of binary quadratic forms", Ph.D. Thesis, Berkeley, 1979.

16. L. Berman, C. Cordes, and R. Ware, Quadratic forms, rigid elements, and formal power series fields, J. of Alg. 66 (1980) 123-133.

17. J. Bochnak, M. Coste, and M.-F. Roy, "Géométrie algébrique réelle", Springer-Verlag, Berlin and New York, 1987.

18. R. Bos, "Quadratic forms, orderings, and abstract Witt rings", Dissertation, Utrecht (1984).

19. L. Bröcker, Zur Theorie der quadratischen Formen über formal reelen Körpern, Math. Ann. 210 (1974) 233-256.

20. L. Bröcker, Characterization of fans and hereditarily pythagorean fields, Math. Zeit. 152 (1976) 149-163.

21. L. Bröcker, Über die Anzahl der Anordnungen eines kommutativen Körpers, Archiv. der Math. 29 (1977) 458-464.

22. L. Bröcker, Minimale Erzeugung von Positivbereichen, Geom. Dedicata 16 (1984) 335-350.

23. L. Bröcker, Spaces of orderings and semi-algebraic sets, in "Quadratic forms and Hermitian forms", CMS Conf. Proceedings 4 (1984) 231-248.

24. L. Bröcker, Spaces of orderings of certain rings, to appear.

25. L. Bröcker, On basic semi-algebraic sets, Expositiones Math., to appear.

26. L. Bröcker, On the stability index of Noetherian rings, to appear.

27. R. Brown, Superpythagorean fields, J. Alg. 42 (1976) 483-494.

28. R. Brown, The reduced Witt ring of a formally real field, Trans. Amer. Math. Soc. 230 (1977) 257-292.

29. R. Brown, Powers of the augmentation ideal in the Witt ring of a field, Pub. Mathematicae, 28 (1981)

30. R. Brown and M. Marshall, The reduced theory of quadratic forms, Rky. Mtn. J. of Math. 11(2) (1981) 161-175.

31. G.W. Brumfiel, Witt rings and K-theory, Rky. Mtn. J. of Math. 14 (1984) 733-765.

32. A. Carson and M. Marshall, Decomposition of Witt rings, Can. J. of Math. 34(6) (1982) 1276-1302.

33. C. Cordes, Quadratic forms over nonformally real fields with a finite number of quaternion algebras, Pac. J. of Math. 63(2) (1976) 357-365.

34. C. Cordes, Quadratic forms over fields with four quaternion algebras, Acta. Arith. 41 (1982) 55-70.

35. C. Cordes and J. Ramsey Jr., Quadratic forms over fields with $u = q/2 < +\infty$, Fund. Math. 99 (1978) 1-10.

36. C. Cordes and J. Ramsey Jr., Quadratic forms over quadratic extensions of fields with two quaternion algebras, Can. J. of Math. 31(5) (1979) 1047-1058.

37. T. Craven, The Boolean space of orderings of a field, Trans. Amer. Math. Soc. 209 (1975) 225-235.

38. T. Craven, Stability in Witt rings, Trans. Amer. Math. Soc. 225 (1977) 227-242

39. T. Craven, Characterizing Reduced Witt rings of fields, J. Alg. 53 (1978) 68-77.

40. T. Craven, Characterizing reduced Witt rings II, Pac. J. of Math. 80(2) (1979) 341-349.

41. T. Craven, An application of Polya's theory of counting to an enumeration problem arising in quadratic form theory, J. of Comb. Th. A 29(2) (1980) 174-181.

42. T. Craven, Orderings of higher level and semi-local rings, Math. Zeit. 176 (1981) 577-588.

43. T. Craven, On the prime ideal of an ordering of higher level, Math. Zeit. 180 (1982) 553-565.

44. T. Craven, Witt rings and orderings of skew fields, J. of Alg. 77 (1982) 74-96.

45. T. Craven, Fields maximal with respect to a set of orderings, J. of Alg. 115(1) (1988) 200-218.

46. T. Craven, Approximation properties for orderings on *-fields, Trans. Amer. Math. Soc. 310(2) (1988) 837-850.

47. T. Craven, Orderings and valuations on *-fields, Rky. Mtn. J. of Math. 19(3) (1989) 629-646.

48. T. Craven, Characterization of fans in *-fields, J. of Pure and Applied Alg. 65 (1990) 15-24.

49. T. Craven, Places on *-fields and the real holomorphy ring, Comm. in Alg. 18(9) (1990) 2791-2820.

50. T. Craven, Witt groups of Hermitian forms over *-fields, to appear.

51. M. Dickmann, A combinatorial geometric structure on the space of orderings of a field, to appear.

52. R. Fitzgerald, Gorenstein Witt rings, Can. J. Math. 40 (1988) 1186-1202.

53. R. Fitzgerald, Linked quaternionic quotients and homomorphisms, Comm. in Alg. 18(12) (1990) 4171-4224.

54. R. Fitzgerald, Combinatorial techniques and abstract Witt rings III, Pac. J. Math. 148(1) (1991) 39-58.

55. R. Fitzgerald and J. Yucas, Local factors of finitely generated Witt rings, Rky. Mtn. J. of Math. 16(3) (1986) 619-627.

56. R. Fitzgerald and J. Yucas, Combinatorial techniques and abstract Witt rings I, J. Alg. 114 (1988) 40-52.

57. R. Fitzgerald and J. Yucas, Combinatorial techniques and abstract Witt rings II, Rky. Mtn. J. Math., to appear.

58. S. Holland, *-valuations and ordered *-fields, Trans. Amer. Math. Soc. 262 (1980) 219-243.

59. S. Holland, Strong orderings of *-fields, J. Alg. 101 (1986) 16-46.

60. T. Iwakami, D. Kijima, and M. Nishi, Kaplansky's radical and Hilbert 90 III, Hiroshima Math. J. 15(1) (1985) 81-88.

61. W. Jacob, The model theory of 'R-formal' fields, Annals of Math. Logic 19 (1980) 263-282.

62. W. Jacob, On the structure of Pythagorean fields, J. Alg. 68(2) (1981) 247-267.

63. W. Jacob and R. Ware, A recursive description of the maximal pro-2 Galois group via Witt rings, Math. Zeit. 200 (1989) 379-396.

64. W. Jacob and R. Ware, Realizing dyadic factors of elementary type Witt rings and pro-2 Galois groups, to appear.

65. F. Kalhoff, Eine Kennzeichnung anordnungsfähiger Ternärkörper, J. Geometry 31 (1988) 100-113.

66. F. Kalhoff, Über Präordnungen von Ternärkörpern, Abh. Math. Sem. Univ. Hamburg 58 (1988) 137-151.

67. F. Kalhoff, Über Stellen und Präordnungen von Ternärkörpern, Geom. Dedicata 27 (1988) 137-151.

68. F. Kalhoff, Spaces of orderings and Witt rings of planar ternary rings, J. Pure and App. Alg. 58 (1989) 169-180.

69. F. Kalhoff, Some local global principles of ordered projective planes, Geom. Dedicata 32 (1989) 59-79.

70. F. Kalhoff, Formal power series over cartesian groups, Geom. Dedicata 36 (1990) 329-345.

71. F. Kalhoff, The holomorphy rings of planar ternary rings with rational prime field, Abh. Math. Sem. Univ. Hamburg 60 (1990) 1-15.

72. F. Kalhoff, Witt rings and semiorderings of planar ternary rings, Topics in Comb. and Graph Th. (1990).

73. F. Kalhoff, Witt rings of weakly orderable double loops and nearfields, Results in Math. 17 (1990) 106-119.

74. F. Kalhoff, Trivialization of fans in planar ternary rings with rational prime field, Aequationes Math. 39 (1990) 149-160.

75. F. Kalhoff, Anordnungsraume unter der Moulton Konstruktion und ebener der Lenz Klasse III, J. Geometry 38 (1990) 59-77.

76. I. Kaplansky, Fröhlich's local quadratic forms, J. reine angew. Math. 239 (1969) 74-77.

77. I. Kaplansky and R. Shaker, Abstract quadratic forms, Can. J. of Math. 21 (1969) 1218-1233.

78. D. Kijima, Quadratic extensions of quasi-Pythagorean fields, Hiroshima Math. J. 15(1) (1985) 145-161.

79. D. Kijima an M. Nishi, Kaplansky's radical and Hilbert 90 I,II, Hiroshima Math. J. 11(2) (1981) 443-456, 13 (1983) 29-37.

80. D. Kijima and M. Nishi, On the space of orderings and the group H, Hiroshima Math. J. 13(1) (1983) 215-225.

81. D. Kijima, M. Nishi, and H. Uda, On the q-dimension of a space of orderings and q-fans, Hirosima Math. J. 14(1) (1984) 159-168.

82. J. Kleinstein and A. Rosenberg, Signatures and semi-signatures of abstract Witt rings and Witt rings of semi-local rings, Can. J. Math. 30(4) (1978) 872-895.

83. J. Kleinstein and A. Rosenberg, Succinct and representational Witt

rings, Pac. J. Math. 86(1) (1980) 99-137.

84. M. Knebusch, On the local theory of signatures and reduced quadratic forms, Abh. Math. Sem. Univ. Hamburg 51 (1981) 149-195.

85. M. Knebusch, A. Rosenberg, and R. Ware, Structure of Witt rings and quotients of abelian group rings, Amer. J. of Math. 94 (1972) 119-155.

86. M. Knebusch, A. Rosenberg, and R. Ware, Grothendieck and Witt rings of Hermitian forms over Dedekind rings, Pac. J. of Math. 43(3) (1972) 657-673.

87. M. Knebusch and C. Scheiderer, "Einführung in die reelle Algebra", Vieweg-Studium, Braunschweig/Wiesbaden, 1989.

88. M. Krüskemper, On real local-global principles, to appear.

89. M. Kula, Fields with prescribed quadratic form schemes, Math. Zeit. 167 (1979) 201-212.

90. M. Kula, Finite sets of quasi-orderings, Prace. Mat. 10, Prace nankowe Uniwersytetu Slaskiego 332 (1980) 69-85.

91. M. Kula, Crystal growth and Witt rings, J. of Alg. 136(1) (1991) 190-196.

92. M. Kula, M. Marshall, and A. Sładek, Direct limits of finite spaces of orderings, Pac. J. of Math. 112(1) (1984).

93. M. Kula, L. Szczepanik, and K. Szymiczek, Quadratic forms over formally real fields with eight square classes, Manuscripta Math. 29 (1979) 295-303.

94. M. Kula, L. Szczepanik, and K. Szymiczek, Quadratic form schemes and quaternionic schemes, Fund. Math. 130 (1988) 181-190.

95. T.Y. Lam, Ten lectures on quadratic forms over fields, in "Conference on quadratic forms" (G. Orzech, Editor) Queen's Papers in Pure and Applied Mathematics 46 (1976) 1-102.

96. T.Y. Lam, The theory of ordered fields, in "Ring theory and algebra III" (B. McDonald, editor) Lecture Notes in Pure and Applied Math. 55, Dekker, (1980) 1-152.

97. T.Y. Lam, "Orderings, valuations, and quadratic forms", CBMS regional conference series in Math. 52 (1983).

98. T.Y. Lam, An introduction to real algebra, Rky. Mtn. J. of Math. 14 (1984) 767-814.

99. T.Y. Lam and D. Shaperio, The square class invariant for Pythagorean fields, in "Ordered fields and real algebraic geometry", Cont. Math. 8 (1982) 327-340.

100. D. Leep and M. Marshall, Isomorphisms and automorphisms of Witt rings, Can. Math. Bull. 31(2) (1988) 250-256.

101. D. Lewis, Units in Witt rings, Comm. in Alg. 18(10) (1990) 3295-3306.

102. L. Mahé, Signatures et composantes connexes, Math. Ann. 260 (1982) 191-210.

103. L. Mahé, Théorème de Pfister pour les variétés et anneaux de Witt reduits, Invent. Math. 85 (1986) 53-72.

104. L. Mahé, Level and Pythagorous number of some geometric rings, to appear.

105. L. Mahé, Une démonstration élementaire du théorème de Bröcker-Scheiderer, to appear.

106. M. Marshall, Some local-global principles for formally real fields, Can. J. of Math. 29(3) (1977) 606-614.

107. M. Marshall, A reduced theory of quadratic forms, in "Conference on quadratic forms", Queen's papers in pure and applied Math. 46 (1977) 569-579, Queen's University.

108. M. Marshall, Classification of finite spaces of orderings, Can. J. Math. 31(2) (1979) 320-330.

109. M. Marshall, Quotients and inverse limits of spaces of orderings, Can. J. of Math. 31(3) (1979) 604-616.

110. M. Marshall, The Witt ring of a space of orderings, Trans. Amer. Math. Soc. 258 (1980) 505-521.

111. M. Marshall, Spaces of orderings IV, Can. J. of Math. 32 (1980) 603-627.

112. M. Marshall, "Abstract Witt rings", Queen's papers in pure and applied Math. 57 (1980), Queen's University.

113. M. Marshall, Spaces of orderings: systems of quadratic forms, local structure, and saturation, Comm. in Alg. 12(6) (1984) 723-743.

114. M. Marshall, Exponentials and logarithms on Witt rings, Pac. J. of Math. 127 (1987) 127-140.

115. M. Marshall, Decomposing Witt rings of characteristic 2, Rky. Mtn. J. of Math. 19(3) (1989) 793-806.

116. M. Marshall, A simple system of axioms for spaces of signatures, J. Pure and Applied Alg. 57 (1989) 159-164.

117. M. Marshall, Orderings and real places on commutative rings, J. of Alg., to appear.

118. M. Marshall, Minimal generation of basic sets in the real spectrum of a commutative ring, to appear.

119. M. Marshall and C. Mulcahy, The Witt ring of a space of signatures, J. Pure and Applied Alg., 67 (1990) 179-188.

120. M. Marshall and L. Walter, Signatures of higher level on rings with many units, Math. Zeit. 204 (1990) 129-142.

121. M. Marshall and J. Yucas, Linked quaternionic mappings and their associated Witt rings, Pac. J. of Math., 95(2) (1981) 411-425.

122. J. Merzel, Quadratic forms over fields with finitely many orderings, in "Ordered fields and real algebraic geometry", Cont. Math. 8 (1982) 185-229.

123. J. Mináč, "Galois groups, order spaces, and valuations", Ph.D. Thesis, Queen's Univ., Kingston, Ontario, 1986.

124. J. Mináč, On fields for which the number of orderings is divisible by a high power of 2, I,II,III, C.R. Math. Rep. Acad. Sci. Canada 7(3) (1985) 183-188, 7(4) 221-226, 7(5) 297-301.

125. J. Mináč, Poincare polynomials and ordered fields, C. R. Math. Rep. Acad. Sci. Canada, 8(6) (1986) 411-416.

126. J. Mináč, Remarks on the set O(n) in the theory of ordered fields, C. R. Math. Rep. Acad. Sci. Canada 11(2) (1987) 125-129.

127. J. Mináč and T. Smith, A characterization of C-fields via Galois groups, J. Alg. 137 (1991) 1-11.

128. J. Mináč and M. Spira, Formally real fields, Pythagorean fields, C-fields, and W-groups, Math. Zeit. 205 (1990) 519-530.

129. J. Mináč and R. Ware, Pro-2-Demuškin groups of rank $\aleph_0$ as Galois groups of maximal 2-extensions of fields, to appear.

130. C. Mulcahy, "Spaces of signatures", Ph.D. Thesis, Cornell Univ., 1985.

131. C. Mulcahy, An abstract approach to higher level forms and rigidity, Comm. in Alg. 16 (1988) 577-612.

132. C. Mulcahy, Representation theorem for higher level reduced Witt rings, J. of Alg. 119 (1988) 105-122.

133. A. Pfister, Quadratische Formen in beliebigen Körpern, Invent. Math. 1 (1966) 116-132.

134. V. Powers, "Finite constructible spaces of signatures", Ph.D. Thesis, Cornell Univ., 1985.

135. V. Powers, Characterizing reduced Witt rings of higher level, Pac. J. Math. 128(2) (1987) 333-347.

136. V. Powers, Higher level reduced Witt rings of skew fields, Math. Zeit. 198 (1988) 545-554.

137. V. Powers, Finite spaces of signatures, Can. J. of Math. 41(5) (1989) 808-829.

138. V. Powers, Holomorphy rings and higher level orders of skew fields, J. Alg. 136(1) (1991) 51-59.

139. A. Prestel, Quadratische Semi-Ordnungen und quadratische Formen, Math. Zeit. 133 (1973) 319-342.

140. A. Prestel, "Lectures on formally real fields", Monografias de Matematica 22 IMPA, Rio de Janerio (1975).

141. A. Prestel, A local-global principle for quadratic forms, Math. Zeit. 142 (1975) 91-95.

142. W. Scharlau, "Quadratic forms and Hermitian forms", Springer-Verlag, Berlin and New York, 1985.

143. W. Scharlau, Generating functions for finitely generated Witt rings, Acta Arith. 54 (1989) 51-59.

144. C. Scheiderer, Stability index of real varieties, Invent. Math. 97 (1989) 467-483.

145. C. Scheiderer, Some remarks on orderings under field extensions, to appear.

146. C. Scheiderer, Spaces of orderings of fields under field extensions, to appear.

147. N. Schwartz, Local stability and saturation in spaces of orderings, Can. J. Math. 35(3) (1983) 454-477.

148. A. Sładek, Abstract theory of quadratic forms, Prace Mat. 10, Prace nankowe Uniwersytetu Slaskiego 332 (1980) 46-57.

149. A. Sładek, "Witt rings of skew fields", Prace Naukowe Uniwersytetu Slaskiego 1079, 1989.

150. T. Smith, "Some 2-groups arising in quadratic form theory and their generalizations", Ph.D. Thesis, Univ. of Calif., Berkeley, 1988.

151. M. Spira, "Witt rings and Galois groups", Ph.D. Thesis, Univ. of Calif., Berkeley, 1987.

152. L. Szczepanik, Quadratic form schemes with non-trivial radical, Colloq. Math. 59 (1985) 143-160.

153. L. Szczepanik, Fields and quadratic form schemes with the index of radical not exceeding 16, Ann. Math. Silesianae 1(13) Katowice (1985)

23-46.

154. K. Szymiczek, Structure of the basic part of a field, J. of Alg., 99(2) (1986) 422-429.

155. K. Szymiczek, Abstract Grothendieck rings, Coll. Math. 53 (1987) 17-25.

156. K. Szymiczek, Generalized rigid elements of fields, Pac. J. of Math. 129(1) (1987) 171-186.

157. A. Tschimmel, "Über Anordnungsräume von Schiefkorpern", Dissertation, Münster (1981).

158. L. Walter, "Quadratic forms, orderings, and quaternion algebras over rings with many units", M.Sc. Thesis, Univ. of Saskatchewan (1988).

159. R. Ware, Quadratic forms and profinite 2-groups, J. Alg. 58 (1979) 227-237.

160. R. Ware, Valuation rings and rigid elements in fields, Can. J. of Math. 33(6) (1981) 1338-1355.

161. R. Ware, Quadratic forms and pro-2 groups II: The Galois group of the Pythagorean closure of a formally real field, J. Pure and Applied Alg. 30 (1983) 95-107.

162. R. Ware, Quadratic forms and pro-2-groups III: Rigid elements and semi-direct products, Comm. in Alg. 13(8), (1985) 1713-1736.

163. R. Ware, Stability in Witt rings and Abelian subgroups of pro-2 Galois groups, Rky. Mtn. J. Math. 19 (1989) 955-985.

164. R. Ware, Witt rings and almost free pro-2 groups, J. Alg. 132(2) (1990) 377-383.

165. J. Yucas, Quadratic forms and rigid elements of fields, Acta Arith. 39 (1981) 313-322.

166. J. Yucas, Witt rings and associated Boolean rings, J. of Alg., 71(1) (1981) 40-49.

167. J. Yucas, Localizations of linked quaternionic mappings, Can. J. of Math. 34(1) (1982) 247-256.

168. J. Yucas, Additive symmetric bijections on geometric structures, J. Comb. Th. A 40(2) (1985) 418-426.

ON THE EXISTENCE OF ORDERINGS OF PRESCRIBED LEVEL

Ralph Berr

Let K be a field. A subset $P \subset K^*$ is called an ordering of level n, if

$$P + P \subset P,\ P \cdot P \subset P,\ K^*/P \cong \mathbb{Z}/2n\mathbb{Z}$$

(see [1], [2]). Orderings of higher level are of importance for the study of sums of 2-nth powers and forms of higher degree (see [1],[2],[4]). In this note we are concerned with the characterization of the fields admitting an ordering of given level $n \in \mathbf{N}$. We will only give an outline of the results. For details see [5].

Let n be a natural number. From [3] resp. [6] we get the following characterization.

Proposition 1: *Let K be a field. Then the following statements are equivalent:*
(1) K admits an ordering of level n.
(2) K admits a real valuation v with $v(K) \neq pv(K)$ for all prime divisors p of n.

The existence of such a real valuation corresponds to arithmetic properties of the field K, as the following result of E. Becker ([1], Satz 2.14) shows:

Theorem 2: *Given $x \in \sum K^{\cdot 2}$, then the following statements are equivalent:*
(1) $x \in \sum K^{\cdot 2n}$.
(2) $v(x) \in 2nv(K)$ for all real valuations v of K.

Applying the last two results we get

Corollary 3: *Let p be a prime number. Then K admits an ordering of level p if and only if*

$$\sum K^{\cdot 2} \neq \sum K^{\cdot 2p}.$$

This result cannot be generalized to arbitrary levels, as the following example shows.

Example 4: Let p, q be odd primes and $n := p \cdot q$. Let $\mathbf{R}(X)$ be a simple transcendental extension of the real numbers $\mathbf{R}$. Now set

$$K := \mathbf{R}(X)\left((X - r_1)^{p^{-k}}, (X - r_2)^{q^{-k}} \mid r_1 < 1 < r_2, r_1, r_2 \in \mathbf{R}, k \in \mathbf{N}\right).$$

K is formally real, as p, q are odd. Given a real valuation v of K, then $v(K)$ is divisible by exactly one of the primes p, q. Applying Proposition 1 we see that K admits no ordering of level n. On the other hand it follows from the construction of K that there exist real valuations v, w of K with $v(K) \neq pv(K)$ and $w(K) \neq qw(K)$. Hence K admits orderings of level p as well as of level q. Thus we have found a field K satisfying

$$\sum K^{\cdot 2} \neq \sum K^{2n}$$

and the stronger condition

$$\sum K^{2} \neq \sum K^{2p}, \ \sum K^{2} \neq \sum K^{2q},$$

but admitting no ordering of level $n = p \cdot q$.

So far we have seen that Corollary 3 does not carry over to the general case. However. an improved version of Theorem 2 will lead to a similar characterization of the existence of orderings of arbitrary level. Let $\mathcal{L}$ be a nonempty set of natural numbers. We denote by

$$\sum_{\mathcal{L}} \sum K^{\cdot *2n}$$

the additive semi-group generated by the $2n$-th powers K^{*2n} with $n \in \mathcal{L}$. These 'sums of mixed powers' can be characterized as follows ([5], Theorem 1.2):

Theorem 5: *Given $\mathcal{L} \subset \mathbf{N}$ and $x \in \sum K^{*2}$. then the following statements are equivalent:*
(1) $x \in \sum_{\mathcal{L}} \sum K^{*2n}$.
(2) $v(x) \in \cup_{n \in \mathcal{L}} 2nv(K)$ for all real valuations v of K.

Now let $n \in \mathbf{N}$ and let $\mathcal{L}$ be the set of prime divisors of n. Applying Proposition 1 and Theorem 5 we get ([5], Proposition 1.5):

Proposition 6: *Given $n \in \mathbf{N}$, then the following statements are equivalent:*
(1) K admits an ordering of level n.
(2) $\sum K^{*2} \neq \sum_{p|n} \sum K^{*2p}$, where p ranges over the prime divisors of n.

This result shows that there is a natural relationship between orderings of higher level and sums of mixed powers in fields. For details see [5].

In [2], E. Becker derived from Theorem 2 the existence of 'Hilbertian identities' of higher degree ([2], Satz 4.1). In view of Theorem 5, the same arguments lead to the following results about sums of mixed powers.

Theorem 8 *Given $k, m \in \mathbf{N}$ and $n_1, \ldots, n_k, s_1, \ldots, s_k \in \mathbf{N}$, there exist natural numbers $l_i = l_i(n_i, s_i, m)$, $i \in \{1, \ldots, k\}$ such that for any field K of characteristiic 0 we have: For $x_{ij_i} \in K$, $i \in \{1, \ldots, k\}$, $j_i \in \{1, \ldots, s_i\}$ there exist $y_{ij_i} \in K$, $i \in \{1, \ldots, k\} j_i \in \{1, \ldots, l_i\}$ such that*

$$\left(\sum_{i=1}^{k} \sum_{j_i=1}^{s_i} x_{ij_i}^{2n_i} \right)^m = \sum_{i=1}^{k} \sum_{j_i=1}^{l_i} y_{ij_i}^{2n_i m}.$$

Let $\mathbf{Q}$ be the field of rational numbers. As an immediate consequence we get the existence of 'Hilbertian identities' for sums of mixed powers.

Corollary 9: *Let $X_1, \ldots, X_k$ be indeterminates. For $m, n_1, \ldots, n_k \in \mathbf{N}$ there exist $l_1, \ldots, l_k \in \mathbf{N}$ and $f_{ij_i} \in \mathbf{Q}(X_1, \ldots, X_k)$, $i = 1, \ldots, k$, $j_i = 1, \ldots, l_i$ such that*

$$(X_1^{2n_1} + \ldots + X_k^{2n_k})^m = \sum_{i=1}^{k} \sum_{j_i=1}^{l_i} f_{ij_i}^{2mn_i}.$$

References

[1] E. Becker, Hereditarily Pythagorean Fields and Orderings of Higher Level, Lecture Notes, Instituto de Mathematica Pura e Aplicade, Rio de Janeiro 1978.

[2] E. Becker, Summen n-ter Potenzen in Körpern, J. reine angew. Math. **307/308**, 8-30 (1979).

[3] E. Becker, J. Harman, A. Rosenberg, Signatures of fields and extension theory, J. reine angew. Math **330**, 53-75 (1982).

[4] E. Becker, A. Rosenberg, Reduced forms and reduced Witt rings of higher level, J. Algebra **92**, No 2, 477-503 (1985).

[5] R. Berr, Sums of mixed powers in fields and orderings of prescribed level, to appear in Math. Z.

[6] N. Schwartz, Chain signatures and real closures, J. reine angew. Math. **347**, 1-20 (1984).

Fachbereich Mathematik
Universität Dortmund
Postfach 500 500
4600 Dortmund
FRG

<u>Prépublications de l'Equipe de Logique</u>.

N° 1. **D. Lascar.**
Les beaux automorphismes, janvier 1990.

N° 2. **F. Delon, M. Dickmann, D. Gondard.**
Séminaire de Structures Algébriques Ordonnées 1988–89, janvier 1990.

N° 3. **J.L. Krivine.**
Opérateurs de mise en mémoire et traduction de Gödel, janvier 1990.

N° 4. **M. Giraudet.**
Groupes à moitié ordonnés. Première partie : Présentation axiomatique des groupes de permutations monotones de chaînes, février 1990.

N° 5. **M. Giraudet et F. Lucas.**
Groupes à moitié ordonnés. Deuxième partie : Sous–groupes de groupes à moitié réticulés, février 1990.

N° 6. **C. Michaux.**
Ordered rings over which output sets are recursively enumerable sets, février 1990.

N° 7. **V. Danos.**
Lambda–calculus and its dynamic algebra 1 : basic results, février 1990.

N° 8. **L. Regnier.**
A propos de l'équivalence opérationnelle.

N° 9. **Ch. Berline.**
Rétrations et Interprétation interne du polymorphisme : Le problème de la rétraction universelle.

N° 10. **D. Gondard–Cozette.**
Axiomatisations simples des théories des corps de Rolle.

N° 11. **A. Fleury et C. Retoré.**
The MIX rule : a proofnet syntax for multiplicatives, their units and (A⊗B)-o(A&B).

N° 12. **P. Malacaria et L. Regnier.**
Lambda–calculus and its dynamic algebra 3 : Weak Nilpotency.

N° 13. **D. Lascar.**
Les automorphismes d'un ensemble fortement minimal.

N° 14. **D. Gluschankof and M. Tilli.**
On some extension theorems in functional analysis and the theory of boolean algebras.

N° 15. **U. Gropp.**
There is no sharp transitivity on q^6 when q is a type of Morley rank 2.

N° 16. **M.A. Dickmann.**
A combinatorial geometric structure on the space of orders of a field.

N° 17. **F. Delon, M. Dickmann, D. Gondard.**
Séminaire de Structures Algébriques Ordonnées 1989–90, décembre 1990.

N° 18. **J.Y. Girard.**
Logic and exceptions : A few remarks.

N° 19. **J.Y. Girard.**
Quantifiers in linear Logic II.

N° 20. **T. Jech.**
Singular cardinal problem : Shelah's theorem on $2^{\aleph_\omega}$.

N° 21. **G. Amiot.**
Sémantique des phases de la Logique Linéaire du second ordre.

N° 22. **T. Ehrhard, P. Malacaria.**
Stone duality for stable functions.

N° 23 **Z. Chatzidakis, L. van den Dries, A. Macintyre.**
Definable sets over finite fields.

N° 24 **J.Y. Girard.**
A new constructive logic : classical logic.

N° 25 **P. Rozière.**
Règles admissibles en calcul propositionnel intuitionniste.

N° 26 **J.Y. Girard.**
On the unity of logic.

N° 27 **S. Kuhlmann.**
Quelques propriétés des espaces vectoriels valués en théorie des modèles.

N° 28 **U. Gropp.**
Action d'un groupe ω–stable sur un ensemble de rang de Morley 2.

N° 29 **A.A. Voronkov.**
On completeness of program synthesis systems.

N° 30 **D. Gondard–Cozette.**
Sur les ordres de niveau 2^n et sur une extension du 17ème problème de Hilbert.

N° 31 **P. Michel.**
Busy Beaver Competition and Collatz–like Problems.

N° 32 **F. Delon, M. Dickmann, D. Gondard.**
Séminaire de Structures Algébriques Ordonnées 1990–91.

N° 33 **M. Dickmann, K. Keimel** (eds).
Workshop "PROCOPE" – Paris, 8–10 novembre 1990.

Hors série. **Y.J. Ringard.**
Mustard watches : an integrated approach to time and food.