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UNIVERSITES PARIS VI et VII

Projets Logique Mathématique et Théorie des Nombres

Institut de Mathématiques de Jussieu – UMR 7586

SEMINAIRE DE STRUCTURES ALGEBRIQUES ORDONNEES

Responsables: F. Delon, M. Dickmann, D. Gondard, T. Servi

2017-2018

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Liste des contributions

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QUAND L’ERGODICITÊ EST UNE PROPRIÊTÊ DU PREMIER ORDRE

TOMÁS IBARLUCÍA

La classe des actions d'un groupe dénombrable donné qui préservent une mesure de probabilité peut être axiomatisée en logique continue. La sous-classe (très importante) des actions ergodiques, par contre, n'est pas toujours élémentaire. Il y a cependant des actions ergodiques dont toute autre action élémentairement équivalente est également ergodique : ce sont les actions *fortement ergodiques*. Il y a même des groupes dont la classe de ses actions ergodiques est bien élémentaire : ce sont précisément les groupes avec la Propriété (T) de Kazhdan.

Dans un travail un commun avec Todor Tsankov [1], nous avons initié une étude modèle-théorique de ces actions. Notamment, nous avons prouvé que les facteurs distaux des actions fortement ergodiques sont contenus dans la clôture algébrique du vide. Cela nous a permis de récupérer des résultats de rigidité de Ioana et Tucker-Drob [2] ainsi que d'aller plus loin, démontrant par exemple que pour les systèmes distaux fortement ergodiques, l'équivalence faible implique l'isomorphisme. Nos méthodes nous ont amené également à une preuve indépendante d'un résultat annoncé par Chifan and Peterson, à savoir que les actions distales des groupes avec la Propriété (T) de Kazhdan sont compactes.

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Hardy-type derivations on fields of generalized power series

Gabriel Lehéricy

Let k be a field and G a totally ordered abelian group. We define

$$k((G)) := \{\text{maps } a : G \to k \mid \text{supp}(a) \text{ is well-ordered}\},$$

where $\text{supp}(a) := \{g \in G \mid a(g) \neq 0\}$ is the support of a. We can define operations $+$ and $.$ on $k((G))$ by $(a+b)(g) := a(g) + b(g)$ and $(a.b)(g) := \sum_{h \in G} a(h)b(g-h)$. This makes $k((G))$ a field, called the field of generalized power series with coefficients in k and exponents in G. Elements of $k((G))$ are usually denoted as formal sums $\sum_{g \in G} a_g t^g$.

Fields of generalized power series play an important role in real algebraic geometry. Any field of power series is naturally endowed with the valuation $v(a) := \min \text{supp}(a)$. Moreover, if k is an ordered field, then this order canonically extends to $k((G))$. Kaplansky showed in [5] that fields of generalized power series are a universal domain for valued fields, in the sense that any valued field of equicharacteristic is embeddable into a field of generalized power series. A natural question which one can then ask is if such a result holds for valued fields endowed with more structure. In particular, we are here interested in knowing whether fields of generalized power series are universal domains for the class of differential-valued fields.

By "differential-valued fields", we mean differential fields of characteristic 0 endowed with a differential valuation as defined by Rosenlicht in [11], and later studied by Aschenbrenner, v.d.Dries and v.d.Hoeven in their works on transseries (see [1],[2],[3], [4]). We recall Rosenlicht's definition: we say that (K, v, D) is a differential-valued field if the two following axioms are satisfied:

(DV1) $\mathcal{O}_v = \mathcal{M}_v + \mathcal{C}$, where $\mathcal{C}$ is the field of constants of (K, D), $\mathcal{O}_v$ is the valuation ring and $\mathcal{M}_v$ the valuation ideal of (K, v).

(DV2) If $a \in \mathcal{O}_v, b \in \mathcal{M}_v$ and $b \neq 0$, then $v(D(a)) > v(\frac{D(b)}{b})$.

A particularly interesting class of differential-valued fields are H-fields, introduced by v.d.Dries and Aschenbrenner in [2]. A H-field is an ordered differential field $(K, D, \leq)$ satisfying the following axioms:

(H1) $\mathcal{O}_v = \mathcal{C} + \mathcal{M}_v$, where $\mathcal{O}_v := \{a \in K \mid \exists c \in \mathcal{C}, |a| \leq c\}$ and $\mathcal{M}_v$ is the maximal ideal of $\mathcal{O}_v$.

(H2) For all $a \in K, a > \mathcal{O}_v \Rightarrow D(a) > 0$.

H-fields are a generalization of Hardy fields and of the field of transseries.

We are interested in the following question:

Question 1
Is there a differential analog of Kaplansky's embedding? More precisely, is any differential-valued field (respectively, any H-field) embeddable into a field of generalized power series?

Answering Question 1 needs first answering the following question:

Question 2
Given a field k of characteristic 0 and G a totally ordered abelian group, can one define a derivation on $k((G))$ so that $k((G))$ equipped with the usual valuation becomes a differential-valued field? If k is moreover ordered, can we require $k((G))$ to be a H-field?

Note that , when $G = \mathbb{Z}$, there is a natural way of defining a derivation on $k((G))$. However, the task becomes more delicate when one considers arbitrary groups. Scanlon gave a way of defining derivations on fields of generalized power series in [12]. However, Scanlon's derivations are in a sense orthogonal to the kind of derivation in which we are interested. Indeed, the field of constants in Scanlon's case is dense in $k((G))$, which contradicts axiom (DV2) above.

Question 2 was already addressed in [7] and [8]. More precisely, the authors of [7] assumed that a map was already defined on the "fundamental monomials" of $k((G))$, and then gave conditions for this map to be extendable to a derivation on $k((G))$. Here we use some of their ideas to give an explicit construction of a derivation on $k((G))$, thus completing the work done in [7].

Our approach uses asymptotic couples. We recall (see [1]) that an asymptotic couple is a pair (G, ψ), where G is a totally ordered abelian group and $\psi : G\backslash\{0\} \to G$ is a map which satisfies the following axioms:

(AC1) $\psi(g + h) \geq \min(\psi(g), \psi(h))$ for any $g, h \neq 0$ with $g + h \neq 0$.

(AC2) $\psi(ng) = \psi(g)$ for any $g \neq 0$, $n \in \mathbb{Z}\backslash\{0\}$.

(AC3) $\psi(g) < \psi(h) + |h|$ for any $g, h \neq 0$.

If moreover (G, ψ) also satisfies the following condition:

$$\text{(ACH)} \quad \forall g, h \neq 0, g \leq h < 0 \Rightarrow \psi(g) \leq \psi(h),$$

then we say that (G, ψ) is a H-type asymptotic couple. Asymptotic couples naturally arise as the value group of differential-valued fields. Indeed, if (K, v, D) is a differential-valued field with value group G, then the logarithmic derivative of K induces a map ψ on $G\backslash\{0\}$, making (G, ψ) an asymptotic couple. If moreover K is a H-field, then (G, ψ) is a H-type asymptotic couple. Asymptotic couples are at the heart of Aschenbrenner's and v.d.Dries's works on transseries.

We will answer Question 2 by answering the following Question:

Question 3
Given a H-type asymptotic couple (G, ψ) and a field k of characteristic 0, can we define a derivation D on $K := k((G))$ such that (K, v, D) is a differential-valued field whose associated asymptotic couple is (G, ψ)? If $(k, \leq)$ is an ordered field, can we do this so that $(K, D, \leq)$ is a H-field?

We now fix a H-type asymptotic couple (G, ψ) and a field k of characteristic 0, we set $K := k((G))$, and we denote by v the usual valuation on K, i.e. $v(a) = \min \operatorname{supp}(a)$. We will show how to answer Question 3. Our method was explained in more details in [9, Section 6] and in [10, Section 5.4].

Our method uses some theory of valued groups, which can for example be found in [6, chapter 0]. Note that, by definition of an asymptotic couple, ψ is what S.Kuhlmann calls a $\mathbb{Z}$-module valuation on G (see [6, chapter 0]). We let $\Lambda \cup \{\infty\}$ be the value chain of this valuation, i.e. Λ is a totally ordered set, $\Lambda := \psi(G \backslash \{0\})$, and ∞ is an extra point with $\Lambda < \infty$. Keep in mind that $\Lambda \subseteq G$. For any $\lambda \in \Lambda$, $G^{\lambda} := \{g \in G \mid \psi(g) \geq \lambda\}$ and $G_{\lambda} := \{g \in G \mid \psi(g) > \lambda\}$ are convex subgroups of G (convexity comes from axiom (ACH)). We set $C_{\lambda} := G^{\lambda}/G_{\lambda}$, and we let $\widehat{C}_{\lambda}$ denote the divisible hull of C_{λ}. The pair $(\Lambda, (C_{\lambda})_{\lambda \in \Lambda})$ is called the skeleton of the valued group (G, ψ). We define the Hahn product $H := \underset{\lambda \in \Lambda}{\mathrm{H}} \widehat{C}_{\lambda}$ of the $\widehat{C}_{\lambda}$'s as the group $H := \{h \in \prod_{\lambda \in \Lambda} \widehat{C}_{\lambda} \mid \operatorname{supp}(h) \text{ is well-ordered}\}$. H is naturally endowed with a valuation $w(h) := \min \operatorname{supp}(h)$. Elements of H are denoted as formal sums: $h = \sum_{\lambda \in \Lambda} h_{\lambda}$, where $h_{\lambda} \in \widehat{C}_{\lambda}$ for all $\lambda \in \Lambda$.

We first consider the case where G is equal to the Hahn product H, and we will explain at the end how to move to the general case. Note that $G = H$ implies in particular that G is divisible, and thus $\widehat{C}_{\lambda} = C_{\lambda}$. Moreover, every element of g can then be written as a formal sum $\sum_{\lambda \in \Lambda} g_{\lambda}$ with $g_{\lambda} \in C_{\lambda}$, and we have $\psi(g) = \min \operatorname{supp}(g)$. Note that, in particular, we can view each C_{λ} as a subgroup of G.

We assume that, for each $\lambda \in \Lambda$, a group embedding $u_{\lambda} : C_{\lambda} \to (k, +)$ is given. We define the derivation on K in 3 steps:

Step 1: Define D on the "fundamental monomials" of K, i.e define $D(t^{g_{\lambda}})$ for each $g_{\lambda} \in C_{\lambda}$ for every $\lambda \in \Lambda$.

Note that, if D is already constructed on K, then we must have $v(D(a)) = v(a) + \psi(v(a))$ for all $a \in K$. Therefore, we want to define $D(t^{g_{\lambda}})$ as an element with valuation $g_{\lambda} + \lambda$. We thus define:

$$D(t^{g_{\lambda}}) := u_{\lambda}(g_{\lambda}) t^{g_{\lambda} + \lambda}.$$

Note that the presence of the coefficient $u_{\lambda}(g_{\lambda})$ is essential to ensure that the usual product rule of derivations is satisfied. Indeed, take $g_{\lambda}, h_{\lambda} \in C_{\lambda}$. We have $t^{g_{\lambda}} t^{h_{\lambda}} = t^{g_{\lambda} + h_{\lambda}}$. By our definition of D, we have $D(t^{g_{\lambda} + h_{\lambda}}) = u_{\lambda}(g_{\lambda}) t^{g_{\lambda} + h_{\lambda} + \lambda} + u_{\lambda}(h_{\lambda}) t^{g_{\lambda} + h_{\lambda} + \lambda} = t^{h_{\lambda}} D(t^{g_{\lambda}}) + t^{g_{\lambda}} D(t^{h_{\lambda}})$, which is what we want. However, $t^{g_{\lambda} + h_{\lambda} + \lambda} + t^{g_{\lambda} + h_{\lambda} + \lambda} \neq t^{g_{\lambda} + h_{\lambda} + \lambda}$.

Step 2: Extend D to all monomials by using a "strong Leibniz rule".

Let $g = \sum_{\lambda\in\Lambda} g_\lambda \in G$. We naively define $D(t^g)$ by assuming that the usual product rule of derivations also holds for infinite products, i.e. we set:

$$D(t^g) := \sum_{\lambda\in\Lambda} D(t^{g_\lambda})t^{g-g_\lambda} = t^g \sum_{\lambda\in\Lambda} u_\lambda(g_\lambda)t^\lambda.$$

Note that the support of the family $(u_\lambda(g_\lambda)t^\lambda)_{\lambda\in\Lambda}$ is isomorphic to a subset of the support of g so it is well-ordered. Therefore, the family $(u_\lambda(g_\lambda)t^\lambda)_{\lambda\in\Lambda}$ is summable, which proves that the above formula makes sense.

Step 3: Extend D to K by strong linearity.

Let $a = \sum_{g\in G} a_g t^g \in K$. We naively define $D(a)$ by assuming that D is strongly linear, so we set:

$$D(a) := \sum_{g\in G} a_g D(t^g) = \sum_{0\neq g\in G} \sum_{\lambda\in\Lambda} a_g u_\lambda(g_\lambda)t^{g+\lambda}. \tag{†}$$

Note that the idea of these three steps comes from [7]. In [7], the authors were working under the assumption that G is a Hahn product of copies of $\mathbb{R}$. They assumed that step 1 was already given, and then gave conditions for the map on fundamental monomials to be extendable by strong Leibniz rule and strong linearity. However, the authors of [7] were not working with an asymptotic couple, but just with a totally ordered abelian group G, and they did not give an explicit construction for the derivation. The idea to use ψ as a valuation, to decompose G using this valuation, and to give an explicit definition of D on the fundamental monomials with the help of ψ, was introduced in [9].

Of course, expression (†) is purely formal, and we need to make sure that this formula acutally makes sense. This is given by the following:

Proposition 1
1. The set $A := \bigcup_{g\in\mathrm{supp}(a)} \mathrm{supp}(D(t^g)) = \{g+\lambda \mid g \in \mathrm{supp}(a), \lambda \in \mathrm{supp}(g)\}$ is well-ordered.
2. For any $h+\mu \in A$, there are only finitely many (g,λ) with $g \in \mathrm{supp}(a)$ and $\lambda \in \mathrm{supp}(g)$ such that $h+\mu = g+\lambda$.

Proof. See [9]. □

Proposition 1 says that the family $(a_g D(t^g))_{g\in\mathrm{supp}(a)}$ is summable. We define its sum as $\sum_{g\in G} c_g t^g$, where $c_g := 0$ if $g \notin A$, and $c_{h+\mu} = \sum\limits_{g+\lambda=h+\mu} a_g u_\lambda(g_\lambda)$ for any $h+\mu \in A$ (this sum is finite thanks to Proposition 1.2).

Therefore, formula (†) above gives us a well-defined map $D : K \to K$. Because of how it was constructed, it is easy to see that D is indeed a derivation. Moreover, we have:

Proposition 2
Let $a = \sum_{g \in G} a_g t^g \in K$ with $v(a) \neq 0$, $g := v(a)$ and $\lambda := \psi(g)$. Then $v(D(a)) \geq g + \lambda$, and the coefficient of $D(a)$ at $g + \lambda$ is $a_g u_\lambda(g_\lambda)$. In particular, if $u_\lambda(g_\lambda) \neq 0$ then $v(D(a)) = v(a) + \psi(v(a))$.

Proof. See [9]. $\square$

It then follows from Proposition 2 that, if each u_λ is injective, then we will have $v(D(a)) = v(a) + \psi(v(a))$. Because of this equality, we can deduce properties (DV1) and (DV2) from the fact that ψ satisfies axioms (AC1),(AC2) and (AC3). If k is an ordered field, then this order canonically lifts to K, and it is easy to see that, if each u_λ is order-reversing, then $(K, D, \leq)$ is a H-field.

This shows that formula (†) gives us the derivation which we want, provided that each u_λ is an embedding of groups (respectively, of ordered groups). We showed it in the case where $G = H$, so let us now explain how to deal with the general case.

We have the following theorem from [6, chapter 0]:

Theorem 1 (Hahn's embedding theorem for valued groups)
There exists an embedding of valued groups $\iota : (G, \psi) \hookrightarrow (H, w)$.

Now note that, because of axiom (ACH), each G_λ is convex in G. It follows that the order on G induces an order on C_λ. It is then easy to see that the orders on the C_λ's can be naturally lifted to an order on H, and that we can choose ι to be an embedding of ordered groups. Finally, if we define $\psi_H(h) := \iota(w(h))$ for any $h \in H$, then we have the following:

Corollary 1
The map $\iota : (G, \psi) \hookrightarrow (H, \psi_H)$ is an embedding of asymptotic couples. Moreover, ι naturally induces an embedding of valued fields $\rho : K \to k((H))$, defined by $\rho(\sum a_g t^g) := \sum a_g t^{\iota(g)}$.

By what we have done before, we know that we can define a derivation D on $k((H))$, defined by $D(\sum_{h \in H} a_h t^h) := \sum_{h \in H} a_h D(t^h) = \sum_{0 \neq h \in H} \sum_{\lambda \in \Lambda} a_h u_\lambda(h_\lambda) t^{h + \iota(\lambda)}$. It is easy to see from this definition that $\rho(K)$ is stable under D, so we can define a derivation D_K on K by setting $D_K := \rho^{-1} \circ D \circ \rho$. It is then easy to see that D_K has the desired properties.

Finally, we can answer Question 3 with the following theorem:

Theorem 2
Let k be a field (respectively, an ordered field) and (G, ψ) a H-type asymptotic couple. Denote by v the usual valuation on $K := k((G))$. Let $(\Lambda, (C_\lambda)_{\lambda \in \Lambda})$ be the skeleton of the valued group (G, ψ). The following conditions are equivalent:

(1) There exists a derivation D on K making (K, v, D) a differential-valued field (respectively, making $(K, \leq, D)$ a H-field) with asymptotic couple (G, ψ).

(2) For every $\lambda \in \Lambda$, the group $(C_\lambda, +)$ is embeddable into the group $(k, +)$ (respectively, the ordered group $(C_\lambda, +, \leq)$ is embeddable into the ordered group $(k, +, \leq)$).

Proof. We have already seen that, if we have a family of group embeddings $(u_\lambda)_{\lambda\in\Lambda}$, we can construct a derivation with the properties which we want via formula (†). We refer to [9] for the converse. $\square$

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EXPONENTIAL FIELDS AND CONWAY'S OMEGA-MAP

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Abstract. Inspired by Conway's surreal numbers, we study real closed fields whose value group is isomorphic to the additive reduct of the field. We call such fields omega-fields and we prove that any omega-field of bounded Hahn series with real coefficients admits an exponential function making it into a model of the theory of the real exponential field. We also consider relative versions with more general coefficient fields.

1. Introduction

We study real closed valued fields $\mathbb{K}$, with a convex valuation ring $\mathrm{O}(1) \subseteq \mathbb{K}$ satisfying the property that the value group $v(\mathbb{K}^\times)$ is isomorphic to the additive reduct $(\mathbb{K}, +, <)$ of the field, where v is the valuation induced by $\mathrm{O}(1)$. We call **omega-field** a field with this property. The name is motivated by the fact that any omega-field admits a map akin to Conway's omega-map $x \mapsto \omega^x$ on the field of surreal numbers $\mathbf{No}$ [3] or its fragments $\mathbf{No}(\lambda)$ studied in [6], where λ is an ϵ-number. We need to recall that any real closed field $\mathbb{K}$ admits a section of the valuation, hence it has a multiplicative subgroup $G \subseteq \mathbb{K}^{>0}$, called a group of **monomials**, given by the image of the section. Since G is a multiplicative copy of $v(\mathbb{K}^\times)$, we have that $\mathbb{K}$ is an omega-field if and only if it admits an isomorphism

$$\omega : (\mathbb{K}, +, 0, <) \cong (G, \cdot, 1, <),$$

and we shall call **omega-map** any such isomorphism. The prototypical example is Conway's omega-map on the surreal numbers, and in analogy with the surreal case, we use the exponential notation ω^x to denote the image of x under ω.

Here we explore the relation between omega-fields and exponential fields, where an **exponential field** is a real closed field $\mathbb{K}$ admitting an **exponential map**, that is an isomorphism $\exp : (\mathbb{K}, +, 0, <) \cong (\mathbb{K}^{>0}, \cdot, 1, <)$. We shall freely write e^x rather than $\exp(x)$ when convenient. Note that ω^x should not be read as $e^{\omega \log(x)}$ (the easiest way to see why is that the map $x \mapsto \omega^x$, if there is such an omega-map, is not continuous in the order topology of $\mathbb{K}$). While in general there are no containments between the class of fields admitting an omega-map and that of fields admitting an exponential map, a non-trivial inclusion of the former in the latter can be obtained by restricting the analysis to κ-bounded Hahn fields, as discussed below.

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In general, any real closed valued field $\mathbb{K}$ with monomials G is isomorphic to a truncation closed subfield (see Definition 2.8 (1)) of the Hahn field $\mathbf{k}((G))$ [13], where $\mathbf{k} \cong \mathrm{O}(1)/o(1)$ is the residue field and we write $o(1)$ for the maximal ideal of $\mathrm{O}(1)$. For the sake of simplicity in this introduction we focus on the typical case $\mathbf{k} = \mathbb{R}$, but our results hold more generally assuming that the residue field $\mathbf{k}$ is a model of $T_{an,\exp}$, the theory of the real exponential field $\mathbb{R}_{\exp}$ with all restricted analytic functions [5]. The full Hahn field $\mathbb{R}((G))$ is always naturally a model of the theory of restricted analytic functions T_{an} [5], but it never admits an exponential function if $G \neq 1$ [10]. However, for every regular uncountable cardinal κ, there is a group G such that the *κ-bounded* Hahn field $\mathbb{R}((G))_\kappa$ does admit an exponential function [12]. We thus restrict our analysis to fields of the form $\mathbb{K} = \mathbb{R}((G))_\kappa$ (without assuming *a priori* that they admit an exponential map). Our first result is the following. The case $G = \mathbf{No}(\kappa)$ with κ regular uncountable is in [6].

Theorem (3.8)**.** *Every omega-field of the form $\mathbb{R}((G))_\kappa$ admits an exponential function making it into a model of $T_{an,\exp}$.*

Our work was partly motivated by the desire to understand the connections between the surreal numbers, with its various subfields studied in [1, 2], and the exponential fields of the form $\mathbb{R}((G))_\kappa$ constructed by S. Kuhlmann and S. Shelah in [12]. We shall prove that the latter are not always omega-fields (Theorem 4.5), but they are omega-fields if and only if G is order-isomorphic to $G^{>1}$ (Theorem 4.1); in this case, given a chain isomorphism $\psi : G \cong G^{>1}$, there is an omega-map satisfying $\omega^g = e^{\psi(g)}$ for all $g \in G$.

Let us now discuss Theorem 3.8 in more detail. We show that given $\mathbb{K} = \mathbb{R}((G))_\kappa$ and an omega-map $\omega : \mathbb{K} \cong G$, we can construct an exponential function directly starting from ω and an auxiliary chain isomorphism

$$h : (\mathbb{K}, <) \cong (\mathbb{K}^{>0}, <),$$

where by **chain** we mean linearly ordered set. Any choice of h yields an exponential field (Theorem 3.4) and at least one choice of h will yield a model of $T_{an,\exp}$ (Theorem 3.8). Varying h we can thus produce a variety of exponential fields; some of them are models of $T_{\exp}$, while all the others are not even o-minimal (Theorem 3.10), depending on the growth properties of h (Definition 2.11).

To define the exponential function, it is more convenient to first define a **logarithm** $\log : \mathbb{K}^{>0} \to \mathbb{K}$ and let exp be the compositional inverse log. To this aim we start by putting

$$\log(\omega^{\omega^x}) = \omega^{h(x)}$$

for $x \in \mathbb{K}$ and $\log(1+\varepsilon) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\varepsilon^n}{n}$ for $\varepsilon \in o(1)$. Note that such infinite sums make sense in the κ-bounded Hahn field $\mathbb{R}((G))_\kappa$.

The extension of log to the whole of $\mathbb{K}^{>0}$ is then carried out guided by the principle that log takes products into sums and ω takes sums into products. We simply extend this to infinite sums. More precisely, log is determined by $\log(\omega^{\sum_{i<\alpha}\omega^{\gamma_i}r_i}) = \sum_{i<\alpha}\omega^{h(\gamma_i)}r_i$ and $\log(\omega^x r(1+\varepsilon)) = \log(\omega^x) + \log(r) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\varepsilon^n}{n}$, where $\varepsilon \in o(1)$ and $r \in \mathbb{R}$. Another way to express the spirit of the construction is that we first define log on the representatives of the multiplicative archimedean classes ω^{ω^x}, then we extend it to the representatives of the additive archimedean classes ω^x, and finally to the whole of $\mathbb{K}$. It is not difficult to show that this construction always yields an exponential field. We now need to show that there is at least one function h such that the exponential field $\mathbb{K}$ arising from ω and h as above is a model of $T_{\exp}$. A necessary condition is that the exponential map grows faster than any polynomial, or equivalently, that its inverse log grows slower than $x^{1/n}$ for all positive $n \in \mathbb{N}$. This translates into the condition $h(x) < r \cdot \omega^x$ for every $x \in \mathbb{K}$ and $r \in \mathbb{R}^{>0}$. We shall abbreviate the above with $h(x) \prec \omega^x$.

Since ω^x is discontinuous (its values are the respresentatives of the archimedean classes), and h is continuous in the order topology of $\mathbb{K}$ (being a chain isomorphism from $\mathbb{K}$ to $\mathbb{K}^{>0}$), the existence of such an h is not immediate. In the case of Gonshor's h on the surreal numbers [7], the condition $h(x) \prec \omega^x$ is forced by the inductive definition of h. However, this cannot be generalized to our more general setting where similar inductive definitions make no sense, and we use instead a bootstrapping procedure (Lemma 3.6). Granted this, the final exp on $\mathbb{K}$ is easily seen to yield a model of $T_{\exp}$ using [15, 5] (Theorem 3.8).

All the logarithms considered in this paper are **analytic** (Definition 2.10): for $\varepsilon \in o(1)$, the function $x \mapsto \log(1+x)$ is given by the familiar Taylor expansion $\log(1+\varepsilon) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\varepsilon^n}{n}$, whereas for $g \in G$, $\log(g)$ is a *purely infinite* element of $\mathbb{R}((G))_\kappa$, and for $r \in \mathbb{R}$, $\log(r)$ is the usual real logarithm.

Theorems 3.4 and 3.8 produce analytic logarithms satisfying two additional restrictions: $\log(\omega^{\omega^x}) \in G$ for all $x \in \mathbb{K}$, and log brings "infinite products" to "infinite sums". It turns out, however, that all analytic logarithms arise in this way, up to changing the omega-map $\omega : \mathbb{K} \cong G$. More precisely, we have the following classification result.

Theorem (Corollary 4.2). *Every analytic logarithm on an omega-field of the form $\mathbb{K} = \mathbb{R}((G))_\kappa$ arises from some omega-map $x \mapsto \omega^x$ and some chain isomorphism $h : \mathbb{K} \cong \mathbb{K}^{>0}$.*

The surreal numbers fit into the above picture if we allow κ to be the proper class of all ordinals and G to be the image of Conway's omega-map $x \mapsto \omega^x$. Gonshor's exponentiation is induced by the omega-map and Gonshor's function h [7]; by the above results, any other analytic logarithm on **No** arises in this way, possibly after replacing Conway's omega-map with another isomorphism from **No** to its group of monomials and Gonshor's h with another chain isomorphism.

2. Preliminaries

2.1. **Valuations.** Let $\mathbb{K}$ be an ordered field (possibly with additional structure) and let $\mathrm{O}(1) \subseteq \mathbb{K}$ be a convex subring. Then $\mathrm{O}(1)$ is the valuation ring of a valuation v and we denote by $o(1)$ the unique maximal ideal of $\mathrm{O}(1)$. If $\mathbb{K}$ is real closed, it has a subfield $\mathbf{k} \subseteq \mathbb{K}$ isomorphic to the residue field $\mathrm{O}(1)/o(1)$ of the valuation, namely we can write $\mathrm{O}(1) = \mathbf{k} + o(1)$. We shall always assume in the sequel that $\mathbb{K}$ is real closed and $\mathrm{O}(1), o(1), \mathbf{k}$ are as above.

Definition 2.1. For $x, y \in \mathbb{K}$ we define:

- $x \preceq y$ if $|x| \leq c|y|$ for some $c \in \mathrm{O}(1)$ (domination);
- $x \asymp y$ if $x \preceq y$ and $y \preceq x$ (comparability);
- $x \prec y$ if $x \preceq y$ and $x \not\asymp y$ (strict domination);
- $x \sim y$ if $x - y \prec x$ (x is asymptotic to y).

With these notations we have $\mathrm{O}(1) = \{x : x \preceq 1\}$ and $o(1) = \{x : x \prec 1\}$.

Definition 2.2. A multiplicative subgroup G of $\mathbb{K}^{<0}$ is a group of **monomials** if it consists in a family of representatives for each $\asymp$-class. In other words a group of monomials is an embedded copy of the value group. It is well known that any real closed field admits a group of monomials.

Remark 2.3. For $x, y \in \mathbb{K}$ we have:

- $x \prec y$ if and only if $c|x| < |y|$ for all $c \in \mathrm{O}(1)$ (or equivalently for all $c \in \mathbf{k}$);
- $x \asymp y$ if and only if $x = cy(1+\varepsilon)$ for some $c \in \mathbf{k}^\times$ and $\varepsilon \in o(1)$;
- $x \sim y$ if and only if $x = y(1+\varepsilon)$ for some $\varepsilon \in o(1)$.
- if $x \neq 0$ there are unique $r \in \mathbf{k}^\times, g \in G, \varepsilon \in o(1)$ such that $x = gr(1+\varepsilon)$.

2.2. **Hahn groups.** By a **chain** we mean a linearly ordered set. We describe a well known procedure to build an ordered group starting from a chain.

Definition 2.4. Given a chain Γ and an ordered abelian group $(C, +, <)$, the **Γ-product of** C is the abelian group of all functions $f : \Gamma \to C$ with *reverse* well-ordered **support** $\{\gamma \in \Gamma : f(\gamma) \neq 0\}$ and pointwise addition, ordered by declaring $f > 0$ if $f(\gamma) > 0$, where γ is the *biggest* element in the support.[1]

If we write G in additive notation, a typical element of G can be written in the form $\sum_{\gamma\in\Gamma} \gamma r_\gamma$, representing the function sending $\gamma \in \Gamma$ to $r_\gamma \in C$, while G itself is denoted $\sum \Gamma C$. We prefer however to use a multiplicative notation and write G as $\prod t^{\Gamma C}$ and a typical element of G as $\prod_{\gamma\in\Gamma} t^{\gamma r_\gamma}$. In this notation the multiplication is given by

$$\left(\prod_{\gamma\in\Gamma} t^{\gamma r_\gamma}\right)\left(\prod_{\gamma\in\Gamma} t^{\gamma r'_\gamma}\right) = \prod_{\gamma\in\Gamma} t^{\gamma(r_\gamma + r'_\gamma)}$$

Since the supports are reverse well-ordered, we can fix a decreasing enumeration $(\gamma_i : i < \alpha)$ of the support, where α is an ordinal, and write an element of $\prod t^{\Gamma C}$ in the form

$$f = \prod_{i<\alpha} t^{\gamma_i r_i} \in \prod t^{\Gamma C}.$$

According to our conventions, $f > 1 \iff r_0 > 0$ and $t^\gamma > t^\beta \iff \gamma > \beta$.

If Γ has only one element, we may identify $\prod t^{\Gamma C}$ with a multiplicative copy t^C of $(C, +, <)$.

When $C = (\mathbb{R}, +, <)$, we obtain the **Hahn group** over Γ, which can be characterized as a maximal ordered group with a set of archimedean classes of the same order type as Γ [8]. Recall that two positive elements are in the same archimedean class if each of them is bounded, in absolute value, by an integer multiple of the other.

Notation 2.5. Let κ be a regular cardinal. If in the definition of the Γ-product we only allow supports of reverse order type $< \kappa$, we obtain the κ-bounded version

$$\left(\prod t^{\Gamma C}\right)_\kappa \subseteq \prod t^{\Gamma C}.$$

We shall also consider the case when Γ is a proper class and $\kappa = \mathbf{On}$, in which case $\left(\prod t^{\Gamma C}\right)_{\mathbf{On}}$ is the ordered group of all functions $f : \Gamma \to C$ whose support is a reverse well ordered *set* (rather than a reverse well ordered class).

[1] Other authors prefer to use well-ordered supports, but one can pass from one version to the other reversing the order of Γ.

2.3. **Hahn fields.** Given a field $\mathbf{k}$ and a multiplicative ordered abelian group G, let $\mathbf{k}((G))$ denote the Hahn field with coefficients in $\mathbf{k}$ and monomials in G. The underlying additive group of $\mathbf{k}((G))$ coincides with the G-product of $\mathbf{k}$: its elements are functions $f : G \to \mathbf{k}$ with reverse well-ordered supports, which we write either in the form $f = \sum_{g \in G} g r_g$, where $r_g = f(g)$, or in in the form

$$f = \sum_{i<\alpha} g_i r_i$$

where α is an ordinal, $(g_i)_{i<\alpha}$ is a decreasing enumeration of the support, and $r_i = f(g_i) \in \mathbf{k}^*$. Addition is defined componentwise and multiplication is given by the usual Cauchy product. We order $\mathbf{k}((G))$ according to the sign of the leading coefficient, namely $f > 0 \iff r_0 > 0$.

Remark 2.6. It can be proved that if $\mathbf{k}$ is real closed and G is divisible, then $\mathbf{k}((G))$ is real closed [9]. Moreover, $\mathbf{k}((G))$ is **spherically complete**: any decreasing intersection of valuation balls has a non-empty intersection.

Notation 2.7. Inside $\mathbf{k}((G))$, we let $\mathrm{O}(1)$ be the valuation ring of all the elements x with $|x| \le r$ for some $r \in \mathbf{k}$, and $o(1)$ be the corresponding maximal ideal. We then have $\mathrm{O}(1) = \mathbf{k} + o(1)$. With respect to this valuation ring, $\mathbf{k}$ is a copy of the residue field and G is a group of monomials. We shall use similar notations for any subfield $\mathbb{K} \subseteq \mathbf{k}((G))$ containing $\mathbf{k}$ and G.

2.4. **Restricted analytic functions.** A family $(f_i)_{i \in I}$ of elements of $\mathbf{k}((G))$ is **summable** if the union of the supports of the elements f_i is reverse well-ordered and, for each $g \in G$, there are only finitely many $i \in I$ such that g is in the support of f_i. In this case $\sum_{i \in I} f_i$ is defined as the element $f = \sum_g g r_g$ of $\mathbf{k}((G))$ whose coefficients are given by $r_g = \sum_{i \in I} r_{g,i} \in \mathbf{k}$ where $r_{g,i}$ is the coefficient of g in f_i. This makes sense since, given $g \in G$, only finitely many $r_{g,i}$ are non-zero.

By Neumann's lemma [14] for any power series $P(x) = \sum_{n \in \mathbb{N}} a_n x^n$ with coefficients in $\mathbf{k}$ and for any $\varepsilon \prec 1$ in $\mathbf{k}((G))$, the family $(a_n \varepsilon^n)_{n \in \mathbb{N}}$ is summable, so we can evaluate $P(x)$ at ε obtaining an element $P(\varepsilon) = \sum_{n \in \mathbb{N}} a_n \varepsilon^n$ of $\mathbf{k}((G))$. Similar considerations apply to power series in several variables.

Definition 2.8. Let $\mathbb{K} \subseteq \mathbf{k}((G))$ be a subfield. We say that $\mathbb{K}$ is an **analytic subfield** if

(1) $\mathbb{K}$ is truncation closed: if $\sum_{i<\alpha} g_i r_i$ belongs to $\mathbb{K}$, then $\sum_{i<\beta} g_i r_i$ belongs to $\mathbb{K}$ for every $\beta \le \alpha$;
(2) $\mathbb{K}$ contains $\mathbf{k}$ and G;
(3) If $P(x) = \sum_{n \in \mathbb{N}} a_n x^n$ is a power series with coefficients in $\mathbf{k}$ and $\varepsilon \prec 1$ is in $\mathbb{K}$, then the element $P(\varepsilon) = \sum_{n \in \mathbb{N}} a_n \varepsilon^n \in \mathbf{k}((G))$ lies in the subfield $\mathbb{K}$. Similarly for power series in several variables.

We recall that T_{an} is the theory of the real field with all analytic functions restricted to the poly-intervals $[-1,1]^n \subseteq \mathbb{K}^n$ [5]. (By rescaling, we can equivalently use any other closed poly-interval.)

Fact 2.9. *We have:*

(1) *The field $\mathbb{R}((G))$ admits a natural intepretation of the analytic functions restricted to the poly-interval $[-1,1]^n \subseteq \mathbb{K}$, making $\mathbb{K}$ into a model of T_{an}.*
(2) *The same holds for any analytic subfield of $\mathbb{R}((G))$, and in particular for $\mathbb{R}((G))_\kappa$ for every regular uncountable κ.*

(3) *More generally, if* $\mathbf{k}$ *is a model of* T_{an}*, then any analytic subfield* $\mathbb{K}$ *of* $\mathbf{k}((G))$ *is naturally a model of* T_{an}*.*

The proof of (1) is in [5] and is based on a quantifier elimination result in the language of T_{an}. The other points follow by the same argument. We interpret the restricted analytic functions in the analytic subfield $\mathbb{K} \subseteq \mathbf{k}((G))$ as follows. Given a real analytic function f converging on a neighbourhood of $[-1,1]^n \cap \mathbb{R}^n$, we need to define $f(x+\varepsilon)$ where $x \in [-1,1]^n \cap \mathbf{k}^n$ and $\varepsilon \in o(1)^n \subseteq \mathbb{K}^n$. We do this by using the Taylor expansion $f(x+\varepsilon) = \sum_i \frac{D^i f(x)}{i!}\,\varepsilon^i$ where i is a multi-index in $\mathbb{N}^n$. Here $D^i f(x) \in \mathbf{k}$ (using the fact that $\mathbf{k}$ is a model of T_{an}) and the infinite sum is in the sense of the Hahn field $\mathbf{k}((G))$.

2.5. **Exponential fields.** A **prelogarithm** on a real closed field $\mathbb{K}$ is a morphism from $(\mathbb{K}^{>0}, \cdot, 1, <)$ to $(\mathbb{K}, +, 0, <)$ and a **logarithm** is a surjective prelogarithm. An **exponential map** is the compositional inverse of a logarithm, that is an isomorphism from $(\mathbb{K}, +, 0, <)$ to $(\mathbb{K}^{>0}, \cdot, 1, <)$. We say that $\mathbb{K}$ is an **exponential** field if it has an exponential map. Given a logarithm log, we write exp for the corresponding exponential map and we write e^x instead of $\exp(x)$ when convenient. Now assume $\mathbf{k}$ has a logarithm and consider the Hahn field $\mathbf{k}((G))$. It turns out that if $G \neq 1$, $\mathbf{k}((G))$ never admits a logarithm extending that on $\mathbf{k}$ [10]. On the other hand if κ is a regular uncountable cardinal, then for suitable choices of G, the logarithm on $\mathbf{k}$ can be extended to $\mathbf{k}((G))_\kappa$, and when $\mathbf{k} = \mathbb{R}$ this can be done in such a way that $\mathbf{k}((G))_\kappa$ is a model of $T_{\exp}$ [12].

Definition 2.10. Let $\mathbf{k}$ be an exponential field and let $\mathbb{K}$ be an analytic subfield of $\mathbf{k}((G))$, for instance $\mathbb{K} = \mathbf{k}((G))_\kappa$ with κ regular uncountable. An **analytic logarithm** on $\mathbb{K}$ is a logarithm $\log : \mathbb{K}^{>0} \to \mathbb{K}$ with the following properties:

(1) $\log : \mathbb{K}^{>0} \to \mathbb{K}$ extends the given logarithm on $\mathbf{k}$.
(2) $\log(1+\varepsilon) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}\,\varepsilon^i$ for all $\varepsilon \prec 1$ in $\mathbb{K}$ (the assumption $\varepsilon \prec 1$ ensures the summability).
(3) $\log(G) = \mathbb{K}^{\uparrow}$, where $\mathbb{K}^{\uparrow} := \mathbf{k}((G^{>1})) \cap \mathbb{K}$ is the group of **purely infinite** elements, namely the series of the form $\sum_{i<\alpha} g_i r_i$ with $g_i \in G^{>1}$ for all i.

Conditions (1) and (2) are rather natural, and ensure that the restrictions of $\log(1+x)$ to small finite intervals agree with the natural T_{an}-interpretations of Fact 2.9. A motivation for (3) is the following. The multiplicative group $\mathbb{K}^{>0}$ admits a direct sum decomposition

$$\mathbb{K}^{>0} = G\,\mathbf{k}^{>0}(1+o(1)),$$

namely any element x of $\mathbb{K}^{>0}$ can be uniquely written in the form $x = gr(1+\varepsilon)$ where $r \in \mathbf{k}^{>0}$, $g \in G$ and $\varepsilon \in o(1)$. Applying log to both sides of the above equation, we get (by (1) and (2)) a direct sum decomposition

$$\mathbb{K} = \log(G) \oplus \mathbf{k} \oplus o(1)$$

of the additive group $(\mathbb{K}, +)$. Indeed by (1) we have $\log(\mathbf{k}^{>0}) = \mathbf{k}$ and $\log(\mathbb{K}^{>0}) = \mathbb{K}$, while (2) implies that the logarithm maps $1 + o(1)$ bijectively to $o(1)$ with inverse given by $\exp(\varepsilon) = \sum_{n\in\mathbb{N}} \frac{\varepsilon^n}{n!}$. We have thus proved that $\log(G)$ is a direct complement of $\mathrm{O}(1) = \mathbf{k} + o(1)$. Although there are several choices for such a complement, the most natural one is $\log(G) = \mathbb{K}^{\uparrow}$, as required in point (3), since it is the unique one closed under truncations.

2.6. **Growth axiom and models of $T_{\exp}$.** Ressayre proved in [15] that an exponential field is a model of $T_{\exp}$ if and only it satisfies the elementary properties of the real exponential restricted to $[0,1]$ and satisfies the growth axiom scheme $x \geq n^2 \to \exp(x) > x^n$ for all $n \in \mathbb{N}$.

Definition 2.11. Given an analytic subfield $\mathbb{K} \subseteq \mathbf{k}((G))$, we say that an analytic logarithm $\log : \mathbb{K}^{>0} \to \mathbb{K}$ satisfies the **growth axiom at infinity** if $\log(x) < x^{1/n}$ for all $x > \mathbf{k}$ and all positive integers n.

Proposition 2.12. *If $\mathbf{k}$ is a model of $T_{an,exp}$ (for instance $\mathbf{k} = \mathbb{R}$) and $\mathbb{K} \subseteq \mathbf{k}((G))$ is an analytic subfield with an analytic logarithm satisfying the growth axiom at infinity, then $\mathbb{K}$ (with the natural intepretation of the symbols) is a model of $T_{an,\exp}$.*

Proof. This follows from [15, 5] but we include some details. The inverse exp of an analytic logarithm is easily seen to satisfy $e^{\varepsilon} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}$ for all $\varepsilon \in o(1)$. Since moreover exp extends the given exponential on $\mathbf{k}$, it follows that the restriction of exp to $[-1,1]$ agrees with the natural T_{an}-interpretation of Fact 2.9. This shows that $\mathbb{K}$ is at least a model of $T_{\exp\,|[-1,1]}$, as it is in fact the restriction of a model of T_{an} to a sublanguage. Since the interpretation of exp grows faster than any polynomial (by the growth axiom at infinity plus the fact that $\mathbf{k}$ is a model of $T_{\exp}$), we can conclude by the axiomatisations of [15, 5]. □

The above result rests on the quantifier elimination result for $T_{an,\exp}$. We do not know whether it suffices that $\mathbf{k}$ is a model of $T_{\exp}$ to obtain that $\mathbf{k}((G))_{\kappa}$ is a model of $T_{\exp}$ (or even $T_{\exp\,|[0,1]}$).

3. Omega fields

Definition 3.1. A real closed field $(\mathbb{K}, +, \cdot, <)$ with a convex valuation ring $\mathrm{O}(1)$ and corresponding group of monomials $G \subseteq \mathbb{K}^{>0}$ is an **omega-field** if $(\mathbb{K}, +, <)$ is isomorphic to $(G, \cdot, <)$ as an ordered group. Given an omega-field $\mathbb{K}$ we shall call **omega-map** any isomorphism of ordered groups

$$\omega : (\mathbb{K}, +, 0, <) \cong (G, \cdot, 1, <).$$

Since the group G of monomials is isomorphic to the value group of $\mathbb{K}$, we have that $\mathbb{K}$ is an omega-field if and only if $(\mathbb{K}, +, <)$ is isomorphic to its value group. The definition of omega-map is inspired by Conway's omega map ω^x on the surreal numbers. We recall that the surreals can be presented in the form $\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}}))_{\mathbf{On}}$, with the image of the omega-map being the group $\omega^{\mathbf{No}}$ of monomials. The subscript $\mathbf{On}$ indicates that we only consider series whose support is a set, rather than a proper class. The surreals should thus be considered as a bounded Hahn field rather than a full Hahn field.

3.1. **Construction of omega-fields.** In the sequel let κ be a regular uncountable cardinal.

Theorem 3.2. *Given an exponential field $\mathbf{k}$, there is a group G such that the field $\mathbb{K} = \mathbf{k}((G))_{\kappa}$ admits an omega-map $\omega : \mathbb{K} \to G$.*

When $\mathbf{k} = \mathbb{R}$ one can take $G = \mathbf{No}(\kappa)$ as in [6]. In the general case the proof is a variant of a similar construction in [12]. Given a chain Γ and an additive ordered group C (in our application $C = (\mathbf{k}, +, <)$), let $H(\Gamma)$ denote, in the following Lemma, the ordered group $\left(\prod t^{\Gamma C}\right)_{\kappa}$.

Lemma 3.3. *Fix a chain Γ_0 and a chain embedding $\eta_0 : \Gamma_0 \to H(\Gamma_0)$ (for instance $\eta_0(\gamma) = t^\gamma$). Then there is a chain $\Gamma \supseteq \Gamma_0$ and a chain isomorphism $\eta : \Gamma \cong H(\Gamma)$ extending η_0.*

Proof. We consider H as a functor from chains to ordered abelian groups: if $j : \Gamma' \to \Gamma''$ is a chain embedding, we define $H(j) : H(\Gamma') \to H(\Gamma'')$ as the group embedding which sends $\prod_i t^{\gamma_i r_i}$ to $\prod_i t^{j(\gamma_i) r_i}$. We do an inductive construction in κ-many steps. At a certain stage $\beta < \kappa$ we are given

$$G_\beta = H(\Gamma_\beta)$$

and a chain embedding $\eta_\beta : \Gamma_\beta \to G_\beta$ together with embeddings $j_{\alpha,\beta} : \Gamma_\alpha \to \Gamma_\beta$ for $\alpha < \beta$. Let $\Gamma_{\beta+1}$ be a chain isomorphic to $(G_\beta, <)$ (for instance G_β itself considered as a chain) and fix a chain isomorphism $f_\beta : G_\beta \to \Gamma_{\beta+1}$. Now let $j_\beta : \Gamma_\beta \to \Gamma_{\beta+1}$ be the composition $f_\beta \circ \eta_\beta$ and let $G_{\beta+1} = H(\Gamma_{\beta+1})_\kappa$. We can then find a commutative diagram of embeddings

(1)
$$\begin{tikzcd} \Gamma_\beta \arrow[r, "\eta_\beta"'] \arrow[d, "j_\beta"'] & H(\Gamma_\beta) \arrow[d, "H(j_\beta)"] \arrow[dl, "f_\beta"'] \\ \Gamma_{\beta+1} \arrow[r, "\eta_{\beta+1}"] & H(\Gamma_{\beta+1}), \end{tikzcd}$$

by letting $\eta_{\beta+1} = H(j_\beta) \circ f_\beta^{-1}$. We can now define $j_{\beta,\beta+1} = j_\beta$ and $j_{\alpha,\beta+1} = j_{\beta,\beta+1} \circ j_{\alpha,\beta}$ for $\alpha < \beta$.

We iterate the contruction along the ordinals. At a limit stage λ, let $\Gamma_\lambda = \varinjlim_{\beta<\lambda} \Gamma_\beta$ and let $j_{\beta,\lambda} : \Gamma_\beta \to \Gamma_\lambda$ be the natural embedding for $\beta < \lambda$.

We then define $\eta_\lambda : \Gamma_\lambda \to H(\Gamma_\lambda)$ as the composition

$$\Gamma_\lambda = \varinjlim_{\beta<\lambda} \Gamma_\beta \to \varinjlim_{\beta<\lambda} H(\Gamma_\beta) \to H(\varinjlim_{\beta<\lambda} \Gamma_\beta) = H(\Gamma_\lambda).$$

More explicitly, for each $\gamma \in \Gamma_\lambda$, pick some $\beta < \lambda$ and $\theta \in \Gamma_\beta$ such that $\gamma = j_{\beta,\lambda}(\theta)$, and define $\eta_\lambda(\gamma) \in G_\lambda$ as the image under $H(j_{\beta,\lambda}) : G_\beta \to G_\lambda$ of $\eta_\beta(\theta) \in G_\beta$. Since κ is regular, when we arrive at stage κ we have an isomorphism

$$\eta_\kappa : \Gamma_\kappa \cong G_\kappa$$

and we can define $\Gamma = \Gamma_\kappa$ and $\eta = \eta_\kappa$. □

Proof of Theorem 3.2. By Lemma 3.3, there is a chain Γ and a chain isomorphism

$$\eta : \Gamma \cong G = H(\Gamma) \tag{2}$$

Now let $\mathbb{K} = \mathbf{k}((G))_\kappa$ and define an omega-map $\omega : \mathbb{K} \to G$ by setting

$$\omega^{\sum_{i<\alpha} g_i r_i} = \prod_{i<\alpha} t^{\gamma_i r_i}. \tag{3}$$

where $g_i = \eta(\gamma_i)$. In particular $\omega^{\eta(\gamma)} = t^\gamma$ for every $\gamma \in \Gamma$. □

3.2. **The logarithm.** In the sequel let κ be a regular uncountable cardinal. Our next goal is to prove the following theorem.

Theorem 3.4. *Every omega-field of the form $\mathbb{K} = \mathbb{R}((G))_\kappa$ admits an analytic logarithm. More generally, if $\mathbf{k}$ is an exponential field, then every omega-field of the form $\mathbb{K} = \mathbf{k}((G))_\kappa$ admits an analytic logarithm.*

Proof. We construct a logarithm depending both on the omega-map and on an auxiliary function h. Let $h : \mathbb{K} \to \mathbb{K}^{>0}$ be a chain isomorphism (any ordered field admits such a function, for instance $h(x) = (-x+1)^{-1}$ for $x \leq 0$ and $h(x) = x+1$ for $x \geq 0$). For $x \in \mathbb{K}$, we let

$$\log(\omega^{\omega^x}) = \omega^{h(x)}.$$

This defines log on the subclass $\omega^{\omega^{\mathbb{K}}}$ of G, which we call the class of **fundamental monomials**. They can be seen as the representatives of the multiplicative archimedean classes.

Next we define $\log(g)$ for an arbitary g in G. Since $G = \omega^{\mathbb{K}}$, we can write $g = \omega^x$ for some $x \in \mathbb{K}$. We then write $x = \sum_{i<\alpha} g_i r_i = \sum_{i<\alpha} \omega^{x_i} r_i$ and set $\log(g) = \sum_{i<\alpha} \omega^{h(x_i)} r_i$. Summing up, the definition of $\log_{|G}$ takes the form

$$\log\left(\omega^{\sum_{i<\alpha} \omega^{x_i} r_i}\right) = \sum_{i<\alpha} \omega^{h(x_i)} r_i. \tag{4}$$

The idea is that $\omega^{\sum_{i<\alpha} g_i r_i}$ should be thought as an infinite product $\prod_{i<\alpha} \omega^{g_i r_i}$, and we stipulate that log maps infinite products into infinite sums.

We can now extend log to the whole of $\mathbb{K}^{>0}$ as follows. For $x \in \mathbb{K}^{>0}$ we write $x = gr(1+\varepsilon)$ with $g \in G, r \in \mathbf{k}^{>0}$ and $\varepsilon \prec 1$, and we define

$$\log(x) = \log(g) + \log(r) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}. \tag{5}$$

The infinite sum makes sense because the terms under the summation sign are summable and the sum belongs to $\mathbf{k}((G))_\kappa$ (becacuse κ is regular and uncountable).

We must verify that with these definitions log is an analytic logarithm (Definition 2.10). It is not difficult to see that log is an increasing morphism from $(\mathbb{K}^{>0}, \cdot, 1, <)$ to $(\mathbb{K}, +, 0, <)$. To prove the surjectivity let us first observe that $\mathbf{k} = \log(\mathbf{k}^{>0}) \subseteq \log(\mathbb{K}^{>0})$. Moreover, for $\varepsilon \prec 1$ we have $\log(1+\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$ with inverse given by $e^\varepsilon = \sum_n \frac{\varepsilon^n}{n!}$, and therefore $\log(1 + o(1)) = o(1)$. Since $\mathbb{K} = \mathbb{K}^{\uparrow} + \mathbf{k} + o(1)$, to finish the proof of the surjectivity it suffices to show that $\log(G) = \mathbb{K}^{\uparrow}$. So let $x = \sum_{i<\alpha} g_i r_i \in \mathbb{K}^{\uparrow}$, namely $g_i \in G^{>1}$ for all i. We must show that x is in the image of log. Since $h : \mathbb{K} \to \mathbb{K}^{>0}$ is surjective and $G = \omega^{\mathbb{K}}$, we have $G^{>1} = \omega^{(\mathbb{K}^{>0})} = \omega^{h(\mathbb{K})}$, so we can choose $x_i \in \mathbb{K}$ so that $g_i = \omega^{h(x_i)}$ for all i. Now by definition $\log\left(\omega^{\sum_{i<\alpha} \omega^{x_i} r_i}\right) = \sum_{i<\alpha} \omega^{h(x_i)} r_i = x$ concluding the proof of surjectivity. □

In the above theorem we have considered $\mathbf{k}((G))_\kappa$, rather than an arbitrary analytic subfield $\mathbb{K}$ of $\mathbf{k}((G))$, because for the proof to work we need to know that whenever $\sum_{i<\alpha} \omega^{x_i} r_i \in \mathbb{K}$, we also have $\sum_{i<\alpha} \omega^{h(x_i)} r_i \in \mathbb{K}$.

Definition 3.5. We call $\log_{\omega,h} : \mathbb{K}^{>0} \to \mathbb{K}$ the analytic logarithm induced by the omega-map $\omega : \mathbb{K} \to G$ and the chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$ as given by (4)-(5) in the proof of Theorem 3.4, and we call $\exp_{\omega,h}$ its compositional inverse.

3.3. **Getting a logarithm satisfying the growth axiom.** The structures constructed so far are exponential fields, but not necessarily models of $T_{\exp}$. In this section we show how to get models of $T_{\exp}$. We need the following lemma to take care of the growth axiom at infinity.

Lemma 3.6. *Let $\mathbb{K} = \mathbf{k}((G))_\kappa$ be equipped with an omega-map $\omega : \mathbb{K} \cong G$. Then there exists a chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$ such that $h(x) \prec \omega^x$ for all $x \in \mathbb{K}$.*

Proof. The idea is a bootstrapping procedure. Given an h we produce a log and an exp, and given the exp we produce a new h. We then glue together a couple of h's obtained in this way to produce the final h.

To begin with, consider the following chain isomorphism $\mathbb{K} \to \mathbb{K}^{>0}$, definable in any ordered field:

$$h_0(x) = \begin{cases} x+1 & \text{for } x \geq 0 \\ \frac{1}{1-x} & \text{for } x < 0, \end{cases} \qquad h_1(x) = \begin{cases} \frac{1}{2}x+1 & \text{for } x \geq 0 \\ \frac{1}{1-x} & \text{for } x < 0. \end{cases}$$

Definition 3.5 yields two logarithmic functions $\log_0 = \log_{\omega,h_0}$ and $\log_1 = \log_{\omega,h_1}$ on $\mathbf{k}((G))_\kappa$ associated with h_0 and h_1 (and the given omega-map). Since $h_1(x) \leq h_0(x)$, we have $\log_1(x) \leq \log_0(x)$ for all $x \in \mathbb{K}^{>1}$. The corresponding exponential functions $\exp_0, \exp_1$ satisfy the opposite inequality: $\exp_0(x) \leq \exp_1(x)$ for $x > 0$.

We claim that

$$\exp_0(x) \prec \omega^x \text{ for } x > \mathbf{k} \quad \text{and} \quad \exp_1(x) \prec \omega^x \text{ for } x \leq -\omega^3.$$

Indeed, note that $h_0(x) > x$ for all $x \in \mathbb{K}$ and $h_1(x) < x$ for $x > 2$. Taking the compositional inverse we obtain $x > h_0^{-1}(x)$ for all $x \in \mathbb{K}$ and $x < h_1^{-1}(x)$ for $x > 2$. Now let $y \in \mathbb{K}^{>\mathbf{k}}$, and let $r\omega^x$ be the leading term of y (where $r \in \mathbf{k}^{>0}$, $x \in \mathbb{K}^{>0}$). Then

$$\exp_0(y) \prec \exp_0(2r\omega^x) = \omega^{2r\omega^{h_0^{-1}(x)}} \prec \omega^{\frac{r}{2}\omega^x} \prec \omega^y,$$

since $2r\omega^x - y > \mathbf{k}$, $y - \frac{r}{2}\omega^x > \mathbf{k}$, and $\omega^{h_0^{-1}(x)} \prec \omega^x$.

Similarly, $h_1(x) < x$ for all $x \in \mathbb{K}^{>2}$. Let $y \in \mathbb{K}^{\geq \omega^3}$, and let $r\omega^x$ be the leading term of y. Then $r \in \mathbf{k}^{>0}$, $x \in \mathbb{K}^{>2}$ and

$$\exp_1(y) \succ \exp_1\left(\frac{r}{2}\omega^x\right) = \omega^{\frac{r}{2}\omega^{h_1^{-1}(x)}} \succ \omega^{2r\omega^x} \succ \omega^y.$$

Letting $z = -y \leq -\omega^3$, we obtain $\exp_1(z) = \frac{1}{\exp_1(y)} \prec \frac{1}{\omega^y} = \omega^z$, and the claim is proved.

We can now build the final chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$ by taking the functions $\exp_0$, $\exp_1$ restricted to suitable convex subsets of $\mathbb{K}$, and defining h on the complement as an increasing function in such a way that globally h is increasing and bijective. A concrete choice can be the following. Let $c = \exp_1(-\omega^3) > 0$. Define

$$h(x) = \begin{cases} \exp_0(x) & \text{for } x > \mathbf{k} \\ 2c + x & \text{for } 0 < x \text{ and } x \preceq 1 \\ 2c + \frac{c}{\omega^3}x & \text{for } -\omega^3 \leq x \leq 0 \\ \exp_1(x) & \text{for } x < -\omega^3. \end{cases}$$

By construction, h is a chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$: it is order preserving because $\exp_0$, $\exp_1$ are themselves chain isomorphisms, and it is surjective since $\exp_0(\mathbb{K}^{>\mathbf{k}}) = \mathbb{K}^{>\mathbf{k}}$, $\exp_1((-\infty, -\omega^3)) = (0, c)$. Moreover, $h(x) \prec \omega^x$ for all $x \in \mathbb{K}$, as desired:

- if $x > \mathbf{k}$, then $h(x) = \exp_0(x) \prec \omega^x$;
- if $0 < x \preceq 1$, then $h(x) = 2c + x \preceq 1 \prec \omega^x$;
- if $-\omega^3 \leq x \leq 0$, then $h(x) \asymp c = \exp_1(-\omega^3) \prec \omega^{-\omega^3} \preceq \omega^x$;
- if $x < -\omega^3$, then $h(x) = \exp_1(x) \prec \omega^x$. □

We next show that an h as constructed above is sufficient to guarantee the growth axiom at infinity.

Lemma 3.7. *Let* $\log = \log_{\omega,h} : \mathbb{K}^{>0} \to \mathbb{K}$ *be as in Definition 3.5. If* h *satisfies* $h(x) \prec \omega^x$ *for every* $x \in \mathbb{K}$*, then* $\log(y) < y^r$ *for all positive* $r \in \mathbf{k}$ *and all* $y > \mathbf{k}$ *(where* y^r *is defined as* $e^{r\log(y)}$*).*

Proof. Assume $h(x) \prec \omega^x$ for all $x \in \mathbb{K}$. This means that $h(x) < \omega^x r$ for all $r \in \mathbf{k}^{>0}$. Let $y = \omega^{\omega^x}$. Then $\log(y) = \log(\omega^{\omega^x}) = \omega^{h(x)} < \omega^{\omega^x r} = y^r$. We have thus proved that $\log(y) < y^r$ for y of the form ω^{ω^x} and $r \in \mathbf{k}^{>0}$.

We now prove the inequality for y of the form ω^x, where $x \in \mathbb{K}^{>0}$. To this aim we write the exponent x in the form $\sum_{i<\alpha} \omega^{x_i} r_i$ and observe that $r_0 > 0$ and $\log(\omega^x) = \log\left(\omega^{\sum_{i<\alpha}\omega^{x_i} r_i}\right) = \sum_{i<\alpha} \log(\omega^{\omega^{x_i}}) r_i$. By the special case we have $\log(\omega^{\omega^{x_i}}) < \omega^{\omega^{x_i} a} \le \omega^{\omega^{x_0} a}$ for every $i < \alpha$ and $a \in \mathbf{k}^{>0}$. Letting $a = rr_0/2$ it follows that

$$\log(\omega^x) < \omega^{\omega^{x_0} a} = \left(\omega^{\omega^{x_0} r_0}\right)^{\frac{a}{r_0}} < \left(\omega^{2x}\right)^{\frac{a}{r_0}} = \omega^{xr}.$$

For a general $y > \mathbf{k}$, write y in the form $\omega^x s(1+\varepsilon)$ with $s \in \mathbf{k}^{>0}$, $x > 0$ and $\varepsilon \prec 1$, and observe that $\log(y) < \log(2s) + \log(\omega^x) < (\omega^x)^{\frac{r}{2}} < y^r$ for any $r \in \mathbf{k}^{>0}$. □

In the case when the residue field $\mathbf{k}$ is archimedian, the statement in the conclusion of Lemma 3.7 is equivalent to the growth axiom at infinity (Definition 2.11). We are now ready for the main result of this section.

Theorem 3.8. *Every omega-field of the form* $\mathbb{K} = \mathbb{R}((G))_\kappa$ *admits an analytic logarithm making it into a model of* $T_{an,\exp}$*. More generally, if* $\mathbf{k}$ *is a model of* $T_{an,\exp}$*, then every omega-field of the form* $\mathbb{K} = \mathbf{k}((G))_\kappa$ *admits an analytic logarithm making it into a model of* $T_{an,\exp}$*.*

Proof. By Proposition 2.12 and Lemma 3.7. □

3.4. **Growth axiom and o-minimality.** We now discuss the connections between the growth axiom and o-minimality (see [4] for the development of the theory of o-minimal structures).

Lemma 3.9. *Let* $\mathbb{K}$ *be an o-minimal exponential field. Note that* exp *must be differentiable and by a linear change of variable, we can assume that* $\exp'(0) = 1$*. Then* $\exp(x) > x^n$ *for all positive* $n \in \mathbb{N}$ *and all* $x > \mathbb{N}$*.*

Proof. Given a definable differentiable unary function $f : \mathbb{K} \to \mathbb{K}$ in an o-minimal expansion of a field, its derivative f' is definable, and if f' is always positive, then f is increasing. It follows that if f, g are definable differentiable functions satisfying $f(a) \le g(a)$ and $f'(x) < g'(x)$ for all $x \ge a$, then $f(x) < g(x)$ for every $x > a$. Starting with $0 < \exp(x)$ and integrating we then inductively obtain that for each positive $k, n \in \mathbb{N}$ there is a positive $c \in \mathbb{N}$ such that $kx^n \le e^x$ for all $x > c$. □

By the above observation and Ressayre's axiomatization [15], an exponential field is a model of $T_{\exp}$ if and only if it satisfies the complete theory of restricted exponentiation and it is o-minimal.

Theorem 3.10. *Assume* $\mathbb{K} = \mathbb{R}((G))_\kappa$ *has an omega-map* $\omega : \mathbb{K} \cong G$*. Fix a chain isomorphism* $h : \mathbb{K} \cong \mathbb{K}^{>0}$ *and put on* $\mathbb{K}$ *the logarithm induced by* ω *and* h *as in Definition 3.5. Then* $\mathbb{K}$ *is either a model of* $T_{\exp}$ *or it is not even o-minimal.*

Proof. We have already seen that if $h(x) \prec \omega^x$ for all $x \in \mathbb{K}$, then $\mathbb{K}$ is a model of $T_{\exp}$ (Theorem 3.8). Now suppose that $h(x) \not\prec \omega^x$ for some x. Then there is some $n \in \mathbb{N}^{>0}$ such that $h(x) \geq \frac{1}{n}\omega^x$. Letting $y = \omega^{\frac{1}{n}\omega^x}$, we have $\log(y) = \frac{1}{n}\log(\omega^{\omega^x}) = \frac{1}{n}\omega^{h(x)} \geq \frac{1}{n}\omega^{\frac{1}{n}\omega^x} = \frac{1}{n}y$, hence $y^n \geq e^y$, contradicting o-minimality by Lemma 3.9 (since exp extends the real exponential function, we have $\exp'(0) = 1$, so the hypothesis of the lemma are satisfied). □

4. Other exponential fields of series

4.1. **Criterion for the existence of an omega-map.** In this section we try to classify all possible analytic logarithms on $\mathbf{k}((G))_\kappa$. We show that in the case of omega-fields every analytic logarithm arises from an omega-map and some h.

Theorem 4.1. *Assume that $\mathbb{K} = \mathbf{k}((G))_\kappa$ has an analytic logarithm* log*. Then:*

(1) *$\mathbb{K}$ has an omega-map $\omega : \mathbb{K} \cong G$ if and only if G is isomorphic to $G^{>1}$ as a chain;*
(2) *moreover, if $G \cong G^{>1}$, there is an omega-map and a chain isomorphism $h : \mathbb{K} \cong \mathbb{K}^{>0}$ such that the logarithm induced by ω and h coincides with the original logarithm.*

Proof. First note that $\mathbb{K}$, being an ordered field, is always isomorphic to $\mathbb{K}^{>0}$ as a chain. If there is an omega-map $\omega : \mathbb{K} \cong G$, we obtain an induced isomorphism from $G = \omega^{\mathbb{K}}$ to $G^{>1} = \omega^{\mathbb{K}^{>0}}$.

For the opposite direction, assume that G is isomorphic to $G^{>1}$ as a chain and let $\psi : G \cong G^{>1}$ be a chain isomorphism. Define $\omega : \mathbb{K} \to G$ by

$$\omega^{\sum_{i<\alpha} g_i r_i} = e^{\sum_{i<\alpha} \psi(g_i) r_i}.$$

In particular we have $\omega^g = e^{\psi(g)}$. Clearly ω is a morphism from $(\mathbb{K}, +, 0, <)$ to $(G, \cdot, 1, <)$ and to prove that it is an omega-map it only remains to verify that it is surjective. To this aim recall that $\log(G) = \mathbb{K}^{\uparrow}$ (by definition of analytic logarithm), so for the corresponding exp we have $G = \exp(\mathbb{K}^{\uparrow})$. Since $e^{\sum_{i<\alpha} \psi(g_i) r_i}$ is an arbitrary element of $\exp(\mathbb{K}^{\uparrow})$, the surjectivity of ω follows. Now since $\psi : G \cong G^{>1}$ and $G = \omega^{\mathbb{K}}$, there is a chain isomorphism $h : \mathbb{K} \to \mathbb{K}^{>0}$ such that

$$\psi(\omega^x) = \omega^{h(x)}.$$

Since $e^{\psi(\omega^x)} = \omega^{\omega^x}$, we obtain $\omega^{\omega^x} = e^{\omega^{h(x)}}$ and thefore $\log(\omega^{\omega^x}) = \omega^{h(x)}$. It then follows that log coincides with the analytic logarithm induced by ω and h. □

Corollary 4.2. *Every analytic logarithm on an omega-field of the form $\mathbb{K} = \mathbf{k}((G))_\kappa$ arises from some omega-map and some chain isomorphism $h : \mathbb{K} \cong \mathbb{K}^{>0}$.*

4.2. **The iota-map.** Our next goal is to show that $\mathbf{k}((G))_\kappa$ may have an analytic logarithm without being an omega-field. This will be proved in the next subsection. Here we recall the following two results from [12] with a sketch of the proofs for the reader's convenience (considering that the notations are different). We use the same notation $H(\Gamma) = \left(\prod t^{\Gamma C}\right)_\kappa$ employed in Lemma 3.3, with $C = (\mathbf{k}, +, <)$.

Fact 4.3 ([12]). *Let $\mathbf{k}$ be an exponential field. Let Γ be a chain and suppose there is an isomorphism of chains $\iota : \Gamma \cong H(\Gamma)^{>1}$. Let $G = H(\Gamma)$ and let $\mathbb{K} = \mathbf{k}((G))_\kappa$. Then:*

(1) *there is an analytic logarithm $\log : \mathbb{K}^{>0} \to \mathbb{K}$ such that $\log(t^\gamma) = \iota(\gamma) \in G$.*

(2) *if* $\mathbf{k}$ *is a model of* $T_{an,\exp}$ *and* $\iota(\gamma) < t^{\gamma r}$ *for each* $r \in \mathbf{k}^{>0}$, *then* log *satisfies the growth axiom at infinity, thus making* $\mathbb{K}$ *into a model of* $T_{an,\exp}$.[2]

Proof. Define $\log = \log_\iota$ on G by

$$\log\Big(\prod_{i<\alpha} t^{\gamma_i r_i}\Big) = \sum_{i<\alpha} \iota(\gamma_i) r_i \in \mathbf{k}((G^{>1}))_\kappa$$

Given $x \in \mathbb{K}^{>0}$, write $x = gr(1+\varepsilon)$ for some $r \in \mathbf{k}^{>0}$, $g \in G$ and $\varepsilon \in o(1)$; now define $\log(x) = \log(g) + \log(r) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\varepsilon^n}{n}$, where $\log(r)$ refers to the given logarithm on $\mathbf{k}$, and observe that since $\varepsilon \prec 1$ and $\kappa > \omega$ the infinite sum belongs to $\mathbb{K} = \mathbf{k}((G))_\kappa$. Clearly log is an analytic logarithm and (1) is proved. The verification of point (2) is as in Theorem 3.4. $\square$

Fact 4.4 ([12])**.** *Fix a chain* Γ_0 *and a chain embedding* $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1}$ *(for instance* $\iota_0(\gamma) = t^\gamma$*). Then:*

(1) *there is a chain* $\Gamma \supseteq \Gamma_0$ *and a chain isomorphism* $\iota : \Gamma \cong H(\Gamma)^{>1}$ *extending* ι_0*;*

(2) *if* $\iota_0(\gamma) < t^{\gamma r}$ *for every* $\gamma \in \Gamma_0$ *and* $r \in C^{>0}$, *then* $\iota(\gamma) < t^{\gamma r}$ *for every* $\gamma \in \Gamma$ *and* $r \in C^{>0}$.

Proof. The proof of (1) is similar to the proof of Lemma 3.3, the only difference is that we use $H(\Gamma)^{>1}$ instead of $H(\Gamma)$. Starting with the initial chain embedding $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1}$ we inductively produce chain embeddings $\iota_\beta : \Gamma_\beta \to H(\Gamma_\beta)^{>1}$ and $j_{\alpha,\beta} : \Gamma_\alpha \to \Gamma_\beta$ for $\alpha < \beta$. The step from β to $\beta+1$ is based on the following diagram

$$\begin{tikzcd} \Gamma_\beta \arrow[r, "\iota_\beta"'] \arrow[d, "j_\beta"'] & H(\Gamma_\beta)^{>1} \arrow[d, "H(j_\beta)"] \arrow[dl, "f_\beta"'] \\ \Gamma_{\beta+1} \arrow[r, "\iota_{\beta+1}"] & H(\Gamma_{\beta+1})^{>1} \end{tikzcd} \tag{6}$$

where $\Gamma_{\beta+1}$ is a chain isomorphic to $H(\Gamma_\beta)^{>1}$, f_β is a chain isomorphism, and the embeddings j_β and $\iota_{\beta+1}$ are defined so that the diagram commutes. Limit stages are handled as in Lemma 3.3. Finally we set $\Gamma = \Gamma_\kappa = \varinjlim_{\beta<\kappa} \Gamma_\beta$ and $\iota = \iota_\kappa$ and observe that $\iota : \Gamma \to H(\Gamma)^{>1}$ is a chain isomorphism.

To prove (2), we show by induction on $\beta < \kappa$ that $\iota_\beta(\gamma) < t^{\gamma r}$ for every $\gamma \in \Gamma_\beta$ and $r \in C^{>0}$, provided this holds for $\beta = 0$. Since limit stages are easy, it suffices to prove the induction step from β to $\beta+1$. So let $\eta \in \Gamma_{\beta+1}$. Then $\eta = f_\beta(x)$ for some $x = \prod_i t^{\gamma_i r_i} \in \left(\prod t^{\Gamma_\beta C}\right)^{>1}_\kappa$. The embedding ι_β sends η to $\prod_i t^{j_\beta(\gamma_i) r_i}$ where $j_\beta = f_\beta \circ \iota_\beta$ is the embedding of Γ_β into $\Gamma_{\beta+1}$. We must prove that $\prod_i t^{j_\beta(\gamma_i) r_i} < t^{\eta r}$ for every $r \in C^{>0}$. This is equivalent to saying $j_\beta(\gamma_0) < \eta$, which in turn is equivalent to $\iota_\beta(\gamma_0) < \prod_i t^{\gamma_i r_i}$. The latter inequality follows from the inductive hypothesis and the proof is complete. $\square$

4.3. **A model without an omega-map.** We can now show that there are fields of the form $\mathbb{R}((G))_\kappa$ which admit an analytic logarithm but not an omega-map.

Theorem 4.5. *Given a regular uncountable cardinal* κ, *there is* G *such that the field* $\mathbb{K} = \mathbb{R}((G))_\kappa$ *has an analytic logarithm making it into a model of* $T_{\exp}$ *but* G *is not isomorphic to* $G^{>1}$ *as a chain (so* $\mathbb{K}$ *is not an omega-field).*

[2] In the cited paper the authors consider $\mathbf{k} = \mathbb{R}$, but the general case is the same.

Proof. Start with the chain $\Gamma_0 = \omega_1 \times \mathbb{Z}$ ordered lexicographically and the initial embedding $\iota_0 : \Gamma_0 \to \left(\prod_\kappa t^{\Gamma_0 \mathbf{k}}\right)^{>1} = H(\Gamma_0)^{>1}$ given by $\iota_0((\alpha, n)) = t^{(\alpha, n-1)}$. Define $\Gamma = \varinjlim_{\beta<\kappa} \Gamma_\beta$ and $\iota : \Gamma \cong H(\Gamma)^{>1}$ as in Fact 4.4 and note that $\iota(\gamma) < t^{\gamma r}$ for every $\gamma \in \Gamma$ and $r \in \mathbf{k}^{>0}$ (since this holds for ι_0 and is preserved at later stages). Now take $G = H(\Gamma)$ and put on the field $\mathbb{K} = \mathbf{k}((G))_\kappa$ the log induced by ι as in Fact 4.3. By the above inequalities the log satisfies the growth axiom at infinity, so $\mathbb{K}$ is a model of $T_{\exp}$. It remains to show that $G \ncong G^{>1}$ as a chain. Note that the image of $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1} = \Gamma_1$ is cofinal and coinitial in $H(\Gamma_0)^{>1}$. It follows that for each $\beta \leq \kappa$, the image of $\iota_\beta : \Gamma_\beta \to H(\Gamma_\beta)^{>1}$ is cofinal and coinitial in $H(\Gamma_\beta)^{>1} = \Gamma_{\beta+1}$. Likewise, by an easy induction, for each $\beta \geq 0$ the image of Γ_0 in Γ_β is initial and cofinal. In particular the image of Γ_0 in the final chain $\Gamma_\kappa = \Gamma \cong H(\Gamma)^{>1}$ is coinitial and cofinal. Since Γ_0 has cofinality ω_1 and coinitiality ω, it follows that Γ and $H(\Gamma)^{>1}$ have cofinality ω_1 and coinitiality ω. Now observe that $1/x$ is an order-reversing bijection from $H(\Gamma)^{<1}$ to $H(\Gamma)^{>1}$, and therefore $H(\Gamma) = H(\Gamma)^{<1} \cup 1 \cup H(\Gamma)^{>1}$ has cofinality and coinitiality both equal to ω_1. We conclude that $G = H(\Gamma)$ cannot be chain isomorphic to $G^{>1}$, because they have different coinitiality. □

5. Omega-groups

A group isomorphic to the value group of an omega-field will be called **omega-group**. It would be interesting to give a characterization of the omega-groups. As a partial result, we characterise those groups G such that $\mathbf{k}((G))_\kappa$ is an omega-field. We also clarify the relation between having an omega-map and having an analytic logarithm.

Proposition 5.1. *Let $\mathbb{K}$ be a field of the form $\mathbf{k}((G))_\kappa$. Then:*

(1) *if $\mathbb{K}$ is an omega-field, then G is isomorphic to $\left(\prod t^{\Gamma \mathbf{k}}\right)_\kappa$, where the chain Γ is order-isomorphic to (the underlying chain of) G itself;*
(2) *if $\mathbb{K}$ has an analytic logarithm, then G is isomorphic to $\left(\prod t^{\Gamma \mathbf{k}}\right)_\kappa$, where Γ is order-isomorphic to $G^{>1}$.*

Proof. (1) The elements of $\mathbb{K}$ can be written in the form $\sum_{i<\alpha} g_i r_i$. So the elements of G are of the form $\omega^{\sum_{i<\alpha} g_i r_i}$. This corresponds to the element $\prod_{i<\alpha} t^{g_i r_i} \in \left(\prod t^{G \mathbf{k}}\right)_\kappa$ via an isomorphism.

(2) Since $\log(G) = \mathbb{K}^\uparrow$, we have $G = \exp(\mathbb{K}^\uparrow)$, and therefore an element g of G can be written in the form $\exp(\sum_{i<\alpha} g_i r_i)$ with $g_i \in G^{>1}$ and $r_i \in \mathbf{k}$. This corresponds to $\prod_{i<\alpha} t^{g_i r_i} \in \left(\prod t^{G^{>1} \mathbf{k}}\right)_\kappa$ via an isomorphism. □

In the following corollary we abstract some of the properties of the groups considered above. We refer to [11] for the definition of the value-set.

Corollary 5.2. *Let $\mathbb{K}$ be a field of the form $\mathbf{k}((G))_\kappa$.*

(1) *If $\mathbb{K}$ has an analytic logarithm, then G is a $\mathbf{k}$-module, the value set Γ of G is order isomorphic to $G^{>1}$, and all the $\mathbf{k}$-archimedean components of G are isomorphic to the additive group of $\mathbf{k}$.*
(2) *If $\mathbb{K}$ is an omega-field, the same properties hold (as in particular $\mathbb{K}$ has an analytic logarithm) and in addition G is isomorphic to $G^{>1}$ as a chain.*

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COUNTING ALGEBRAIC POINTS IN EXPANSIONS OF O-MINIMAL STRUCTURES BY A DENSE SET

PANTELIS E. ELEFTHERIOU

Abstract. The Pila-Wilkie theorem states that if a set $X \subseteq \mathbb{R}^n$ is definable in an o-minimal structure $\mathcal{R}$ and contains 'many' rational points, then it contains an infinite semialgebraic set. In this paper, we extend this theorem to an expansion $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ of $\mathcal{R}$ by a dense set P, which is either an elementary substructure of $\mathcal{R}$, or it is independent, as follows. If X is definable in $\widetilde{\mathcal{R}}$ and contains many rational points, then it is dense in an infinite semialgebraic set. Moreover, it contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$, where $\overline{\mathbb{R}}$ is the real field.

1. Introduction

Point counting theorems have recently occupied an important part of model theory, mainly due to their pivotal role in applications of o-minimality to number theory and Diophantine geometry. Arguably, the biggest breakthrough was the Pila-Wilkie theorem [21], which roughly states that if a definable set in an o-minimal structure contains "many" rational points, then it contains an infinite semialgebraic set. Pila employed this result together with the so-called Pila-Zannier strategy to give an unconditional proof of certain cases of the André-Oort Conjecture [20]. An excellent survey on the subject is [22]. Although several strengthenings of these theorems have since been established within the o-minimal setting, the topic remains largely unexplored in more general tame settings. In this paper, we establish the first point counting theorems in tame expansions of o-minimal structures by a dense set.

Recall that, for a set $X \subseteq \mathbb{R}^n$, the *algebraic part* X^{alg} of X is defined as the union of all infinite connected semialgebraic subsets of X. Pila in [20], generalizing [21], proved that if a set X is definable in an o-minimal structure, then $X \setminus X^{alg}$ contains "few" algebraic points of fixed degree (see definitions below and Fact 2.3). This statement immediately fails if one leaves the o-minimal setting. For example, the set $\mathcal{A}$ of algebraic points itself contains many algebraic points, but $\mathcal{A}^{alg} = \emptyset$. However, adding $\mathcal{A}$ as a unary predicate to the language of the real field results in a well-behaved model theoretic structure, and it is desirable to retain point counting theorems in that setting. We achieve this goal by means of the following definition.

Definition 1.1. Let $X \subseteq \mathbb{R}^n$. The *algebraic trace part* of X, denoted by X^{alg}_t, is the union of all traces of infinite connected semialgebraic sets in which X is dense.

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That is,

$$X_t^{alg} = \bigcup\{X\cap T : T\subseteq \mathbb{R}^n \text{ infinite connected semialgebraic, and } T\subseteq cl(X\cap T)\}.$$

The density requirement $T \subseteq cl(X \cap T)$ is essential: without it, we would always have $X_t^{alg} = X$, as witnessed by $T = \mathbb{R}^n$.

We first show in Section 2 that the above notion is a natural generalization of the usual notion of the algebraic part of a set, in the following sense.

Proposition 1.2. *Suppose $X \subseteq \mathbb{R}^n$ is definable in an o-minimal expansion of the real field. Then $X^{alg} = X_t^{alg}$.*

Then, in Sections 3 and 4, we establish point counting theorems in two main categories of tame structures that go beyond the o-minimal setting: dense pairs and expansions of o-minimal structures by a dense independent set. Indeed, we prove that if X is a definable set in these settings, then $X \setminus X_t^{alg}$ contains few algebraic points of fixed degree (Theorem 1.3 below). We postpone a discussion about the general tame setting until later in this introduction, as we now proceed to fix our notation and state the precise theorem. Some familiarity with the basic notions of model theory, such as definability and elementary substructures, is assumed. The reader can consult [11, 17, 19]. An example of an elementary substructure of the real field is the field $\mathcal{A}$ of algebraic numbers.

For the rest of this paper, and unless stated otherwise, we fix an o-minimal expansion $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, \dots\rangle$ of the real field $\overline{\mathbb{R}} = \langle \mathbb{R}, <, +, \cdot\rangle$, and let $\mathcal{L}$ be the language of $\mathcal{R}$. We fix an expansion $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P\rangle$ of $\mathcal{R}$ by a set $P \subseteq \mathbb{R}$, and let $\mathcal{L}(P) = \mathcal{L} \cup \{P\}$ be the language of $\widetilde{\mathcal{R}}$. By 'A-definable' we mean 'definable in $\widetilde{\mathcal{R}}$ with parameters from A', and by '$\mathcal{L}_A$-definable' we mean 'definable in $\mathcal{R}$ with parameters from A'. We omit the index A if we do not want to specify the parameters. For a subset $A \subseteq \mathbb{R}$, we write $\mathrm{dcl}(A)$ for the definable closure of A in $\mathcal{R}$, and $\mathrm{dcl}_{\mathcal{L}(P)}(A)$ for the definable closure in $\widetilde{\mathcal{R}}$. We call a set $X \subseteq \mathbb{R}$ dcl-*independent over* A, if for every $x \in X$, $x \notin \mathrm{dcl}((X \setminus \{x\}) \cup A)$, and simply dcl-*independent* if it is dcl-independent over $\emptyset$. An example of a dcl-independent set in the real field is a transcendence basis over $\mathbb{Q}$.

Following [19], we define the *(multiplicative) height* $H(\alpha)$ of an algebraic point α as $H(\alpha) = \exp h(\alpha)$, where $h(\alpha)$ is the *absolute logarithmic height* from [6, page 16]. For a set $X \subseteq \mathbb{R}^n$, $k \in \mathbb{Z}^{>0}$ and $T \in \mathbb{R}^{>1}$, we define

$$X(k,T) = \{(\alpha_1,\dots,\alpha_n) \in X : \max_i [\mathbb{Q}(\alpha_i) : \mathbb{Q}] \le k, \max_i H(\alpha_i) \le T\}$$

and

$$N_k(X,T) = \#X(k,T).$$

We say that X *has few algebraic points* if for every $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$,

$$N_k(X,T) = O_{X,k,\epsilon}(T^\epsilon).$$

We say that it *has many algebraic points*, otherwise.

Our main result is the following.

Theorem 1.3. *Suppose $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, \dots\rangle$ is an o-minimal expansion of the real field, and $P \subseteq R$ a dense set such that one of the following two conditions holds:*

(A) *$P \preccurlyeq \mathcal{R}$ is an elementary substructure.*

(B) *P is a* dcl-*independent set.*

Let $X \subseteq \mathbb{R}^n$ be definable in $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P\rangle$. Then $X \setminus X_t^{alg}$ has few algebraic points.

Note that if $\mathcal{R} = \overline{\mathbb{R}}$, Theorem 1.3 is trivial. Indeed, in both cases (A) and (B), if X is a definable set, then $cl(X)$ is $\mathcal{L}$-definable ([14, Section 2]). So, in this case, $cl(X)$ is semialgebraic and hence $X_t^{alg} = X$. In fact, whenever $\widetilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, P \rangle$ satisfies Assumption III from [14], the conclusion of Theorem 1.3 holds. An example of such $\widetilde{\mathcal{R}}$ is an expansion of the real field by a multiplicative group with the Mann property.

The contrapositive of Theorem 1.3 implies that if a definable set contains many algebraic points, then it is dense in an infinite semialgebraic set. We strengthen this result as follows.

Theorem 1.4. *Let X be as in Theorem 1.3. If X has many algebraic points, then it contains an infinite set Y which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$.*

Note that such X is dense in $cl(Y)$, which is semialgebraic by [14, Section 2].

A few words about the general tame setting are in order. As o-minimality can only be used to model phenomena that are locally finite, many authors have early on sought expansions of o-minimal structures which escape from the o-minimal context, yet preserve the tame geometric behavior on the class of all definable sets. These expansions have recently seen significant growth ([1, 2, 5, 8, 10, 12, 16, 18]) and are by now divided into two important categories of structures: those where every open definable set is already definable in the o-minimal reduct and those where an infinite discrete set is definable. Cases (A) and (B) from Theorem 1.3 belong to the first category. Further examples of this sort can be found in [8] and [14]. Certain point counting theorems in the second category have recently appeared in [7]. In both categories, sharp *cone decomposition theorems* are by now at our disposal ([14] and [23]), in analogy with the cell decomposition theorem known for o-minimal structures.

Expansions $\mathcal{R}$ of type (A) are called *dense pairs* and were first studied by van den Dries in [10], whereas expansions of type (B) were recently introduced by Dolich-Miller-Steinhorn in [9]. These two examples are representative of the first category and are often thought of as "orthogonal" to each other, mainly because in the former case $\mathrm{dcl}(\emptyset) \subseteq P$, whereas in the latter, $\mathrm{dcl}(\emptyset) \cap P = \emptyset$. This orthogonality is vividly reflected in our proof of Theorem 1.3. Indeed, since the set $\mathcal{A}$ of algebraic points is contained in $\mathrm{dcl}(\emptyset)$, we have $\mathcal{A} \subseteq P$ in the case of dense pairs and $\mathcal{A} \cap P = \emptyset$ in the case of dense independent sets. Based on this observation, the proof for (A) becomes almost immediate, assuming facts from [10], whereas the proof for (B) makes an essential use of the aforementioned cone decomposition theorem from [14].

The current work provides an extension of the influential Pila-Wilkie theorem to the above two settings. The next step is, of course, to explore any potential applications to number theory and Diophantine geometry. Even though it is currently unclear whether the exact setting of Theorem 1.3 will yield any, the machinery used in our proofs is also available in other settings, or it may be possible to develop therein. Two far reaching generalizations of our two settings are lovely pairs [3] and H-structures [4], respectively. Those settings can also accommodate structures coming from geometric stability theory, such as pairs of algebraically closed fields, or SU-rank 1 structures, and point counting theorems in them are wildly unknown.

Notation. The topological closure of a set $X \subseteq \mathbb{R}^n$ is denoted by $cl(X)$. If $X, Z \subseteq \mathbb{R}^n$, we call X dense in Z, if $Z \subseteq cl(X \cap Z)$. Given any subset $X \subseteq \mathbb{R}^m \times \mathbb{R}^n$ and

$a \in \mathbb{R}^m$, we write X_a for

$$\{b \in \mathbb{R}^n \ : \ (a,b) \in X\}.$$

If $m \leq n$, then $\pi_m : \mathbb{R}^n \to \mathbb{R}^m$ denotes the projection onto the first m coordinates. We write π for π_{n-1}, unless stated otherwise. A family $\mathcal{J} = \{J_g\}_{g \in S}$ of sets is called definable if $\bigcup_{g\in S}\{g\} \times J_g$ is definable. We often identify $\mathcal{J}$ with $\bigcup_{g\in S}\{g\} \times J_g$. If $X, Y \subseteq \mathbb{R}$, we sometimes write XY for $X \cup Y$. By $\mathcal{A}$ we denote the set of real algebraic points. If $M \subseteq \mathbb{R}$, by $M \preccurlyeq \mathcal{R}$ we mean that M is an elementary substructure of $\mathcal{R}$ in the language of $\mathcal{R}$.

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2. The algebraic trace part of a set

In this section, we introduce the notion of the *algebraic trace part* of a set, and prove that it generalizes the notion of the algebraic part of a set definable in an o-minimal structure. We also state a version of Pila's theorem [19], Fact 2.3 below, suitable for our purposes.

The proof of Theorem 1.3, in both cases (A) and (B), is by reducing it to Pila's theorem, Fact 2.3 below. The formulation of that fact involves a refined version of the usual algebraic part of a set, which prompts the following definitions.

Definition 2.1. Let $A \subseteq \mathbb{R}$ be a set. An *A-set* is an infinite connected semialgebraic set definable over A. If it is, in addition, a cell, we call it an *A-cell*.

We are mainly interested in $\mathbb{Q}$-sets. One important observation is that the set $\mathcal{A}$ of algebraic points is dense in every $\mathbb{Q}$-set. This fact will be crucial in the proofs of Lemma 3.2 and Theorem 4.15 below.

Definition 2.2. Let $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}$. The *algebraic part of X over A*, denoted by X^{alg_A}, is the union of all A-subsets of X. That is,

$$X^{alg_A} = \bigcup\{T \subseteq X : T \text{ is an } A\text{-set}\}.$$

It is an effect of the proof in [19] that the following statement holds.

Fact 2.3. *Let $X \subseteq \mathbb{R}^n$ be $\mathcal{L}$-definable. Then $X \setminus X^{alg_\mathbb{Q}}$ has few algebraic points.*

Let us now also refine Definition 1.1 from the introduction, as follows.

Definition 2.4. Let $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}$. The *algebraic trace part of X over A*, denoted by $X_t^{alg_A}$ is the union of all traces of A-sets in which X is dense. That is,

$$X_t^{alg_A} = \bigcup\{X \cap T : T \text{ an } A\text{-set, } X \text{ dense in } T\}$$

Remark 2.5.

(1) An $\mathbb{R}$-set is exactly an infinite connected semialgebraic set. Also, $X^{alg_\mathbb{R}} = X^{alg}$ and $X_t^{alg_\mathbb{R}} = X_t^{alg}$.

(2) In Theorems 3.3 and 4.15 below, we prove Theorem 1.3 after replacing X_t^{alg} by $X_t^{alg_\mathbb{Q}}$. Since the latter set is contained in the former, these are stronger statements.

Remark 2.6. An alternative expression for $X_t^{alg_A}$ is the following:

$$X_t^{alg_A} = \bigcup\{Y \subseteq X : cl(Y) \text{ is an } A\text{-set}\}.$$

$\subseteq$. Let T be an A-set such that X is dense in T. Set $Y = X \cap T \subseteq X$. Then $T \subseteq cl(Y) \subseteq cl(T)$, and hence $cl(Y) = cl(T)$ is an A-set, as required.

$\supseteq$. Let $Y \subseteq X$ such that $cl(Y)$ is an A-set. Set $T = cl(Y)$. Then $Y \subseteq X \cap T$ and $T \subseteq cl(X \cap T)$, as required.

The goal of this section is to prove the following proposition. This result is not essential for the rest of the paper, but we include it here as it provides canonicity of our definitions. Observe also that it is independent of the expansion $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ we consider.

Proposition 2.7. *Let $X \subseteq \mathbb{R}^n$ be an $\mathcal{L}$-definable set. Then*

$$X^{alg} = X_t^{alg}.$$

The main idea for proving ($\supseteq$) is as follows. Let Z be an $\mathbb{R}$-set with $Z \subseteq cl(Z\cap X)$. We need to prove that every point $x \in Z \cap X$ is contained in an $\mathbb{R}$-set W contained in X. If one applies cell decomposition directly to $Z \cap X$, then the resulting cells need not be semialgebraic, as X is not. So we apply cell decomposition only to Z, deriving an $\mathbb{R}$-cell $Z_0 \subseteq Z$ with $x \in cl(Z_0)$ and of maximal dimension. We then show that close enough to x, the set $T = Z_0 \setminus X$ has dimension strictly smaller than $\dim Z_0$. We use Lemma 2.10 to express this fact properly. Finally, by Lemma 2.11, we find an $\mathbb{R}$-set $W_0 \subseteq Z_0 \setminus T$ with $x \in \mathrm{cl}(W_0)$. We set $W = W_0 \cup \{x\}$.

The first lemma asserts that, under certain assumptions, the property of being dense in a set passes to suitable subsets.

Lemma 2.8. *Let $X, Z \subseteq \mathbb{R}^n$ be $\mathcal{L}$-definable sets, with $Z \subseteq cl(Z \cap X)$. Suppose that $Z_0 \subseteq Z$ is a cell with $\dim Z_0 = \dim Z$. Then $Z_0 \subseteq cl(Z_0 \cap X)$.*

Proof. Let $x \in Z_0$, and suppose towards a contradiction that $x \notin cl(Z_0 \cap X)$. Then there is an open box $B \subseteq \mathbb{R}^n$ containing x such that $B \cap Z_0 \cap X = \emptyset$. It follows that for every $x' \in B \cap Z_0$, $x' \notin cl(Z_0 \cap X)$. Since $Z \subseteq cl(Z \cap X)$,

$$B \cap Z_0 \subseteq cl((Z \setminus Z_0) \cap X) \subseteq cl(Z \setminus Z_0)$$

and, hence,

$$B \cap Z_0 \subseteq cl(Z \setminus Z_0) \setminus (Z \setminus Z_0),$$

and thus $\dim(B \cap Z_0) < \dim(Z \setminus Z_0)$. Moreover, since Z_0 is a cell and $B \cap Z_0 \neq \emptyset$, $\dim(Z_0) = \dim(B \cap Z_0)$. All together,

$$\dim(Z_0) < \dim(Z \setminus Z_0) \leq \dim Z,$$

a contradiction. □

We will need a local version of Lemma 2.8. First, a definition.

Definition 2.9. Let $Z \subseteq \mathbb{R}^n$ be an $\mathcal{L}$-definable set and $x \in Z$. The *local dimension of Z at x* is defined to be

$$\dim_x(Z) = \min\{\dim(B \cap Z) : B \subseteq \mathbb{R}^n \text{ an open box containing } x\}.$$

Lemma 2.10. *Let $X, Z \subseteq \mathbb{R}^n$ be infinite $\mathcal{L}$-definable sets with $Z \subseteq cl(Z \cap X)$, and $x \in Z$. Suppose $Z_0 \subseteq Z$ is an $\mathbb{R}$-cell with $\dim_x(Z) = \dim Z_0$ and $x \in cl(Z_0)$. Then there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that $B \cap Z_0 \subseteq cl(Z_0 \cap X)$. Moreover, $B \cap Z_0$ is an $\mathbb{R}$-cell.*

Proof. Let $Z \setminus Z_0 = Z_1 \cup \cdots \cup Z_m$ be a decomposition into cells. It is not hard to see from the definition of $\dim_x(Z)$, that there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that for every $1 \leq i \leq m$, if $B \cap Z_i \neq \emptyset$, then $\dim_x(Z) \geq \dim B \cap Z_i$. We may shrink B if needed so that $B \cap Z_0$ becomes an $\mathbb{R}$-cell. Let I be the set of indices $1 \leq i \leq m$ such that $B \cap Z_i \neq \emptyset$. Set

$$Z' := B \cap Z.$$

Since $Z \subseteq cl(Z \cap X)$, we easily obtain that $Z' \subseteq cl(Z' \cap X)$. Moreover, since $x \in cl(Z)$, we have

$$Z' = (B \cap Z_0) \cup \bigcup_{i \in I} (B \cap Z_i),$$

and hence $\dim Z' = \dim(B \cap Z_0)$. Therefore, by Lemma 2.8 (for Z' and $B \cap Z_0 \subseteq Z'$),

$$B \cap Z_0 \subseteq cl(B \cap Z_0 \cap X) \subseteq cl(Z_0 \cap X),$$

as needed. $\square$

We also need the following lemma.

Lemma 2.11. *Let $Z \subseteq \mathbb{R}^n$ be an $\mathbb{R}$-cell, $T \subseteq Z$ a definable set, and $x \in cl(Z) \setminus T$. Suppose that $\dim T < \dim Z$. Then there is an $\mathbb{R}$-set $W \subseteq Z \setminus T$ with $x \in cl(W)$.*

Proof. We work by induction on $n > 0$. For $n = 0$, it is trivial. Let $n > 0$. We split into two cases:

Case I: $\dim Z = n$. Since $\dim T < \dim Z$, it follows easily, by cell decomposition, that there is a line segment $W \subseteq Z$ with initial point x, staying entirely outside T.

Case II: $\dim Z = k < n$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a suitable coordinate projection such that $\pi_{\restriction Z}$ is injective. Then $\pi(Z)$ is an $\mathbb{R}$-cell, $\pi(T) \subseteq \pi(Z)$, $\dim \pi(T) < \dim \pi(Z)$ and $\pi(x) \in cl(\pi(Z))$. By inductive hypothesis, there is an $\mathbb{R}$-set $W_1 \subseteq \pi(Z) \setminus \pi(T)$, such that $\pi(x) \in cl(W_1)$. Let

$$W = \pi^{-1}(W_1) \cap Z.$$

Then W is clearly an $\mathbb{R}$-set with $W \subseteq Z \setminus T$, and it is also easy to check that $x \in cl(W)$. $\square$

We are now ready to prove Proposition 2.7.

Proof of Proposition 2.7. We need to show $X_t^{alg} \subseteq X^{alg}$. Let Z be an $\mathbb{R}$-set with $Z \subseteq cl(Z \cap X)$. We need to prove that every point $x \in Z \cap X$ is contained in an $\mathbb{R}$-set W contained in X. By cell decomposition in the real field, there is a semialgebraic cell $Z_0 \subseteq Z$ over A, such that $\dim_x(Z) = \dim Z_0$ and $x \in cl(Z_0)$. By Lemma 2.10, there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that $B \cap Z_0$ is an $\mathbb{R}$-cell and $B \cap Z_0 \subseteq cl(Z_0 \cap X)$. Let

$$T = (B \cap Z_0) \setminus (Z_0 \cap X) \subseteq cl(Z_0 \cap X) \setminus (Z_0 \cap X).$$

Then

$$\dim T < \dim(Z_0 \cap X) \leq \dim Z_0 = \dim(B \cap Z_0).$$

Also, $x \in Z \setminus T$. Therefore, by Lemma 2.11 (for $Z = B \cap Z_0$), there is an $\mathbb{R}$-set $W_0 \subseteq (B \cap Z_0) \setminus T$ with $x \in cl(W_0)$. But

$$(B \cap Z_0) \setminus T = B \cap Z_0 \cap X,$$

so $W_0 \subseteq X$. Since $x \in cl(W_0)$, the set $W = W_0 \cup \{x\}$ is connected, and hence the desired $\mathbb{R}$-set. $\square$

Remark 2.12. If we specify parameters in Proposition 2.7, then the proposition need not be true. Indeed

$$X^{alg_{\mathbb{Q}}} \neq X_t^{alg_{\mathbb{Q}}}.$$

For example, fix a dcl-independent tuple $a = (a_1, a_2) \in \mathbb{R}^2$, and let

$$X = \mathbb{R}^2 \setminus \{(a_1, y) : y > a_2\}.$$

Then $a \in X \subseteq X_t^{alg_{\mathbb{Q}}}$, since $cl(X) = \mathbb{R}^2$ is a $\mathbb{Q}$-set. However, $a \notin X^{alg_{\mathbb{Q}}}$. Indeed, no open box around a can be contained in X. Hence if $a \in X^{alg_{\mathbb{Q}}}$, there must be some 1-dimensional semialgebraic set over $\emptyset$ that contains a, contradicting the dcl-independence of a. Note that in the proof of Proposition 2.7, unless $x \in \mathrm{dcl}(\emptyset)$, we cannot conclude that W is semialgebraic over $\emptyset$.

We do not know whether $X^{alg_A} = X_t^{alg_A}$ is true if X is A-definable.

Remark 2.13. The proof of Proposition 2.7 uses nothing in particular about the real field. In other words, if we fix an expansion $\widetilde{\mathcal{M}}$ of any real closed field $\mathcal{M}$, and define the notions of X^{alg} and X_t^{alg} in the same way as in the introduction after replacing 'semialgebraic' by '$\mathcal{M}$-definable', and 'connected' by '$\mathcal{M}$-definably connected', then for every $\mathcal{M}$-definable set X, we have $X^{alg} = X_t^{alg}$.

We conclude this section with an easy fact.

Fact 2.14. *Let $X, Y \subseteq \mathbb{R}^n$ be two definable sets.*

(1) *If $X \subseteq Y$, then $X_t^{alg_{\mathbb{Q}}} \subseteq Y_t^{alg_{\mathbb{Q}}}$.*
(2) *(a) If $X \subseteq Y$ and Y has few algebraic points, then so does X.*
(b) If X and Y have few algebraic points, then so does $X \cup Y$.
(3) *If $X \setminus X_t^{alg_{\mathbb{Q}}}$ and $Y \setminus Y_t^{alg_{\mathbb{Q}}}$ have few algebraic points, then so does $(X \cup Y) \setminus (X \cup Y)_t^{alg_{\mathbb{Q}}}$.*

Proof. (1) and (2) are obvious. For (3), we have:

$$(X\cup Y)\setminus(X\cup Y)_t^{alg_{\mathbb{Q}}} \subseteq (X\setminus(X\cup Y)_t^{alg_{\mathbb{Q}}})\cup(Y\setminus(X\cup Y)_t^{alg_{\mathbb{Q}}}) \subseteq (X\setminus X_t^{alg_{\mathbb{Q}}})\cup(Y\setminus Y_t^{alg_{\mathbb{Q}}}),$$

and we are done by (2). □

3. Dense pairs

In this section, we let $\widetilde{\mathcal{R}} = \langle \mathbb{R}, P\rangle$ be a dense pair. As mentioned in the introduction, since $P \preccurlyeq \mathcal{R}$, we have $\mathcal{A} \subseteq \mathrm{dcl}(\emptyset) \subseteq P$. In this setting, Theorem 1.4 has a short and illustrative proof, and we include it first.

Theorem 3.1. *For every definable set X, if X has many algebraic points, then it contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$.*

Proof. Since $\mathcal{A} \subseteq P$, $X \cap P^n$ also contains many algebraic points. By [10, Theorem 2], there is an $\mathcal{L}$-definable $Y \subseteq \mathbb{R}^n$, such that $X = Y \cap P^n$. So Y also contains many algebraic points. By Fact 2.3, there is a $\mathbb{Q}$-set $Z \subseteq Y$. Then the set $Z \cap P^n$ is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$ and it is contained in $Y \cap P^n = X$. Since the set of algebraic points $\mathcal{A}^n$ is dense in Z, we have $Z \subseteq cl(Z \cap \mathcal{A}^n) \subseteq cl(Z \cap P^n)$, and hence $Z \cap P^n$ is infinite. □

We now proceed to the proof of Theorem 1.3.

Lemma 3.2. *Let $X = Y \cap P^n$, for some $\mathcal{L}$-definable set $Y \subseteq \mathbb{R}^n$. Then*

$$X \cap Y^{alg_{\mathbb{Q}}} \subseteq X_t^{alg_{\mathbb{Q}}}.$$

Proof. Let $x \in X \cap Y^{alg_{\mathbb{Q}}}$. So x is contained in a $\mathbb{Q}$-set $Z \subseteq Y$. We prove that X is dense in Z. Observe that $Z \cap X = Z \cap P^n$. Since $\mathcal{A}^n \subseteq P^n$, we have

$$Z \subseteq cl(Z \cap \mathcal{A}^n) \subseteq cl(Z \cap P^n) = cl(Z \cap X),$$

and hence X is dense in Z. □

Theorem 3.3. *For every definable set X, $X \setminus X_t^{alg_{\mathbb{Q}}}$ has few algebraic points.*

Proof. Let $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$. We first observe that if the statement holds for $X \cap P^n$, then it holds for X. Of course, $X \setminus X_t^{alg_{\mathbb{Q}}} \subseteq X \setminus (X \cap P^n)_t^{alg_{\mathbb{Q}}}$. Since $\mathcal{A}^n \subseteq P^n$, the set X has the same algebraic points as $X \cap P^n$, and hence if $(X \cap P^n) \setminus (X \cap P^n)_t^{alg_{\mathbb{Q}}}$ has few algebraic points, then so does $X \setminus (X \cap P^n)_t^{alg_{\mathbb{Q}}}$, and therefore also $X \setminus X_t^{alg_{\mathbb{Q}}}$.

We may thus assume that $X \subseteq P^n$. By [10, Theorem 2], there is an $\mathcal{L}$-definable $Y \subseteq \mathbb{R}^n$, such that $X = Y \cap P^n$. By Fact 2.3, $Y \setminus Y^{alg_{\mathbb{Q}}}$ has few algebraic points. By Lemma 3.2,

$$X \cap Y^{alg_{\mathbb{Q}}} \subseteq X_t^{alg_{\mathbb{Q}}}.$$

Hence

$$X \setminus X_t^{alg_{\mathbb{Q}}} \subseteq X \setminus Y^{alg_{\mathbb{Q}}} \subseteq Y \setminus Y^{alg_{\mathbb{Q}}}$$

has few algebraic points. □

4. Dense independent sets

In this section, $P \subseteq \mathbb{R}$ is a dense dcl-independent set. The proof of Theorem 4.15 runs by induction on the *large dimension* of a definable set X (Definition 4.8), by making use of the *cone decomposition theorem* from [14] (Fact 4.5). As mentioned in the introduction, since P contains no elements in $\operatorname{dcl}(\emptyset)$, we have $P \cap \mathcal{A} = \emptyset$. The base step of the aforementioned induction is to show a generalization of this fact; namely, that for a *small* set X (Definition 4.1), $X \cap \mathcal{A}$ is finite (Corollary 4.12).

4.1. Cone decomposition theorem.

In this subsection we recall all necessary background from [14]. The following definition is taken essentially from [12].

Definition 4.1. Let $X \subseteq \mathbb{R}^n$ be a definable set. We call X *large* if there is some m and an $\mathcal{L}$-definable function $f : \mathbb{R}^{nm} \to \mathbb{R}$ such that $f(X^m)$ contains an open interval in $\mathbb{R}$. We call X *small* if it is not large.

The notion of a cone is based on that of a supercone, which in its turn generalizes the notion of being co-small in an interval. Both supercones and cones are unions of special families of sets, which not only are definable, but they are so in a very uniform way. Although this uniformity is not fully exploited in this paper, we include it here to match the definitions from [14].

Definition 4.2 ([14]). A *supercone* $J \subseteq \mathbb{R}^k$, $k \geq 0$, and its *shell* $sh(J)$ are defined recursively as follows:

- $\mathbb{R}^0 = \{0\}$ is a supercone, and $sh(\mathbb{R}^0) = \mathbb{R}^0$.
- A definable set $J \subseteq \mathbb{R}^{n+1}$ is a supercone if $\pi(J) \subseteq \mathbb{R}^n$ is a supercone and there are $\mathcal{L}$-definable continuous $h_1, h_2 : sh(\pi(J)) \to \mathbb{R} \cup \{\pm\infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, J_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it. We let $sh(J) = (h_1, h_2)_{sh(\pi(J))}$.

Note that, $sh(J)$ is an open cell in $\mathbb{R}^k$ and $cl(sh(J)) = cl(J)$.

Recall that in our notation we identify a family $\mathcal{J} = \{J_g\}_{g\in S}$ with $\bigcup_{g\in S}\{g\} \times J_g$. In particular, $cl(\mathcal{J})$ and $\pi_n(\mathcal{J})$ denote the closure and a projection of that set, respectively.

Definition 4.3 (Uniform families of supercones [14])**.** Let $\mathcal{J} = \bigcup_{g\in S}\{g\} \times J_g \subseteq \mathbb{R}^{m+k}$ be a definable family of supercones. We call $\mathcal{J}$ *uniform* if there is a cell $V \subseteq \mathbb{R}^{m+k}$ containing $\mathcal{J}$, such that for every $g \in S$ and $0 < j \leq k$,

$$cl(\pi_{m+j}(\mathcal{J})_g) = cl(\pi_{m+j}(V)_g).$$

We call such a V a *shell* for $\mathcal{J}$.

Remark 4.4. A shell for a uniform family of supercones $\mathcal{J}$ need not be unique. Also, one can identify a supercone $J \subseteq \mathbb{R}^k$ with a uniform family of supercones $\mathcal{J} \subseteq M^{m+k}$ with $\pi_m(\mathcal{J})$ a singleton; in that case, a shell for $\mathcal{J}$ is unique and equals that of J.

Definition 4.5 (Cones [14] and H-cones[1])**.** A set $C \subseteq \mathbb{R}^n$ is a *k-cone*, $k \geq 0$, if there are a definable small $S \subseteq \mathbb{R}^m$, a uniform family $\mathcal{J} = \{J_g\}_{g\in S}$ of supercones in $\mathbb{R}^k$, and an $\mathcal{L}$-definable continuous function $h : V \subseteq \mathbb{R}^{m+k} \to \mathbb{R}^n$, where V is a shell for $\mathcal{J}$, such that

(1) $C = h(\mathcal{J})$, and
(2) for every $g \in S$, $h(g, -) : V_g \subseteq \mathbb{R}^k \to \mathbb{R}^n$ is injective.

We call C a *k-H-cone* if, in addition, $S \subseteq P^m$ and $h : \mathcal{J} \to \mathbb{R}^n$ is injective. An *(H-)cone* is a k-(H-)cone for some k.

The cone decomposition theorem [14, Theorem 5.1] is a statement about definable sets and functions. Here we are only interested in a decomposition of sets into H-cones. Before stating the *H-cone decomposition theorem*, we need the following fact.

Fact 4.6. *Let $S \subseteq \mathbb{R}^n$ be an A-definable small set. Then S is a finite union of sets of the form $f(X)$, where*

- *$f : Z \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is an $\mathcal{L}_A$-definable continuous map,*
- *$X \subseteq P^m \cap Z$ is A-definable, and*
- *$f : X \to \mathbb{R}^l$ is injective.*

Proof. By [14, Lemma 3.11], there is an $\mathcal{L}_A$-definable map $h : \mathbb{R}^m \to \mathbb{R}^n$ such that $X \subseteq h(P^m)$. The result follows from [15, Theorem 2.2]. □

Fact 4.7 (H-cone decomposition theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be an A-definable set. Then X is a finite union of A-definable H-cones.*

Proof. By [14, Theorem 5.12] and [15, Theorem 2.2], X is a finite union of A-definable cones $h(\mathcal{J})$ with $h : \mathcal{J} \to \mathbb{R}^n$ injective (such $h(\mathcal{J})$ is called *strong cone* in the above references). By Fact 4.6, it is not hard to see that $h(\mathcal{J})$ is a finite union of A-definable H-cones. □

We next recall the notion of 'large dimension' from [14].

[1]The letter 'H' derives from 'Hamel basis' - see [9] for the motivating example $\langle \mathbb{R}, <, +, H\rangle$.

Definition 4.8 (Large dimension [14]). Let $X \subseteq \mathbb{R}^n$ be definable. If $X \neq \emptyset$, the *large dimension* of X is the maximum $k \in \mathbb{N}$ such that X contains a k-cone. The large dimension of the empty set is defined to be $-\infty$. We denote the large dimension of X by $\mathrm{ldim}(X)$.

Some basic properties of the large dimension that will be used in the sequel are the following (see [14, Lemma 6.11]): for every two definable sets $X, Y \subseteq \mathbb{R}^n$,

- if $X \subseteq Y$, then $\mathrm{ldim}X \leq \mathrm{ldim}Y$.
- if X is $\mathcal{L}$-definable, then $\mathrm{ldim}X = \dim X$.
- X is small if and only if $\mathrm{ldim}X = 0$.

4.2. **Point counting.** We now proceed to the proof of Theorem 1.3 (B). We need several preparatory lemmas. First, a very useful fact.

Fact 4.9. *For every $A \subseteq \mathbb{R}$ with $A \backslash P$ dcl-independent over P, we have $\mathrm{dcl}_{\mathcal{L}(P)}(A) = \mathrm{dcl}(A)$.*

Proof. Take $x \in \mathrm{dcl}_{\mathcal{L}(P)}(A)$. That is, the set $\{x\}$ is A-definable in $\langle \mathcal{R}, P \rangle$. By [14, Assumption III], since $A \setminus P$ is dcl-independent over P, we have that $cl(\{x\})$ is $\mathcal{L}_A$-definable. But $cl(\{x\}) = \{x\}$. So $x \in \mathrm{dcl}(A)$. □

The following lemma is crucial and relies on the fact that P is dcl-independent.

Lemma 4.10. *Let $h : Z \subseteq P^m \times \mathbb{R}^k \to \mathbb{R}^n$ be a definable injective map. Let $B \subseteq \mathbb{R}$ be a finite set. Then there is a finite set $S_0 \subseteq P^m$ such that*

$$h\left(\bigcup_{g \in P^m \setminus S_0} \{g\} \times Z_g\right) \cap \mathrm{dcl}(B)^n = \emptyset.$$

Proof. Suppose h is A-definable, with A finite. Let $A_0 \subseteq A \cup B$ and $P_0 \subseteq P$ be finite so that $A \cup B \subseteq \mathrm{dcl}(A_0 P_0)$ and A_0 is dcl-independent over P. Suppose $q = h(g, t)$, where $g \in P^m$, $t \in Z_g$ and $q \in \mathrm{dcl}(B)$. By injectivity of h, all coordinates of g are in

$$\mathrm{dcl}_{\mathcal{L}(P)}(Aq) \subseteq \mathrm{dcl}_{\mathcal{L}(P)}(AB) \subseteq \mathrm{dcl}_{\mathcal{L}(P)}(A_0 P_0) = \mathrm{dcl}(A_0 P_0).$$

Since P is dcl-independent, there can be at most $|A_0|$ many such g's, and hence so can q's. □

Two particular cases of the above lemma are the following (recall, $\mathcal{A} \subseteq \mathrm{dcl}(\emptyset)$).

Corollary 4.11. *Let $C = h\left(\bigcup_{g \in S} \{g\} \times J_g\right)$ be an H-cone. Then there is a finite set $S_0 \subseteq S$ such that $h\left(\bigcup_{g \in S \setminus S_0} \{g\} \times J_g\right)$ contains no algebraic points.*

Corollary 4.12. *Every small set contains only finitely many algebraic points.*

Proof. By Lemma 4.10, for $k = 0$, and Fact 4.6. □

The key lemma in the inductive step of the proof of Theorem 4.15 is the following.

Lemma 4.13. *Let $J \subseteq \mathbb{R}^k$ be a supercone with shell Z, and $B \subseteq \mathbb{R}$ finite. Then there is an $\mathcal{L}$-definable set $F \subseteq Z$ with $\dim(F) < k$, such that*

$$Z \cap \mathrm{dcl}(B)^k \subseteq J \cup F.$$

Proof. By induction on k. For $k = 0$, the statement is trivial. For $k > 0$, assume $J = \bigcup_{g\in\Gamma}\{g\} \times J_g$, where $\Gamma \subseteq \mathbb{R}^{k-1}$ is a supercone. By inductive hypothesis, there is $F_1 \subseteq \pi(Z)$, such that

$$\pi(Z) \cap \mathrm{dcl}(B)^{k-1} \subseteq \Gamma \cup F_1.$$

Since $\dim(F_1 \times \mathbb{R}) < k$, it suffices to write $\left(\bigcup_{g\in\Gamma}\{g\} \times Z_g\right) \cap \mathrm{dcl}(B)^k$ as a subset of $J \cup F_2$, for some $F_2 \subseteq Z$ with $\dim(F_2) < k$. Let

$$X = \bigcup_{g\in\Gamma}\{g\} \times (Z_g \setminus J_g).$$

So we need to prove that $X \cap \mathrm{dcl}(B)^k$ is contained in an $\mathcal{L}$-definable set $F_2 \subseteq Z$ with $\dim(F_2) < k$. By [15, Theorem 2.2] and [14, Corollary 5.11], X is a finite union of sets $X_1, \dots, X_l$, each of the form

$$X_i = f\left(\bigcup_{g\in S}\{g\} \times U_g\right),$$

where

- $f : V \subseteq \mathbb{R}^{m+k-1} \to \mathbb{R}^k$ is an $\mathcal{L}$-definable continuous map,
- $U \subseteq (S \times \Gamma) \cap V$ is a definable set, and
- $f_{\restriction U}$ is injective.

Using Fact 4.6, we may further assume that $S \subseteq P^m$. By Lemma 4.10, for $h = f$, there is a finite set $S_0 \subseteq P^m$ such that

$$f\left(\bigcup_{g\in S\setminus S_0}\{g\} \times U_g\right) \cap \mathrm{dcl}(B)^k = \emptyset.$$

For each $i = 1, \dots, l$, and X_i as above, set

$$D_i = f\left(\bigcup_{g\in S_0}\{g\} \times U_g,\right).$$

Then $F_2 = \bigcup_{i=1}^l D_i$ satisfies the required properties. □

Corollary 4.14. *Let $C = h(J) \subseteq \mathbb{R}^n$, where $J \subseteq \mathbb{R}^k$ is a supercone with shell Z, and $h : Z \to \mathbb{R}^n$ an $\mathcal{L}$-definable and injective map. Then there is a definable set $F \subseteq Z$ with $\dim(F) < k$, such that all algebraic points of $h(Z)$ are contained in $h(J \cup F)$.*

Proof. Suppose h is $\mathcal{L}_B$-definable, and take F be as in Lemma 4.13. Let $x = h(y) \in h(Z)$ be an algebraic point. In particular, $x \in \mathrm{dcl}(\emptyset)$. Since h is $\mathcal{L}$-definable and injective, $y \in \mathrm{dcl}(B) \subseteq J \cup F$. □

Theorem 4.15. *For every definable set X, $X \setminus X_t^{alg_{\mathbb{Q}}}$ has few algebraic points.*

Proof. Let $X \subseteq \mathbb{R}^n$ be a definable set. We work by induction on the large dimension of X. If $\mathrm{ldim}(X) = 0$, then X is small and the statement follows from Corollary 4.12. Assume $\mathrm{ldim}(X) = k > 0$. By Facts 4.7 and 2.14(3), we may assume that X is a k-H-cone, say $h(\mathcal{J})$ with $\mathcal{J} \subseteq \mathbb{R}^{m+k}$. By Corollary 4.11, we may further assume that $\pi_m(\mathcal{J})$ is a singleton, and hence, that $X = h(J) \subseteq \mathbb{R}^n$, where $J \subseteq \mathbb{R}^k$ is a supercone. Let Z be the shell of J, and $F \subseteq Z \setminus J$ as in Corollary 4.14. We have that $X \subseteq h(Z \setminus F) \cup h(F)$. By Fact 2.14(3), it suffices to show the statement for each of $X \cap h(Z \setminus F)$ and $X \cap h(F)$.

$X \cap h(F)$. We have

$$\operatorname{ldim}(X \cap h(F)) \leq \operatorname{ldim} h(F) = \dim h(F) < k,$$

and hence we conclude by inductive hypothesis.

$X \cap h(Z \setminus F)$. Observe that

$$h(Z \setminus F)^{alg_{\mathbb{Q}}} \subseteq (X \cap h(Z \setminus F))_t^{alg_{\mathbb{Q}}}.$$

Indeed, let $T \subseteq h(Z \setminus F)$ be a $\mathbb{Q}$-set. We need to show that $T \subseteq cl(X \cap T)$. By the conclusion of Corollary 4.13, $T \cap \mathcal{A}^n \subseteq T \cap X$. Since the set of algebraic points $\mathcal{A}$ is dense in Y, we obtain that

$$T \subseteq cl(T \cap \mathcal{A}^n) \subseteq cl(T \cap X),$$

as required. Hence, by Fact 2.3, the sets

$$(X \cap h(Z \setminus F)) \setminus (X \cap h(Z \setminus F))_t^{alg_{\mathbb{Q}}} \subseteq h(Z \setminus F) \setminus h(Z \setminus F)^{alg_{\mathbb{Q}}}$$

has few algebraic points. □

We now turn to the proof of Theorem 1.4. Note that Theorem 4.15 implies that if a definable set X contains many algebraic points, then it is dense in an infinite semialgebraic set. However, the last conclusion by itself does not guarantee that X contains an infinite set definable in $\langle \overline{\mathbb{R}}, P \rangle$. For example, let $\mathcal{R} = \langle \overline{\mathbb{R}}, \exp \rangle$ and $X = e^P$. Then X is definable (in $\langle \mathcal{R}, P \rangle$), and dense in $\mathbb{R}$. Suppose, towards a contradiction, that it contains an infinite set Y definable in $\langle \overline{\mathbb{R}}, P \rangle$. Then Y must be small in the sense of $\langle \overline{\mathbb{R}}, P \rangle$. Indeed, e^P is small in the sense of $\widetilde{\mathcal{R}}$, and smallness is preserved under reducts, by [14, Corollary 3.12]. Now, since Y is small in the sense of $\langle \overline{\mathbb{R}}, P \rangle$, by [13], there is a semialgebraic $h : \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq P^n$, such that $h_{\restriction S}$ is injective and $h(S) = Y \subseteq e^P$. We leave it to the reader to verify that this statement contradicts the dcl-independence of P.

We need two preliminary lemmas.

Lemma 4.16. *Let $J \subseteq \mathbb{R}^k$ be a supercone. Then there is $b \in \mathcal{A}^k$, such that*

$$(b + P^k) \cap sh(J) \subseteq J.$$

In particular, J contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$.

Proof. Denote $Z = sh(J)$. We work by induction on k. For $k = 0$, $J = P^0 = \mathbb{R}^0 = \{0\}$, and the statement holds. Now let $k > 1$. By inductive hypothesis, there is $b_1 \in \mathcal{A}^{k-1}$, such that

$$(b_1 + P^{k-1}) \cap \pi(Z) \subseteq \pi(J).$$

Let $S = (b_1 + P^{k-1}) \cap \pi(Z)$. For every $t \in S$, the set $(Z_t \setminus J_t) - P$ is small, and hence $\bigcup_{t \in S} (Z_t \setminus J_t) - P$ is also small. By Lemma 4.12, the last set contains only finitely many algebraic points. So there is

$$b_2 \in \mathcal{A} \setminus \bigcup_{t \in S} ((Z_t \setminus J_t) - P).$$

But then for every $p \in P$ and $t \in S$, if $b_2 + p \in Z_t$, then $b_2 + p \in J_t$. That is, $(b_2 + P) \cap Z_t \subseteq J_t$. Therefore, for $b = (b_1, b_2) \in \mathcal{A}^k$, we have that

$$(b + P) \cap Z \subseteq J.$$

For the "in particular" clause, let $B \subseteq sh(J)$ be any $\emptyset$-definable open box, and b as above. Then $(b + P^k) \cap B \subseteq J$ is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$. It is also infinite, by density of P in $\mathbb{R}$. $\square$

Question 4.17. *Let $J \subseteq \mathbb{R}^k$ be a supercone. Does J contain a set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$ and has large dimension k?*

Lemma 4.18. *Let $X \subseteq \mathbb{R}^n$ be a definable set and $T \subseteq \mathbb{R}^n$ a $\mathbb{Q}$-set, such that $\mathcal{A}^n \cap T \subseteq X$. Then* $\mathrm{ldim}(X \cap T) = \dim T$.

Proof. Clearly, $\mathrm{ldim}(X \cap T) \leq \mathrm{ldim} T = \dim T$. Let $k = \dim T$. The set $X \cap T$ is a finite union of H-cones. By Corollary 4.11, there are finitely many cones $h_i(J_i)$ contained in $X \cap T$ and containing all algebraic points of $X \cap T$. Since $\mathcal{A}^n \cap T \subseteq X$, $\mathcal{A}^n \cap T$ is contained in the union of those cones. So

$$T \subseteq cl(\mathcal{A}^n \cap T) \subseteq \bigcup_i cl(h_i(J_i)),$$

implying that for some i, $\dim cl(h_i(J_i)) \geq k$. Therefore, some J_i is a supercone in $\mathbb{R}^k$, implying that $\mathrm{ldim}(X \cap T) \geq k$. $\square$

Theorem 4.19. *Let $X \subseteq \mathbb{R}^n$. If X contains many algebraic points, then it contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$.*

Proof. The beginning of the proof is similar to that of Theorem 4.15, and thus we are brief. We work by induction on $\mathrm{ldim}(X) = 0$. If $\mathrm{ldim} X = 0$, then X is small and the statement holds trivially by Corollary 4.12. For $\mathrm{ldim} X = k > 0$, we may assume that $X = h(J)$ is a k-cone, with $J \subseteq \mathbb{R}^k$. Let Z be the shell of J, and $F \subseteq Z \setminus J$ as in Corollary 4.14. So one of $X \cap h(F)$ and $X \cap h(Z \setminus F)$ must contain many algebraic points. If the former one does, then we can conclude by inductive hypothesis. If the latter one does, then by Fact 2.3, there is a $\mathbb{Q}$-cell $T \subseteq h(Z \setminus F)$. By the conclusion of Corollary 4.12, $\mathcal{A}^n \cap T \subseteq X$. By Lemma 4.18, $\mathrm{ldim} X \cap T = \dim T$. Also,

$$T \subseteq \mathrm{cl}(\mathcal{A}^n \cap T) \subseteq cl(X \cap T),$$

and hence if follows easily that

$$\dim cl(X \cap T) = \mathrm{ldim} X \cap T.$$

Now, if T is open, then $\mathrm{ldim} X \cap T = n$, and hence $X \cap T$ contains a supercone in $\mathbb{R}^n$ (by [14, Theorem 5.7(1)]). By Lemma 4.16, $X \cap T$ contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$. Suppose $T = \Gamma(f)$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a coordinate projection that is injective on T. Then $\mathrm{ldim} \pi(X \cap T) = k$ and hence $\pi(X \cap T)$ contains a supercone in $\mathbb{R}^k$, and thus, by Lemma 4.16, an infinite set S which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$. Then $\Gamma(f_{\restriction S})$ is contained in X and is as desired. $\square$

We conclude with a remark that goes also beyond the scope of this section.

Remark 4.20. Let $X \subseteq \mathbb{R}^n$ and $P \subseteq \mathbb{R}$ be as in Theorem 1.3. Define

$$X_P^{alg} = \bigcup \{Y \subseteq X : Y \text{ infinite } \emptyset\text{-definable in } \langle \overline{\mathbb{R}}, P\rangle\}.$$

It is natural to ask whether $X \setminus X_P^{alg}$ has few algebraic points. An affirmative answer to this question would strengthen Theorem 1.3, and its contrapositive would imply Theorem 1.4. For the case of dense pairs, it is actually not too hard to adjust the proofs of Lemma 3.2 and Theorem 3.3 and obtain an affirmative answer. For the case of dense independent sets, the question is open, and it is possible that an affirmative answer to Question 4.18 could be relevant.

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[16] A. Günaydın, P. Hieronymi, *The real field with the rational points of an elliptic curve*, Fund. Math. 215 (2011) 167-175.

[17] D. Marker, MODEL THEORY: AN INTRODUCTION, Springer (Graduate Texts in Mathematics 217).

[18] C. Miller, P. Speissegger, *Expansions of the real line by open sets: o-minimality and open cores*, Fund. Math. 162 (1999), 193-208.

[19] J. Pila, *On the algebraic points of a definable set*, Selecta Math. N.S. 15 (2009), 151-170.

[20] J. Pila, *O-minimality and the André-Oort conjecture for* $\mathbb{C}^n$, Ann. Math. 173 (2011), 1779–1840.

[21] J. Pila and A. J. Wilkie, *The rational points of a definable set*, Duke Math. J. 133 (2006), 591–616.

[22] T. Scanlon, *Counting special points: logic, Diophantine geometry, and transcendence theory*, Bull. Amer. Math. Soc. (N.S.), 49(1):5171, 2012.

[23] M. Tychonievich, *Tameness results for expansions of the real field by groups*, Ph.D. Thesis, Ohio State University (2013).

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Special day in honor of Paulo Ribenboim
On occasion of his 90th birthday - 20 Mars 2018

Institut Henri Poincaré - Amphi Hermite
11, rue Pierre et Marie Curie 75005 PARIS

Event supported by IMJ-PRG, Projets Logique and Théorie des nombres, and by the GDR-STN.

ORGANIZERS: Zoé Chatzidakis, Françoise Delon, Max Dickmann, Danielle Gondard and Tamara Servi.

Programme

Welcome 10:30

10:45 - 11:30 Lou van den Dries: *Hardy fields, transseries, and surreal numbers*
This is about joint work with Matthias Aschenbrenner and Joris van der Hoeven. I will discuss how the three topics in the title are related. Partly this is still conjectural. Valuation theory as in Paulo's *Théorie des Valuations* plays an important role in dealing with these structures.

11:45 - 12:30 Franziska Jahnke: *Definable Henselian Valuations*
Although the study of the definability of henselian valuations has a long history starting with J. Robinson, most of the results in this area were proven during the last few years. We survey a number of these results which address the definability of concrete henselian valuations, the existence of definable henselian valuations on a given field, and questions of uniformity and quantifier complexity.

Lunch 12:30 -14:30

14:30 - 15:15 Sibylla Priess-Crampe: *Asymptotic approximations - a short insight into my joint work with Paulo for our book on Ultrametric Spaces*
A strictly contracting mapping from a spherically complete ultrametric space into itself has a unique fixed point. This fixed point can be approximated by a family of elements which are determined by the mapping. We will show how this general method can be applied to the case of ordinary differential equations of polynomial type over Hardy fields.

15:30 - 16:15 Paulo Ribenboim: *Roots of Polynomials in Ultranormed Rings*

Coffee and tea break 16:15 - 17:00

17:00 - 17:45 Daniel Bertrand: *Pell equations over polynomial rings*
Interest in these equations goes back to Abel. I will describe the new development brought by Masser and Zannier in the case of square free discriminants, in relation with their work on unlikely intersections in abelian schemes. The case of non separable discriminants leads to similar problems on generalized jacobians, where unexpected solutions sometimes appear. I will also discuss generalized Pell equations and their link with relative versions of the Mordell-Lang problem.

Birthday cocktail 18:00 - 20:00

PAULO RIBENBOIM
Some highlights of his mathematical life

Paulo Ribenboim – Born in Recife, Pernambuco, Brazil, March 13, 1928.
1951 – Married to Huguette Ribenboim, née Demangelle.

EDUCATION.
---1936 - 1937 - Grupo Escolar João Barbalho, Recife, Brazil.
---1938 - 1945 - Primary & Secondary School, Colégio Anglo-Americano, Colégio Andrews.
---1946 - 1948 – Mathematics, Faculdade Nacional de Filosofia da Universidade do Brasil.
---1948 - Bachelor Degree in Mathematics.
---1957 - Ph.D in Mathematics, Universidade de São Paulo.

APPOINTMENTS AND FELLOWSHIPS.
---1949 – Department of Mathematics, Brazilian Center for Research in Physics.
---April 1950 - June 1951 – Fellowship from the French Government to study Mathematics in Nancy, France.
---August 1953 – Research assistant, Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil.
---June 1953 - June 1956 – Fellowship from the German Academic Interchange Service to work with W. Krull in Bonn, Germany.
---August 1956 - August 1959 – Research chief at IMPA.
---August 1959 - August 1962 – Awarded Fulbright Fellowship. Associate Professor, University of Illinois at Urbana-Champaign, USA.
---1962 - 1993 – Professor, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada.
---1993 to present – Emeritus Professor, Queen's University.

SHORT APPOINTMENTS.
In Uruguay – Instituto de Matemática, Montevideo.
In Argentina – Universidad Nacional del Sur in Bahia Blanca, Patagonia.
In USA: University of Michigan, Ann Arbor, Mich.; State University of Pennsylvania, State College, Penn.; Harvard University, Cambridge, Mass.; Northeastern University, Boston, Mass.; Institute for Advanced Study, Princeton, N.J.
In France: Université Pierre et Marie Curie, Paris; Université Paris-Sud at Orsay, Paris; Université Paris-Nord; Université de Sciences et Techniques at Lille; Université de Lyon I; Université Paul Sabatier at Toulouse; Université Aix-Marseille at Aix-en-Provence; Université de Provence, Marseille; Université de Marseille II.
In Germany: Ludwig Maximilian University, Munich.
In Belgium: Université Catholique de Louvain at Louvain-la-Neuve.
In Spain: Mathematics Research Center, Bellaterra.
In Portugal: Faculdade de Ciências da Universidade de Lisbôa, Lisbon.
In Japan: Yoshida College at University of Kyoto; Science University of Tokyo, Tokyo; Science University of Tokyo, Noda (Chiba); Suda College, Takanodai.

TALKS IN MEETINGS, COLLOQUIA AND SEMINARS.
These talks took place in 41 different countries: Canada, USA, Mexico, Curaçao. Brazil, Uruguay, Argentina, Colombia. Tunisia, Morocco, Egypt, Sudan. Israel, Iran, Japan, Hong Kong, Taiwan. Portugal, Spain, France, Germany, Italy, Belgium, The Netherlands, Luxembourg, Austria, Croatia, Slovakia, Czech Republic, Poland, Hungary, Greece, Romania, England, Scotland, Denmark, Norway, Sweden, Finland, Switzerland.

SHORT COURSES.

---1957 – ”Krull Valuations” in Montevideo, Uruguay.

---1958 – “Teoria de Grupos Ordenados” in Bahia Blanca, Argentina.

---1964 – “Théorie des Valuations” at the Séminaire de Mathématique Supérieures, Université de Montréal, Canada.

---1967 – One year course: “Commutative Fields and Galois Theory”, at Northeastern University, Boston.

---1969 – Half year course “Arithmétique des Corps”, Université P.et M. Curie, Paris.

---1972 – Cours de 3ième cycle in Switzerland, at Lausanne, on “Arithmétique des Corps”.

---1981 – Greek Lectures on Fermat’s Last Theorem, at Delphi, Greece.

---1985 – Recent Advances on Fermat’s Last Theorem, at Université Paris-Sud, Orsay, Paris.

PH. D. STUDENTS.

1. Malcolm P. Griffin (from New Zealand), 1965: Rings of Krull Type.
2. Tenkasi M. Viswanathan (from India), 1967: Studies on Ordered Rings and Ordered Modules.
3. Syed M. Fakhruddin (from India), 1969: Modules Over a Valuation Ring.
4. Tor H. Gulliksen (from Norway), 1969: Homological Invariants of Local Rings.
5. Murray Marshall (from Canada), 1969: The Ramification Filters of Abelian Extensions of Local Fields.
6. George Maxwell (from Australia), 1970: Forms Overs Rings with Anti-Automorphism.
7. Gordon D.I. Edwards (from Canada), 1972: Primitive and Group-Like Elements in Symmetric Algebras.
8. Aron Simis (from Brazil), 1972: 1 - Projective Modules Over Sets of Inert Prime Ideals, 2 – Orders Over Valuation Rings.
9. Paul Jean Cahen (from France), 1972: Torsion Theories and Commutative Algebra.
10. Danielle Gondard-Cozette (from France), 1973: sur le 17ème Problème de Hilbert (as I had no link to the Université Paris VII, the official supervisor was Pierre Samuel).
11. Mostafa Guennoun (from Morocco), 1980: Espaces et Structures Hiérarchimétriques.
12. Karl Dilcher (from Germany), 1983: Zeros of Bernoulii Polynomials.
13. Ján Minác (from Czechoslovakia), 1986: Galois Groups, Order Spaces and Valuations.
14. Andrew J. Grandville (from England), 1987: Diophantine Equations with Varying Exponents (with Special Reference to Fermat’s Last Theorem).

SOME PAPERS.

The full list of papers will require more than 10 pages. Complete information may be obtained from MathSciNet. The topics are: Ordered Structures, Valuation Theory, Hilbert’s 17th Problem, Higher Order Derivations, Generalized Power Series, Diophantine Equations; Recurrent Sequences, Prime Numbers, Ultrametric Spaces and Miscellanea. Some of the numerous papers are detailed below :

I. **Papers on Valuation Theory.**

---1954 – La conjecture de Krull sur les anneaux primaires complètement intégralement clos, Nagoya J. Math. 9, 1955. This gave the first counter example to Krull’s conjecture constructing a domain which is not a valuation ring, but it is primary and completely integrally closed.

---1964 – **Théorie des Valuations** – Séminaire de Mathématiques Supérieures, Université de Montréal. It contains a coherent presentation of the theory of Krull valuations. The volume became standard, and was for long the unique book on the subject.

---A paper on finite extensions of fields with Krull Valuations, relating the degree, the ramification index and the inertial degree, was published in Transactions AMS 105, 1962.

II. **Hilbert’s 17th Problem**.

Two papers co-authored with D. Gondard were fundamental for the study of Hilbert’s 17th Problem and opened the way to applications to real algebraic geometry:

---1974 – Fonctions définies positives sur varietés réelles, Bull. Sci. Math. 98, 1974;

---1974 – Le 17ième. problème de Hilbert pour les matrices, Bull. Sci. Math. 98, 1974.

The proofs in these papers make use of methods of model theory.

III. **Diophantine Equations and Prime Numbers**.

---1985 – An extension of the method of Sophie Germain for a wide class of Diophantine equations, J. Reine Angew. Math. 356.

---1985 – Note on a paper by M. Filaseta on Fermat's Last Theorem, Ann. Univ. Turku. Série AI, 187. This paper was written with B. Powell. The paper uses Faltings' famous theorem and has very interesting calculations.

---1993 – Density results on families of Diophantine equations with finitely many solutions, L'Enseign. Math. 39. In this paper is obtained an asymptotic and uniform density result for families of certain types of Diophantine equations. The main tool is Faltings' theorem.

---2009 – A remark on Polignac's Conjecture, Proceedings AMS, 137. Polignac had conjectured that every even number is, in infinitely many ways, the difference of two prime numbers. In particular there exists infinitely many pairs of twin primes.

---2011 – Multiple patterns of k-tuples of integers, International Journal of Number Theory, 7. The pigeon-hole principle is applied to show that a wide variety of sequences display arbitrarily large patterns of sums, differences, higher differences, etc.

Paper to appear: Hilbert's theory of inertia and ramification for extensions of Krull valuations. Submitted to Annali della Scuola Normale Superiore di Pisa.

SELECTED BOOKS AND LECTURE NOTES.

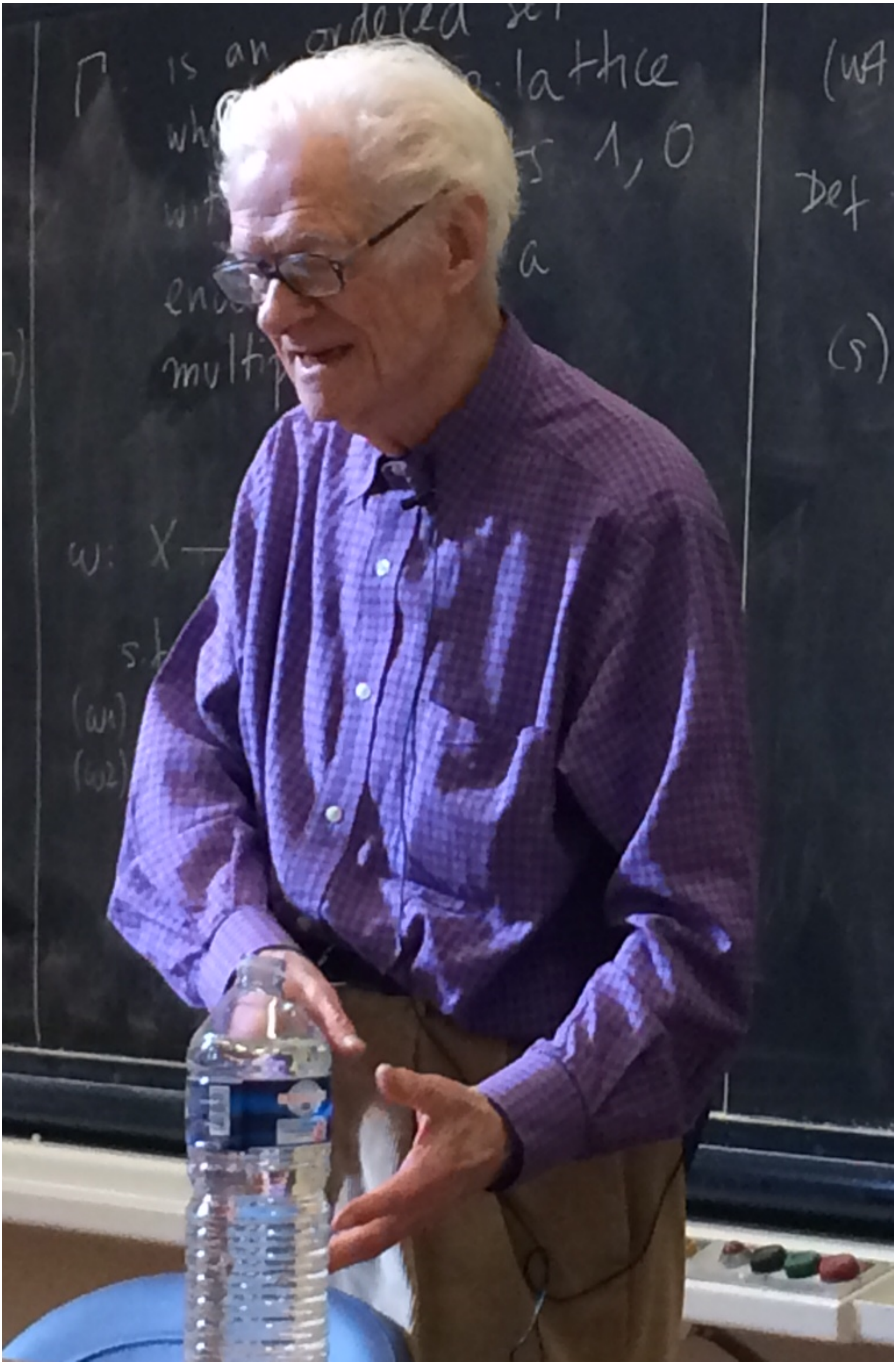
1. Ideais em Anéis de Tipo Infinito (1958), IMPA, Rio de Janeiro.
2. O Teorema de Riemann-Roch para Curvas Algébricas (Tesc) (1959), Faculdade Nacional de Filosofia, Rio de Janeiro.
3. Théorie des Groupes Ordonées (1963), Universidad Nacional del Sur, Bahia Blanca.
4. Functions, Limits and Continuity (1964), Wiley, New York.
5. Théorie des Valuations (1964), Université de Montréal.
6. Tópicos de Teoria dos Números, Seminário with collaborations of O. Endler, A. Azevedo, A. Micali – (1965), IMPA, Rio de Janeiro.
7. Linear Representations of Finite Groups (1966), Queen's University, Kingston.
8. La Conjecture d'Artin sur les Équations Diophantiennes (1968), Queen's University, Kingston.
9. Rings and Modules (1969), Wiley, New York.
10. Algebraic Numbers (1972), Wiley, New York.
11. L'Arithmétique des Corps (1972), Hermann, Paris
12. Lectures on Fermat's Last Theorem (1979), Springer-Verlag, New York.
13. The Book of Prime Number Records (1988), Springer-Verlag, New York.
14. The Little Book of Big Primes (1991), Springer-Verlag, New York.
15. Catalan's Conjecture (1994), Academic Press, Boston.
16. The New Book of Prime Numbers Records (1995), Springer-Verlag, New York.
17. Paulo Ribenboim. Career up to 1995 (1995), Queen's University, Kingston.
18. Fermat's Last Theorem for Amateurs (1999), Springer-Verlag, New York.
19. Theory of Classical Valuations (1999), Springer-Verlag, New York.
20. My Numbers, My Friends (2000), Springer-Verlag, New York.
21. The Classical Theory of Algebraic Numbers (2001), Springer-Verlag, New York.
22. The Little Book of Bigger Primes (2003), Springer-Verlag, New York.
23. Die Welt der Primzahlen: Geheimnisse und Rekorde (2006), Springer-Verlag, Heidelberg.
24. Prime Numbers, Friends Who Give Problems: a Trialogue with Papa Paulo (2016), World Scientific Publishing Co., Singapore.

BOOKS IN PREPARATION.

25. Theory of Ultrametric Spaces – In collaboration with S. Priess-Crampe (to appear in 2018), Springer, New York.
26. Collected Works in Ordered Structures and Mathematical Logic (2 volumes) (to appear in 2018), Springer, New York.

BOOKS EDITED.
List of collected papers in the series Queen's Papers in Pure and Applied Mathematics (QPPAM). 1989 - T. Morishima - No. 84; 1992 - K.E. Aubert – No. 89; 1992 - K. Inkeri - No. 91; 1994 - S. Susuki - No. 97; 1995 - P. Samuel (2 vols) - No. 99;1995 - R. Torelli - No. 101; 1996 - G. Albanese - No. 103; 1997 - P. Ribenboim (7 vols) - No. 106; 1998 - N. Alling - No. 107; 1999 - - M. Fiorentini - No. 113; 2001 - W. L. McDaniel - No. 122; 2002 - P. Roquette (3 vols) - No. 118; 2002 - T. Nagell (4 vols) - No. 121; 2003 - W. Ljunggren (2 vols) - No. 115; 2005 - L. Moser - No. 125; W. Krull (2vols) W. de Gruyter, Bonn; 2000

SPECIAL DISTINCTIONS.
---1938 - 1942 – Awarded the Gold Medal for Best Pupil in 5 consecutive years in Colégio Anglo-Americano.
---1957 – At age 29, elected member of the Brazilian Academy of Sciences.
---1961 – Bolzano Medal.
---1969 – At age 41, elected Fellow of the Royal Society of Canada (FRSC).
---1971 – Appointed Associate Member of the “Centro Superiore di Logica e Scienze Comparate”, in Bologna, Italy.
---1977 – Received Honoris Causa Doctoral Degree from the Université de Caen, France.
---1983 – Prize for Excellence in Research from Queen's University.
---1985 – Medal from the University Ján Mazaryk in Brno, at the occasion of the creation of the Czech Republic.
---1988 – The magazine Choice designated the book “The Book of Prime Records” as the best published mathematical book in the year 1988.
---1990 – Medal from Sociedade Portuguesa de Matemática.
---1991 – Special Session at the occasion of retirement, during the Third Meeting of the Canadian Number Theory Association, at Queen's University.
---1993 – Medal from the Universidad Complutense de Madrid.
---1998 – The Biennial Prize of the Canadian Number Theory Association was named the Ribenboim Prize. It is a prize for young number theorists who are Canadian or have connection to Canadian Mathematics. The first winner (1999) was Andrew Granville, then Henri Darmon (2002), Michael Bennett (2004), Vinayak Vatsal (2006), Adrian Iovita (2008), Valentin Blomer (2010), Dragos Ghioca (2012), Florian Herzig (2014), and Jacob Tsimerman (2016).
---1995 – Awarded the George Pólya Award from the Mathematical Association of America, given for the 1994 paper, “Prime Number Records”.
---2006 – Paulo Ribenboim was included in the list “Mathematicians of the Day” from the MacTutor Site from University of Saint Andrews, in Scotland.
---2008 – On the occasion of Paulo's 80th birthday, the Maine-Québec Number Theory. Conference was held in his honour at the Université Laval in Quebec City.


Def

HARDY FIELDS, THE INTERMEDIATE VALUE PROPERTY, AND ω-FREENESS

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

Abstract. We discuss the conjecture that every maximal Hardy field has the Intermediate Value Property for differential polynomials, and its equivalence to the statement that all maximal Hardy field are elementarily equivalent to the differential field of transseries. As a modest but essential step towards establishing the conjecture we show that every maximal Hardy field is ω-free.

Introduction

Du Bois Reymond's "orders of infinity" were put on a firm basis by Hardy [8] and Hausdorff [9], leading to the notion of a Hardy field (Bourbaki [6]). A Hardy field is a field H of germs at $+\infty$ of differentiable real-valued functions on intervals $(a, +\infty)$ such that the germ of the derivative of any differentiable function whose germ is in H is also in H. (See Section 2 for more precision.) A Hardy field is naturally a differential field, and is an ordered field with the germ of f being > 0 if and only if $f(t) > 0$, eventually.

If H is a Hardy field, then so is $H(\mathbb{R})$ (obtained by adjoining the germs of the constant functions) and for any $h \in H$, the germ e^h generates a Hardy field $H(\mathrm{e}^h)$ over H, and so does any differentiable germ with derivative h [13]. Each Hardy field H has a unique Hardy field extension that is algebraic over H and real closed [12]. The ultimate extension result of this kind would be the following:

Conjecture. *Let H be a Hardy field, $P(Y) \in H\{Y\}$ a differential polynomial and $f < g$ in H such that $P(f) < 0 < P(g)$. Then there is an element ϕ in a Hardy field extension of H such that $f < \phi < g$ and $P(\phi) = 0$.*

In [7] this is proved for P of order 1. In [11] it is shown that there do exist Hardy fields with the intermediate value property for all differential polynomials. Every Hardy field extends to a maximal Hardy field, by Zorn, and so the Conjecture above is equivalent to maximal Hardy fields having the intermediate value property for differential polynomials. By the results mentioned earlier, maximal Hardy fields contain $\mathbb{R}$ as a subfield and are Liouville closed in the sense of [2]. At the end of Section 1 we show that for Liouville closed Hardy fields containing $\mathbb{R}$ the intermediate value property is equivalent to the conjunction of two other properties, ω-*freeness* and *newtonianity*. These two notions are central in [2] in a more general setting. Roughly speaking, ω-freeness controls the solvability of second-order homogeneous linear differential equations in suitable extensions, and newtonianity is a very strong version of differential-henselianity. (We did not consider the intermediate value property in [2] and mention it here mainly for expository reasons: it

Date: March 2019.
The first-named author was partially supported by NSF Grant DMS-1700439.

is easier to grasp than the more subtle and more fundamental notions of $\upomega$-freeness and newtonianity.)

The main result in these seminar notes is that any Hardy field has an $\upomega$-free Hardy field extension: Theorem 3.1. We do not present it here, but we also have a detailed outline for showing that any $\upomega$-free Hardy field extends to a newtonian $\upomega$-free Hardy field. At this stage (March 2019) that proof is not yet finished. If we finish the proof, it would follow that every maximal Hardy field is an $\upomega$-free newtonian Liouville closed H-field with small derivation, in the terminology of [2]. Now the elementary theory $T^{\text{nl}}_{\text{small}}$ of $\upomega$-free newtonian Liouville closed H-fields with small derivation is *complete*, by [2, Corollary 16.6.3]. Thus finishing the proof alluded to would give that any two maximal Hardy fields are indistinguishable as to their elementary properties, that is, any two maximal Hardy fields would be elementarily equivalent as ordered differential fields.

The present seminar notes prove some results announced in our exposition [3]. There we also discuss another fundamental conjecture and partial results towards it, namely that the underlying ordered set of any maximal Hardy field is η_1. Our plan for proving it does depend on first establishing the conjecture that maximal Hardy fields are newtonian. The two conjectures together imply: all maximal Hardy fields are isomorphic as ordered differential fields, assuming the continuum hypothesis (CH); for more on this, see [3].

Let us add here a remark about maximal Hardy fields that is more set-theoretic in nature. Every Hardy field is contained in the ring $\mathcal{C}$ of germs at $+\infty$ of continuous real-valued functions on half-lines $(a, +\infty)$, and so there *at most* $2^{\mathfrak{c}}$ many Hardy fields, where $\mathfrak{c} = 2^{\aleph_0}$ is the cardinality of the continuum; note that $\mathfrak{c}$ is also the cardinality of $\mathcal{C}$. It is worth mentioning that the two conjectures above imply that there are in fact $2^{\mathfrak{c}}$ many different maximal Hardy fields: In an email to one of the authors, Ilijas Farah showed that there are $2^{\mathfrak{c}}$ many maximal Hausdorff fields, Hausdorff fields being the subfields of the ring $\mathcal{C}$ (without differentiability assumptions as in the case of Hardy fields). Farah's proof can easily be modified to give the same conclusion about the number of maximal Hardy fields assuming these conjectures.

Throughout we use the algebraic and valuation-theoretic tools from [2]. We need in addition analytic facts about real and complex solutions of linear differential equations; these facts and various generalities about Hardy fields are in Section 2.

The second-named author gave a talk about the above material in the *Séminaire de structures algébriques ordonnées* in honor of Paulo Ribenboim's 90th birthday. We dedicate this paper to Paulo in gratitude for his fundamental contributions to the theory of valuations, which is indispensable in our work.

Notations and terminology. Throughout, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. Given an additively written abelian group A we let $A^{\neq} := A \setminus \{0\}$. Rings are commutative with identity 1, and for a ring R we let $R^{\times}$ be the multiplicative group of units (consisting of the $a \in R$ such that $ab = 1$ for some $b \in R$). A *differential ring* will be a ring R containing (an isomorphic copy of) $\mathbb{Q}$ as a subring and equipped with a derivation $\partial\colon R \to R$; note that then $C_R := \big\{a \in R : \partial(a) = 0\big\}$ is a subring of R, called the ring of constants of R, and that $\mathbb{Q} \subseteq C_R$. If R is a field, then so is C_R. An *ordered differential field* is an ordered field equipped with a derivation; such an ordered differential field is in particular a differential ring.

Let R be a differential ring and $a \in R$. When its derivation ∂ is clear from the context we denote $\partial(a), \partial^2(a), \ldots, \partial^n(a), \ldots$ by $a', a'', \ldots, a^{(n)}, \ldots$, and if $a \in R^\times$, then $a^\dagger$ denotes a'/a, so $(ab)^\dagger = a^\dagger + b^\dagger$ for $a, b \in R^\times$. In Sections 1 and 3 we need to consider the function $\omega = \omega_R \colon R \to R$ given by $\omega(z) = -2z' - z^2$, and the function $\sigma = \sigma_R \colon R^\times \to R$ given by $\sigma(y) = \omega(z) + y^2$ for $z := -y^\dagger$.

We have the differential ring $R\{Y\} = R[Y, Y', Y'', \ldots]$ of differential polynomials in an indeterminate Y. We say that $P = P(Y) \in R\{Y\}$ has order at most $r \in \mathbb{N}$ if $P \in R[Y, Y', \ldots, Y^{(r)}]$; in this case $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$, as in [2, Section 4.2], with $\boldsymbol{i}$ ranging over tuples $(i_0, \ldots, i_r) \in \mathbb{N}^{1+r}$, $Y^{\boldsymbol{i}} := Y^{i_0}(Y')^{i_1} \cdots (Y^{(r)})^{i_r}$, coefficients $P_{\boldsymbol{i}}$ in R, and $P_{\boldsymbol{i}} \neq 0$ for only finitely many $\boldsymbol{i}$. For $P \in R\{Y\}$ and $a \in R$ we let $P_{\times a}(Y) := P(aY)$. For $\phi \in R^\times$ we let R^ϕ be the *compositional conjugate of R by ϕ*: the differential ring with the same underlying ring as R but with derivation $\phi^{-1}\partial$ instead of ∂. We have an R-algebra isomorphism $P \mapsto P^\phi \colon R\{Y\} \to R^\phi\{Y\}$ such that $P^\phi(y) = P(y)$ for all $y \in R$; see [2, Section 5.7].

For a field K we have $K^\times = K^{\neq}$, and a (Krull) valuation on K is a surjective map $v \colon K^\times \to \Gamma$ onto an ordered abelian group Γ (additively written) satisfying the usual laws, and extended to $v \colon K \to \Gamma_\infty := \Gamma \cup \{\infty\}$ by $v(0) = \infty$, where the ordering on Γ is extended to a total ordering on Γ_∞ by $\gamma < \infty$ for all $\gamma \in \Gamma$. A *valued field* K is a field (also denoted by K) together with a valuation ring $\mathcal{O}$ of that field, and the corresponding valuation $v \colon K^\times \to \Gamma$ on the underlying field is such that $\mathcal{O} = \{a \in K : va \geqslant 0\}$ as explained in [2, Section 3.1].

Let K be a valued field with valuation ring $\mathcal{O}_K$ and valuation $v \colon K^\times \to \Gamma_K$. Then $\mathcal{O}_K$ is a local ring with maximal ideal $\smallo_K = \{a \in K : va > 0\}$ and residue field $\operatorname{res}(K) = \mathcal{O}_K/\smallo_K$. If $\operatorname{res}(K)$ has characteristic zero, then K is said to be of equicharacteristic zero. When the ambient valued field K is clear from the context, then we denote Γ_K, $\mathcal{O}_K$, $\smallo_K$, by Γ, $\mathcal{O}$, $\smallo$, respectively, and for $a, b \in K$ we set

$$\begin{aligned}
&a \asymp b \ :\Leftrightarrow\ va = vb, && a \preccurlyeq b \ :\Leftrightarrow\ va \geqslant vb, && a \prec b \ :\Leftrightarrow\ va > vb,\\
&a \succcurlyeq b \ :\Leftrightarrow\ b \preccurlyeq a, && a \succ b \ :\Leftrightarrow\ b \prec a, && a \sim b \ :\Leftrightarrow\ a - b \prec a.
\end{aligned}$$

It is easy to check that if $a \sim b$, then $a, b \neq 0$, and that $\sim$ is an equivalence relation on $K^\times$. We use *pc-sequence* to abbreviate *pseudocauchy sequence*, and $a_\rho \rightsquigarrow a$ indicates that the pc-sequence (a_ρ) pseudoconverges to a; see [2, Sections 2.2, 3.2]. As in [2], a *valued differential field* is a valued field K of equicharacteristic zero that is also equipped with a derivation $\partial \colon K \to K$, and an *ordered valued differential field* is a valued differential field K equipped with an ordering on K making K an ordered field.

1. H-Fields and IVP

We recall from [2, Introduction] that an *H-field* is an ordered differential field K with constant field C such that:

(H1) $\partial(a) > 0$ for all $a \in K$ with $a > C$;

(H2) $\mathcal{O} = C + \smallo$, where $\mathcal{O}$ is the convex hull of C in the ordered field K, and $\smallo$ is the maximal ideal of the valuation ring $\mathcal{O}$.

Let K be an H-field, and let $\mathcal{O}$ and $\smallo$ be as in (H2). Thus K is a valued field with valuation ring $\mathcal{O}$. The residue morphism $\mathcal{O} \to \operatorname{res}(K) = \mathcal{O}/\smallo$ restricts to an isomorphism $C \xrightarrow{\cong} \operatorname{res}(K)$. The valuation topology on K equals its order topology if $C \neq K$. We consider K as an $\mathcal{L}$-structure, where

$$\mathcal{L} := \{0, 1, +, -, \times, \partial, <, \preccurlyeq\}$$

is the language of ordered valued differential fields. The symbols $0, 1, +, -, \times, \partial, <$ are interpreted as usual in K, and $\preccurlyeq$ encodes the valuation: for $a, b \in K$,

$$a \preccurlyeq b \quad \Longleftrightarrow \quad a \in \mathcal{O}b.$$

An H-field K is said to be *Liouville closed* if it is real closed and for all $a \in K$ there exists $b \in K$ with $a = b'$ and also $b \in K^{\times}$ with $a = b^{\dagger}$.

Remarks on IVP. Ordered valued differential subfields of H-fields are called pre-H-fields, and are characterized in [2, Section 10.5]. Below we assume some familiarity with the H-asymptotic couple (Γ, ψ) of a pre-H-field K, as explained in [2], and properties of K based on those of (Γ, ψ), such as K having *asymptotic integration* and K having a *gap* [2, Sections 9.1, 9.2].

Let K be a pre-H-field. We say that K has *IVP* (the *Intermediate Value Property*) if for all $P(Y) \in K\{Y\}$ and $f < g$ in K with $P(f) < 0 < P(g)$ there is a $\phi \in K$ such that $f < \phi < g$ and $P(\phi) = 0$. Restricting this to P of order $\leqslant r$, where $r \in \mathbb{N}$, gives the notion of r-IVP. Thus K having 0-IVP is equivalent to K being real closed as an ordered field. In particular, if K has 0-IVP, then the H-asymptotic couple (Γ, ψ) of K is divisible. From [2, Section 2.4] recall our convention that $K^{>} = \{a \in K : a > 0\}$, and similarly with $<$ replacing $>$.

Lemma 1.1. *Suppose $\Gamma \neq \{0\}$ and K has* 1-IVP. *Then $\partial K = K$, $(K^{>})^{\dagger} = (K^{<})^{\dagger}$ is a convex subgroup of K, $\Psi := \{\psi(\gamma) : \gamma \in \Gamma^{\neq}\}$ has no largest element, and Ψ is convex in Γ.*

Proof. We have $y' = 0$ for $y = 0$, and y' takes arbitrarily large positive values in K as y ranges over $K^{>\mathcal{O}} = \{a \in K : a > \mathcal{O}\}$, since by [2, Lemma 9.2.6] the set $(\Gamma^{<})'$ is coinitial in Γ. Hence y' takes all positive values on $K^{>}$, and therefore also all negative values on $K^{<}$. Thus $\partial K = K$. Next, let $a, b \in K^{>}$, and suppose $s \in K$ lies strictly between $a^{\dagger}$ and $b^{\dagger}$. Then $s = y^{\dagger}$ for some $y \in K^{>}$ strictly between a and b; this follows by noting that for $y = a$ and $y = b$ the signs of $sy - y'$ are opposite.

Let $\beta \in \Psi$ and take $a \in K$ with $v(a') = \beta$. Then $a \succ 1$, since $a \preccurlyeq 1$ would give $v(a') > \Psi$. Hence for $\alpha = va < 0$ we have $\alpha + \alpha^{\dagger} = \beta$, so $\alpha^{\dagger} > \beta$. Thus Ψ has no largest element. Therefore the set Ψ is convex in Γ. $\square$

Thus the ordered differential field $\mathbb{T}_{\log}$ of logarithmic transseries, [2, Appendix A], does not have 1-IVP, although it is a newtonian ω-free H-field.

Does IVP imply that K is an H-field? No: take an $\aleph_0$-saturated elementary extension of $\mathbb{T}$ and let Δ be as in [2, Example 10.1.7]. Then the Δ-coarsening of K is a pre-H-field with IVP and nontrivial value group, and has a gap, but it is not an H-field. On the other hand:

Lemma 1.2. *Suppose K has* 1-IVP *and has no gap. Then K is an H-field.*

Proof. In [2, Section 11.8] we defined

$$\mathrm{I}(K) := \{y \in K : \ y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}.$$

Since K has no gap, we have

$$\partial \smallo \subseteq \mathrm{I}(K) = \{y \in K : \ y \preccurlyeq f' \text{ for some } f \in \smallo\}.$$

Also $\Gamma \neq \{0\}$, and so (Γ, ψ) has asymptotic integration by Lemma 1.1. We show that K is an H-field by proving $\mathrm{I}(K) = \partial\smallo$, so let $g \in \mathrm{I}(K)$, $g < 0$. Since $(\Gamma^{>})'$ has no least element we can take positive $f \in \smallo$ such that $f' \succ g$. Since $f' < 0$, this gives $f' < g$. Since $(\Gamma^{>})'$ is cofinal in Γ we can also take positive $h \in \smallo$ such that $h' \prec g$, which in view of $h' < 0$ gives $g < h'$. Thus $f' < g < h'$, and so 1-IVP yields $a \in \smallo$ with $g = a'$. □

We refer to Sections 11.6 and 14.2 of [2] for the definitions of *λ-freeness* and *r-newtonianity* ($r \in \mathbb{N}$). From the introduction we recall that $\omega(z) := -2z' - z^2$.

Lemma 1.3. *Suppose K is an H-field, $\Gamma \neq \{0\}$, and K has* 1-IVP. *Then K is λ-free and* 1*-newtonian, and the subset $\omega(K)$ of K is downward closed.*

Proof. First we note that K has (asymptotic) integration, by Lemma 1.1. Assume towards a contradiction that K is not λ-free. We can arrange that K has small derivation, and thus K has an element $x \succ 1$ with $x' = 1$, and so $x > C$. This leads to a pc-sequence (λ_ρ) and an element $s \in K$ such that $\lambda_\rho \rightsquigarrow -s$ with $\lambda_\rho \sim x^{-1}$ for all ρ. Hence $s \sim -x^{-1}$, and s creates a gap over K by [2, Lemma 11.5.14]. Now note that for $P := Y' + sY$ we have $P(0) = 0$ and $P(x^2) = 2x + sx^2 \sim x$, so by 1-IVP we have $P(y) = 1$ for some $y \in K$, contradicting [2, Lemma 11.5.12].

Let $P \in K\{Y\}$ of order at most 1 have Newton degree 1; we have to show that P has a zero in $\mathcal{O}$. We know that K is λ-free, so by [2, Proposition 13.3.6] we can pass to an elementary extension, compositionally conjugate, and divide by an element of $K^\times$ to arrange that K has small derivation and $P = D + R$ where $D = cY + d$ or $D = cY'$ with $c, d \in C$, $c \neq 0$, and where $R \prec^\flat 1$. Then $R(a) \prec^\flat 1$ for all $a \in \mathcal{O}$. If $D = cY + d$, then we can take $a, b \in C$ with $D(a) < 0$ and $D(b) > 0$, which in view of $R(a) \prec D(a)$ and $R(b) \prec D(b)$ gives $P(a) < 0$ and $P(b) > 0$, and so P has a zero strictly between a and b, and thus a zero in $\mathcal{O}$. Next, suppose $D = cY'$. Then we take $t \in \smallo^{\neq}$ with $v(t^\dagger) = v(t)$, that is, $t' \asymp t^2$, so

$$P(t) \ = \ ct' + R(t), \quad P(-t) \ = \ -ct' + R(-t), \qquad R(t),\ R(-t) \ \prec \ t'.$$

Hence $P(t)$ and $P(-t)$ have opposite signs, so P has a zero strictly between t and $-t$, and thus P has a zero in $\mathcal{O}$.

From $\omega(z) = -z^2 - 2z'$ we see that $\omega(z) \to -\infty$ as $z \to +\infty$ and as $z \to -\infty$ in K, so $\omega(K)$ is downward closed by 1-IVP. □

For results involving 2-IVP we need a minor variant of [2, Lemma 11.8.31]. Here $\Gamma(K) = \{a^\dagger : a \in K \setminus \mathcal{O}\}$ as in [2, Section 11.8], and the superscripts ↑, ↓ indicate upward, respectively downward, closure, as in [2, Section 2.1].

Lemma 1.4. *Let K be an H-field with asymptotic integration. Then*

$$K^{>} \ = \ \mathrm{I}(K)^{>} \cup \Gamma(K)^{\uparrow}, \qquad \sigma\big(K^{>} \setminus \Gamma(K)^{\uparrow}\big) \ \subseteq \ \omega(K)^{\downarrow}.$$

Proof. If $a \in K$, $a > \mathrm{I}(K)$, then $a \geqslant b^\dagger$ for some $b \in K^{\succ 1}$, and thus $a \in \Gamma(K)^{\uparrow}$. The inclusion involving σ now follows as in the proof of [2, Lemma 11.8.13]. □

The concept of *ω-freeness* is introduced in [2, Section 11.7].

Lemma 1.5. *Suppose K is an H-field, $\Gamma \neq \{0\}$, and K has* 2-IVP. *Then K is* 2*-newtonian, the operator $\partial^2 - a$ splits over $K[\mathrm{i}]$ for all $a \in K$, and K is ω-free.*

Proof. Let $P \in K\{Y\}$ of order at most 2 have Newton degree 1; we have to show that P has a zero in $\mathcal{O}$. Lemma 1.3 tells us that K is λ-free, and in view of [2, Corollary 13.3.7] and 2-IVP this allows us to repeat the argument in the proof of that lemma for differential polynomials of order at most 1 so that it applies to our P of order at most 2. Thus K is 2-newtonian.

By [2, Section 5.2] it remains to show that $K = \omega(K) \cup \sigma(K^\times)$. In view of Lemma 1.1 we can arrange by compositional conjugation that $a^\dagger = -1$ for some $a \prec 1$ in $K^>$. Below we fix such a. Let $f \in K$; our job is to show that $f \in \omega(K) \cup \sigma(K^\times)$. Since $\omega(0) = 0$, we do have $f \in \omega(K)$ if $f \leqslant 0$, by Lemma 1.3. So assume $f > 0$; we show that then $f \in \sigma(K^>)$. Now for $y \in K^>$, $f = \sigma(y)$ is equivalent (by multiplying with y^2) to $P(y) = 0$, where

$$P(Y) \ := \ 2YY'' - 3(Y')^2 + Y^4 - fY^2 \in K\{Y\}.$$

See also [2, Section 13.7]. We have $P(0) = 0$ and $P(y) \to +\infty$ as $y \to +\infty$ (because of the term y^4). In view of 2-IVP it will suffice to show that for some $y > 0$ in K we have $P(y) < 0$. Now with $y \in K^>$ and $z := -y^\dagger$ we have

$$P(y) = y^2\big(\sigma(y) - f\big) = y^2\big(\omega(z) + y^2 - f\big),$$

hence

$$P(a) = a^2\big(\omega(1) + a^2 - f\big) = a^2(-1 + a^2 - f) < 0.$$

As to ω-freeness, this now follows from Lemma 1.4 and [2, Corollary 11.8.30]. $\square$

It follows that Liouville closed H-fields having 2-IVP are Schwarz closed as defined in [2, Section 11.8]. (There exist H-fields with a non-trivial derivation that have IVP but are not Liouville closed; see [1, Section 14].)

Corollary 1.6. *Suppose K is an H-field, $\Gamma \neq \{0\}$, and K has* IVP. *Then K is ω-free and newtonian.*

Proof. Showing that every $P \in K\{Y\}$ of Newton degree 1 has a zero in $\mathcal{O}$ is done just as in the proof of Lemma 1.3. $\square$

Corollary 1.7. *Let K be a Liouville closed H-field. Then*

$$K \text{ has IVP} \iff K \text{ is } \omega\text{-free and newtonian.}$$

Proof. The forward direction is part of Corollary 1.6. For the backward direction we appeal to the main results from the book [10] to the effect that $\mathbb{T}_{\mathrm{g}}$, the ordered differential field of grid-based transseries (cf. [2, Appendix A]), is a newtonian Liouville closed H-field with small derivation, and has IVP. In particular, it is a model of the theory $T^{\mathrm{nl}}_{\mathrm{small}}$, which we mentioned in the introduction. This theory is complete by [2, Corollary 16.6.3], so every model of it has IVP. If K is ω-free and newtonian but its derivation is not small, then it nevertheless has IVP: some compositional conjugate K^ϕ with $\phi \in K^>$ has small derivation and is Liouville closed, ω-free and newtonian. $\square$

2. Preliminaries on Hardy Fields

We begin with some results from Boshernitzan [5] on ordered fields of germs of continuous functions. Next we prove some easy facts about extending ordered fields inside an ambient partially ordered ring, as needed later.

Germs of continuous functions. As in [2, Section 9.1] we let $\mathcal{G}$ be the ring of germs at $+\infty$ of real-valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a,+\infty)$, $a\in\mathbb{R}$; the domain may vary and the ring operations are defined as usual. If $g\in\mathcal{G}$ is the germ of a real-valued function on a subset of $\mathbb{R}$ containing an interval $(a,+\infty)$, $a\in\mathbb{R}$, then we simplify notation by letting g also denote this function if the resulting ambiguity is harmless. With this convention, given a property P of real numbers and $g\in\mathcal{G}$ we say that $P\big(g(t)\big)$ *holds eventually* if $P\big(g(t)\big)$ holds for all sufficiently large real t. We identify each real number r with the germ at $+\infty$ of the function $\mathbb{R}\to\mathbb{R}$ that takes the constant value r. This makes the field $\mathbb{R}$ into a subring of $\mathcal{G}$. We call a germ $g\in\mathcal{G}$ *continuous* if it is the germ of a continuous function $(a,+\infty)\to\mathbb{R}$ for some $a\in\mathbb{R}$, and we let $\mathcal{C}\supseteq\mathbb{R}$ be the subring of $\mathcal{G}$ consisting of the continuous germs $g\in\mathcal{C}$. We let x denote the germ at $+\infty$ of the identity function on $\mathbb{R}$.

Asymptotic relations on $\mathcal{C}$. Note that the multiplicative group $\mathcal{C}^\times$ of $\mathcal{C}$ consists of the $f\in\mathcal{C}$ such that $f(t)\neq 0$, eventually. Thus for $f\in\mathcal{C}^\times$, either $f(t)>0$, eventually, or $f(t)<0$, eventually. Although $\mathcal{C}$ is not a valued field, it will be convenient to equip $\mathcal{C}$ with the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ (which are defined on any valued field) as follows: for $f,g\in\mathcal{C}$,

$$\begin{aligned}
f\preccurlyeq g \;&:\Longleftrightarrow\; \text{there exists } c\in\mathbb{R}^{>} \text{ such that eventually } |f(t)|\leqslant c|g(t)|,\\
f\prec g \;&:\Longleftrightarrow\; g\in\mathcal{C}^\times \text{ and } \lim_{t\to\infty} f(t)/g(t)=0,\\
f\sim g \;&:\Longleftrightarrow\; g\in\mathcal{C}^\times \text{ and } \lim_{t\to\infty} f(t)/g(t)=1\\
&\;\Longleftrightarrow\; f-g\prec g.
\end{aligned}$$

Thus $\preccurlyeq$ is a transitive and reflexive binary relation on $\mathcal{C}$, and $\sim$ is an equivalence relation on $\mathcal{C}^\times$. Moreover, for $f,g,h\in\mathcal{C}$ we have

$$f\prec g\;\Rightarrow\; f\preccurlyeq g,\qquad f\preccurlyeq g\prec h\;\Rightarrow\; f\prec h,\qquad f\prec g\preccurlyeq h\;\Rightarrow\; f\prec h.$$

Note that $\prec$ is a transitive binary relation on $\mathcal{C}$. For $f,g\in\mathcal{C}$ we also set

$$f\asymp g :\Leftrightarrow\; f\preccurlyeq g \text{ and } g\preccurlyeq f,\qquad f\succcurlyeq g :\Leftrightarrow\; g\preccurlyeq f,\qquad f\succ g :\Leftrightarrow\; g\prec f,$$

so $\asymp$ is an equivalence relation on $\mathcal{C}$.

Subfields of $\mathcal{C}$. Let K be a subfield of $\mathcal{C}$, that is, a subring of $\mathcal{C}$ that happens to be a field. (In the introduction we called such K a *Hausdorff field*.) Then K itself has the subfield $K\cap\mathbb{R}$. Every nonzero $f\in K$ has a multiplicative inverse in K, so eventually $f(t)\neq 0$, hence either eventually $f(t)<0$ or eventually $f(t)>0$ (by eventual continuity of f). We make K an ordered field by declaring

$$f>0\;:\Longleftrightarrow\; f(t)>0,\text{ eventually.}$$

We now have [5, Propositions 3.4 and 3.6]:

Lemma 2.1. *Let K^{rc} consist of the $y\in\mathcal{C}$ with $P(y)=0$ for some $P(Y)\in K[Y]^{\neq}$. Then K^{rc} is the unique real closed subfield of $\mathcal{C}$ that extends K and is algebraic over K. In particular, K^{rc} is a real closure of the ordered field K.*

In [5] this lemma assumes $K\supseteq\mathbb{R}$, but this is not really needed in the proof. The ordered field K has a convex subring

$$\mathcal{O}\;=\;\big\{f\in K:\;|f|\leqslant n\text{ for some } n\big\},$$

which is a valuation ring of K, and we consider K accordingly as a valued ordered field. Restricting $\preccurlyeq$, $\prec$, $\sim$ from the previous subsection to K gives exactly the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on K that it comes equipped with as a valued field.

Composition and compositional inversion. Let $g \in \mathcal{C}$ be eventually strictly increasing with $\lim_{t\to+\infty} g(t) = +\infty$. Then its compositional inverse $g^{\text{inv}} \in \mathcal{C}$ is given by $g^{\text{inv}}\big(g(t)\big) = t$, eventually, and the composition operation

$$f \mapsto f \circ g \ : \ \mathcal{C} \to \mathcal{C}, \qquad (f \circ g)(t) \ := \ f\big(g(t)\big) \ \text{ eventually},$$

is an automorphism of the ring $\mathcal{C}$ that is the identity on the subring $\mathbb{R}$, with inverse $f \mapsto f \circ g^{\text{inv}}$. In particular, $g \circ g^{\text{inv}} = g^{\text{inv}} \circ g = x$, and $f \mapsto f \circ g$ maps each subfield K of $\mathcal{C}$ isomorphically (as an ordered field) onto the subfield $K \circ g$ of $\mathcal{C}$. Note that if the subfield K of $\mathcal{C}$ contains x, then $K \circ g$ contains g.

Extending ordered fields inside an ambient partially ordered ring. Let R be a commutative ring with $1 \neq 0$, equipped with a translation-invariant partial ordering $\leqslant$ such that $r^2 \geqslant 0$ for all $r \in R$, and $rs \geqslant 0$ for all $r, s \in R$ with $r, s \geqslant 0$. It follows that for $a, b, r \in R$ we have: if $a \leqslant b$ and $r \geqslant 0$, then $ar \leqslant br$; if a is a unit and $a > 0$, then $a^{-1} = a \cdot (a^{-1})^2 > 0$; if a, b are units, and $0 < a \leqslant b$, then $0 < b^{-1} \leqslant a^{-1}$. Relevant cases: $R = \mathcal{G}$ and $R = \mathcal{C}$, with partial ordering given by

$$f \leqslant g \ :\Longleftrightarrow \ f(t) \leqslant g(t), \ \text{eventually}.$$

Call a subset K of R *totally ordered* if the partial ordering of R induces a total ordering on K. An *ordered subfield of* R is a subfield K of R that is totally ordered as a subset of R; note that then K equipped with the induced partial ordering is indeed an ordered field, in the usual sense of that term. (Thus any subfield of $\mathcal{C}$ with the above partial ordering is an ordered subfield of $\mathcal{C}$.)

We identify $\mathbb{Z}$ with its image in R via the unique ring embedding $\mathbb{Z} \to R$, and this makes $\mathbb{Z}$ with its usual ordering into an ordered subring of R.

Lemma 2.2. *Assume D is a totally ordered subring of R and every nonzero element of D is a unit of R. Then D generates an ordered subfield* $\operatorname{Frac} D$ *of R.*

Proof. It is clear that D generates a subfield $\operatorname{Frac} D$ of R. For $a \in D$, $a > 0$, we have $a^{-1} > 0$. It follows that $\operatorname{Frac} D$ is totally ordered. □

Thus if every $n \geqslant 1$ is a unit of R, then we may identify $\mathbb{Q}$ with its image in R via the unique ring embedding $\mathbb{Q} \to R$, making $\mathbb{Q}$ into an ordered subfield of R.

Lemma 2.3. *Suppose K is an ordered subfield of R, all $g \in R$ with $g > K$ are units of R, and $K < f \in R$. Then we have an ordered subfield $K(f)$ of R.*

Proof. For $P(Y) \in K[Y] \setminus K$ of degree $d \geqslant 1$ with leading coefficient $a > 0$ we have $P(f) = af^d(1 + \varepsilon)$ with $-1/n < \varepsilon < 1/n$ for all $n \geqslant 1$, in particular, $P(f) > K$ is a unit of R. It remains to appeal to Lemma 2.2. □

Lemma 2.4. *Assume K is a real closed ordered subfield of R. Let A be a nonempty downward closed subset of K such that A has no largest element and $B := K \setminus A$ is nonempty and has no least element. Let $f \in R$ be such that $A < f < B$. Then the subring $K[f]$ has the following properties:*

(i) *$K[f]$ is a domain;*
(ii) *$K[f]$ is totally ordered;*
(iii) *K is cofinal in $K[f]$;*

(iv) *for all* $g \in K[f] \setminus K$ *and* $a \in K$, *if* $a < g$, *then* $a < b < g$ *for some* $b \in K$, *and if* $g < a$, *then* $g < b < a$ *for some* $b \in K$.

Proof. Let $P \in K[Y] \setminus K$; to obtain (i) and (ii) it suffices to show that then $P(f) < 0$ or $P(f) > 0$. We have

$$P(Y) \;=\; cQ(Y)(Y - a_1)\cdots(Y - a_n)$$

where $c \in K^{\neq}$, $Q(Y)$ is a product of monic quadratic irreducibles in $K[Y]$, and $a_1, \dots, a_n \in K$. This gives $\delta \in K^{>}$ such that $Q(r) \geqslant \delta$ for all $r \in R$. Assume $c > 0$. (The case $c < 0$ is handled similarly.) We can arrange that $m \leqslant n$ is such that $a_i \in A$ for $1 \leqslant i \leqslant m$ and $a_j \in B$ for $m < j \leqslant n$. Take $\varepsilon > 0$ in K such that $a_i + \varepsilon \leqslant f$ for $1 \leqslant i \leqslant m$ and $f \leqslant a_j - \varepsilon$ for $m < j \leqslant n$. Then

$$P(f) \;=\; cQ(f)(f - a_1)\cdots(f - a_m)(f - a_{m+1})\cdots(f - a_n),$$

and $(f - a_1)\cdots(f - a_m) \geqslant \varepsilon^m$. If $n - m$ is even, then $(f - a_{m+1})\cdots(f - a_n) \geqslant \varepsilon^{n-m}$, so $P(f) \geqslant a\delta\varepsilon^n > 0$. If $n - m$ is odd, then $(f - a_{m+1})\cdots(f - a_n) \leqslant -\varepsilon^{n-m}$, so $P(f) \leqslant -a\delta\varepsilon^n < 0$. These estimates also yield (iii) and (iv). □

Lemma 2.5. *With* K, A, f *as in Lemma 2.4, suppose all* $g \in R$ *with* $g \geqslant 1$ *are units of* R. *Then we have an ordered subfield* $K(f)$ *of* R *such that* (iii) *and* (iv) *of Lemma 2.4 go through for* $K(f)$ *in place of* $K[f]$.

Proof. Note that if $g \in R$ and $g \geqslant \delta \in K^{>}$, then $g\delta^{-1} \geqslant 1$, so g is a unit of R and $0 < g^{-1} \leqslant \delta^{-1}$. For $Q \in K[Y]^{\neq}$ with $Q(f) > 0$ we can take $\delta \in K^{>}$ such that $Q(f) \geqslant \delta$, and thus $Q(f)$ is a unit of R and $0 < Q(f)^{-1} \leqslant \delta^{-1}$. Thus we have an ordered subfield $K(f)$ of R by Lemma 2.2, and the rest now follows easily. □

Adjoining pseudolimits and increasing the value group. Let K be a real closed subfield of $\mathcal{C}$, and view K as an ordered valued field as before. Let (a_ρ) be a strictly increasing divergent pc-sequence in K. Set

$$A \;:=\; \{a \in K:\; a < a_\rho \text{ for some } \rho\}, \qquad B \;:=\; \{b \in K:\; b > a_\rho \text{ for all } \rho\},$$

so A is nonempty and downward closed without a largest element. Moreover, $B = K \setminus A$ is nonempty and has no least element, since a least element of B would be a limit and thus a pseudolimit of (a_ρ). Let $f \in \mathcal{C}$ satisfy $A < f < B$. Then we have an ordered subfield $K(f)$ of $\mathcal{C}$, and:

Lemma 2.6. $K(f)$ *is an immediate valued field extension of* K *with* $a_\rho \rightsquigarrow f$.

Proof. We can assume that $v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$ for all indices $\tau > \sigma > \rho$. Set $d_\rho := a_{s(\rho)} - a_\rho$ ($s(\rho) :=$ successor of ρ). Then $a_\rho + 2d_\rho \in B$ for all indices ρ; see the discussion preceding [2, Lemma 2.4.2]. It then follows from that lemma that $a_\rho \rightsquigarrow f$. Now (a_ρ) is a divergent pc-sequence in the henselian valued field K, so it is of transcendental type over K, and thus $K(f)$ is an immediate extension of K. □

Lemma 2.7. *Suppose* K *is a subfield of* $\mathcal{C}$ *with divisible value group* $\Gamma = v(K^\times)$. *Let* P *be a nonempty upward closed subset of* Γ, *and let* $f \in \mathcal{C}$ *be such that* $a < f$ *for all* $a \in K^{>}$ *with* $va \in P$, *and* $f < b$ *for all* $b \in K^{>}$ *with* $vb < P$. *Then* f *generates a subfield* $K(f)$ *of* $\mathcal{C}$, *with* $P > vf > Q$, $Q := \Gamma \setminus P$.

Proof. For any positive $a \in K^{\text{rc}}$ there is $b \in K^{>}$ with $a \asymp b$ and $a < b$, and also an element $b \in K^{>}$ with $a \asymp b$ and $a > b$. Thus we can replace K by K^{rc} and arrange in this way that K is real closed. Set

$$A \;:=\; \{a \in K:\; a \leqslant 0 \text{ or } va \in P\}, \qquad B := K \setminus A.$$

Then we are in the situation of Lemma 2.4 for $R = \mathcal{C}$, so by that lemma and Lemma 2.5 we have an ordered subfield $K(f)$ of $\mathcal{C}$. Clearly then $P > vf > Q$. $\square$

Notational conventions on functions and germs. Let r range over $\mathbb{N} \cup \{\infty\}$, and let U be a nonempty open subset of $\mathbb{R}$. Then $\mathcal{C}^r(U)$ denotes the $\mathbb{R}$-algebra of r-times continuously differentiable functions $U \to \mathbb{R}$, with the usual pointwise defined algebra operations. (We use "$\mathcal{C}$" instead of "C" since C will often denote the constant field of a differential field.) For $r = 0$ this is the $\mathbb{R}$-algebra $\mathcal{C}(U)$ of continuous real-valued functions on U, so

$$\mathcal{C}(U) \ = \ \mathcal{C}^0(U) \ \supseteq \ \mathcal{C}^1(U) \ \supseteq \ \mathcal{C}^2(U) \ \supseteq \ \cdots \ \supseteq \ \mathcal{C}^\infty(U).$$

For $r \geqslant 1$ we have the derivation $f \mapsto f' \colon \mathcal{C}^r(U) \to \mathcal{C}^{r-1}(U)$ (with $\infty - 1 := \infty$). This makes $\mathcal{C}^\infty(U)$ a differential ring, with its subalgebra $\mathcal{C}^\omega(U)$ of real-analytic functions $U \to \mathbb{R}$ as a differential subring. The algebra operations on the algebras below are also defined pointwise.

Let a range over $\mathbb{R}$. Then $\mathcal{C}^r_a$ denotes the $\mathbb{R}$-algebra of functions $[a, +\infty) \to \mathbb{R}$ that extend to a function in $\mathcal{C}^r(U)$ for some open $U \supseteq [a, +\infty)$. Thus $\mathcal{C}^0_a$ is the $\mathbb{R}$-algebra of real-valued continuous functions on $[a, +\infty)$, and

$$\mathcal{C}^0_a \ \supseteq \ \mathcal{C}^1_a \ \supseteq \ \mathcal{C}^2_a \ \supseteq \ \cdots \ \supseteq \mathcal{C}^\infty_a.$$

We also have the subalgebra $\mathcal{C}^\omega_a$ of $\mathcal{C}^\infty_a$, consisting of the functions $[a, +\infty) \to \mathbb{R}$ that extend to a real-analytic function $U \to \mathbb{R}$ for some open $U \supseteq [a, +\infty)$. For $r \geqslant 1$ we have the derivation $f \mapsto f' \colon \mathcal{C}^r_a \to \mathcal{C}^{r-1}_a$. This makes $\mathcal{C}^\infty_a$ a differential ring with $\mathcal{C}^\omega_a$ as a differential subring.

For each of the algebras A above we also consider its complexification $A[\mathrm{i}]$ which consists by definition of the $\mathbb{C}$-valued functions $f = g + h\mathrm{i}$ with $g, h \in A$, so $g = \operatorname{Re} f$ and $h = \operatorname{Im} f$ for such f. We consider $A[\mathrm{i}]$ as a $\mathbb{C}$-algebra with respect to the natural pointwise defined algebra operations. We identify each complex number with the corresponding constant function to make $\mathbb{C}$ a subfield of $A[\mathrm{i}]$ and $\mathbb{R}$ a subfield of A. (This justifies the notation $A[\mathrm{i}]$.) For $r \geqslant 1$ we extend $g \mapsto g' \colon \mathcal{C}^r_a \to \mathcal{C}^{r-1}_a$ to the derivation

$$g + h\mathrm{i} \mapsto g' + h'\mathrm{i} \ : \ \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}^{r-1}_a[\mathrm{i}] \qquad (g, h \in \mathcal{C}^r_a[\mathrm{i}]),$$

which for $r = \infty$ makes $\mathcal{C}^\infty_a$ a differential subring of $\mathcal{C}^\infty_a[\mathrm{i}]$. We also use the map

$$f \mapsto f^\dagger := f'/f \ : \ \mathcal{C}^1_a[\mathrm{i}]^\times = \big(\mathcal{C}^1_a[\mathrm{i}]\big)^\times \to \mathcal{C}^0_a[\mathrm{i}],$$

with

$$(fg)^\dagger = f^\dagger + g^\dagger \qquad \text{for } f, g \in \mathcal{C}^1_a[\mathrm{i}]^\times,$$

in particular the fact that $f \in \mathcal{C}^1_a[\mathrm{i}]^\times$ and $f^\dagger \in \mathcal{C}^0_a[\mathrm{i}]$ are related by

$$f(t) \ = \ f(a) \exp\left[\int_a^t f^\dagger(s)\, ds\right] \qquad (t \geqslant a).$$

Let $\mathcal{C}^r$ be the partially ordered subring of $\mathcal{C}$ consisting of the germs at $+\infty$ of the functions in $\bigcup_a \mathcal{C}^r_a$; thus $\mathcal{C}^0 = \mathcal{C}$ consists of the germs at $+\infty$ of the continuous real valued functions on intervals $[a, +\infty)$, $a \in \mathbb{R}$. Note that $\mathcal{C}^r$ with its partial ordering satisfies the conditions on R from the previous subsection. Also, every $g \geqslant 1$ in $\mathcal{C}^r$ is a unit of $\mathcal{C}^r$, so Lemmas 2.3 and 2.5 apply to ordered subfields of $\mathcal{C}^r$. We have

$$\mathcal{C}^0 \ \supseteq \ \mathcal{C}^1 \ \supseteq \ \mathcal{C}^2 \ \supseteq \ \cdots \ \supseteq \ \mathcal{C}^\infty,$$

and we set $\mathcal{C}^{<\infty} := \bigcap_r \mathcal{C}^r$. Thus $\mathcal{C}^{<\infty}$ is naturally a differential ring with $\mathbb{R}$ as its ring of constants. Note that $\mathcal{C}^{<\infty}$ has $\mathcal{C}^\infty$ as a differential subring. The differential ring $\mathcal{C}^\infty$ has in turn the differential subring $\mathcal{C}^\omega$, whose elements are the germs at $+\infty$ of the functions in $\bigcup_a \mathcal{C}^\omega_a$.

Second-order differential equations. Let $f \in \mathcal{C}^0_a$, that is, $f\colon [a,\infty) \to \mathbb{R}$ is continuous. We consider the differential equation

$$Y'' + fY \;=\; 0.$$

The solutions $y \in \mathcal{C}^2_a$ form an $\mathbb{R}$-linear subspace $\operatorname{Sol}(f)$ of $\mathcal{C}^2_a$. The solutions $y \in \mathcal{C}^2_a[\mathrm{i}]$ are the $y_1 + y_2\mathrm{i}$ with $y_1, y_2 \in \operatorname{Sol}(f)$ and form a $\mathbb{C}$-linear subspace $\operatorname{Sol}_{\mathbb{C}}(f)$ of $\mathcal{C}^2_a[\mathrm{i}]$. For any complex numbers c, d there is a unique solution $y \in \mathcal{C}^2_a[\mathrm{i}]$ with $y(a) = c$ and $y'(a) = d$, and the map that assigns to $(c,d) \in \mathbb{C}^2$ this unique solution is an isomorphism $\mathbb{C}^2 \to \operatorname{Sol}_{\mathbb{C}}(f)$ of $\mathbb{C}$-linear spaces; it restricts to an $\mathbb{R}$-linear bijection $\mathbb{R}^2 \to \operatorname{Sol}(f)$. Induction on $r \in \mathbb{N}$ shows: $f \in \mathcal{C}^r_a \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}^{r+2}_a$. Thus $f \in \mathcal{C}^\infty_a \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}^\infty_a$. It is also well-known that $f \in \mathcal{C}^\omega_a \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}^\omega_a$. From [4, Chapter 2, Lemma 1] we recall:

Lemma 2.8 (Gronwall's Lemma). *Let the constant $C \in \mathbb{R}^{\geqslant}$ and the functions $v, y \in \mathcal{C}^0_a$ be such that $v(t), y(t) \geqslant 0$ for all $t \geqslant a$ and*

$$y(t) \;\leqslant\; C + \int_a^t v(s)y(s)\,ds \quad \text{for all } t \geqslant a.$$

Then

$$y(t) \;\leqslant\; C\exp\left[\int_a^t v(s)\,ds\right] \quad \text{for all } t \geqslant a.$$

In the rest of this subsection we assume that $a \geqslant 1$ and that $c \in \mathbb{R}^{>}$ is such that $|f(t)| \leqslant c/t^2$ for all $t \geqslant a$. Under this hypothesis, the lemma above yields the following bound on the growth of the solutions $y \in \operatorname{Sol}(f)$; the proof we give is similar to that of [4, Chapter 6, Theorem 5].

Proposition 2.9. *Let $y \in \operatorname{Sol}(f)$. Then there is $C \in \mathbb{R}^{\geqslant}$ such that $|y(t)| \leqslant Ct^{c+1}$ and $|y'(t)| \leqslant Ct^c$ for all $t \geqslant a$.*

Proof. Let t range over $[a, +\infty)$. Integrating $y'' = -fy$ twice between a and t, we obtain constants c_1, c_2 such that for all t,

$$y(t) \;=\; c_1 + c_2 t - \int_a^t \int_a^{t_1} f(t_2)y(t_2)\,dt_2\,dt_1 \;=\; c_1 + c_2 t - \int_a^t (t-s)f(s)y(s)\,ds$$

and hence, with $C := |c_1| + |c_2|$,

$$|y(t)| \;\leqslant\; Ct + t\int_a^t |f(s)| \cdot |y(s)|\,ds,$$

so

$$\frac{|y(t)|}{t} \;\leqslant\; C + \int_a^t s|f(s)| \cdot \frac{|y(s)|}{s}\,ds.$$

Hence by the lemma above,

$$\frac{|y(t)|}{t} \;\leqslant\; C\exp\left[\int_a^t s|f(s)|\,ds\right] \;\leqslant\; C\exp\left[\int_1^t c/s\,ds\right] \;=\; Ct^c$$

and thus $|y(t)| \leqslant Ct^{c+1}$. Now

$$y'(t) = c_2 - \int_a^t f(s)y(s)\,ds$$

and thus

$$|y'(t)| \ \leqslant\ |c_2| + \int_a^t |f(s)y(s)|\,ds \ \leqslant\ C + Cc\int_1^t s^{c-1}\,ds \ =\ C + Cc\left[\frac{t^c}{c} - \frac{1}{c}\right] \ =\ Ct^c.$$

□

Let $y_1, y_2 \in \operatorname{Sol}(f)$ be $\mathbb{R}$-linearly independent. The Wronskian $w := y_1y_2' - y_1'y_2$ satisfies $w' = 0$ (Abel's identity), so $w \in \mathbb{R}^\times$. It follows that y_1 and y_2 cannot be simultaneously very small:

Lemma 2.10. *There is a positive constant d such that*

$$\max\big(|y_1(t)|, |y_2(t)|\big) \ \geqslant\ dt^{-c} \qquad \text{for all } t \geqslant a.$$

Proof. Proposition 2.9 yields $C \in \mathbb{R}^>$ such that $|y_i'(t)| \leqslant Ct^c$ for $i = 1, 2$ and all $t \geqslant a$. Hence $|w| \leqslant 2\max\big(|y_1(t)|, |y_2(t)|\big)Ct^c$ for $t \geqslant a$, so

$$\max\big(|y_1(t)|, |y_2(t)|\big) \ \geqslant\ \frac{|w|}{2C}t^{-c} \qquad (t \geqslant a).$$

□

Corollary 2.11. *Set $y := y_1 + y_2\mathrm{i}$ and $z := y^\dagger$. Then for some $D \in \mathbb{R}^>$,*

$$|z(t)| \ \leqslant\ Dt^{2c} \qquad \text{for all } t \geqslant a.$$

Proof. Take C as in the proof of Lemma 2.10, and d as in that lemma. Then

$$|z(t)| \ =\ \frac{|y_1'(t) + y_2'(t)\mathrm{i}|}{|y_1(t) + y_2(t)\mathrm{i}|} \ \leqslant\ \frac{|y_1'(t)| + |y_2'(t)|}{\max\big(|y_1(t)|, |y_2(t)|\big)} \ \leqslant\ \left(\frac{2C}{d}\right)t^{2c}$$

for $t \geqslant a$. □

Changing variables. Let now K be a differential field, $f \in K$, and consider the differential polynomial $P(Y) := 4Y'' + fY$. (The factor 4 is to simplify certain expressions, in conformity with [2, Section 9.2].) Which "changes of variable" preserve the general form of P? Here is an answer:

Lemma 2.12. *For $g \in K^\times$ and $\phi := g^{-2}$ we have*

$$g^3 P^{\phi}_{\times g}(Y) \ =\ 4Y'' + g^3P(g)Y.$$

Proof. Let $g, \phi \in K^\times$. Then

$$\begin{aligned} P_{\times g}(Y) \ &=\ 4gY'' + 8g'Y' + (4g'' + fg)Y \ =\ 4gY'' + 8g'Y' + P(g)Y, \quad \text{so} \\ P^{\phi}_{\times g}(Y) \ &=\ 4g(\phi^2Y'' + \phi'Y') + 8g'\phi Y' + P(g)Y \\ &=\ 4g\phi^2Y'' + (4g\phi' + 8g'\phi)Y' + P(g)Y. \end{aligned}$$

Now $4g\phi' + 8g'\phi = 0$ is equivalent to $\phi^\dagger = -2g^\dagger$, which holds for $\phi = g^{-2}$. For this ϕ we get $P^{\phi}_{\times g}(Y) = g^{-3}\big(4Y'' + g^3P(g)Y\big)$, that is, $g^3P^{\phi}_{\times g}(Y) = 4Y'' + g^3P(g)Y$. □

Hardy fields. A *Hardy field* is a subfield of $\mathcal{C}^{<\infty}$ that is closed under the derivation of $\mathcal{C}^{<\infty}$. A Hardy field H is considered as an ordered valued differential field in the obvious way, and has $\mathbb{R} \cap H$ as its field of constants. Hardy fields are pre-H-fields, and H-fields if they contain $\mathbb{R}$. Here are some well-known extension results:

Proposition 2.13. *Any Hardy field H has the following Hardy field extensions:*

(i) *$H(\mathbb{R})$, the subfield of $\mathcal{C}^{<\infty}$ generated by H and $\mathbb{R}$;*
(ii) *H^{rc}, the real closure of H as defined in Lemma 2.1;*
(iii) *$H(\mathrm{e}^f)$ for any $f \in H$;*

(iv) *$H(f)$ for any $f \in \mathcal{C}^1$ with $f' \in H$;*
(v) *$H(\log f)$ for any $f \in H^>$.*

If H is contained in $\mathcal{C}^\infty$, then so are the Hardy fields in (i), (ii), (iii), (iv), (v)*; likewise with $\mathcal{C}^\omega$ instead of $\mathcal{C}^\infty$.*

Note that (v) is a special case of (iv), since $(\log f)' = f'/f \in H$ for $f \in H^>$. Another special case of (iv) is that $H(x)$ is a Hardy field. A consequence of the Proposition is that any Hardy field H has a smallest real closed Hardy field extension H^* with $\mathbb{R} \subseteq H^*$ such that for all $f \in H^*$ we have $e^f \in H^*$ and $g' = f$ for some $g \in H^*$. Note that then H^* is Liouville closed as defined in [2, Section 10.6].

We also have the following more general extension result from Rosenlicht [13], attributed there to M. Singer:

Proposition 2.14. *Let H be a Hardy field and $p(Y), q(Y) \in H[Y]$. Suppose $f \in \mathcal{C}^1$ is a solution of the differential equation $y'q(y) = p(y)$ and $q(f)$ is a unit of $\mathcal{C}^1$. Then f generates a Hardy field $H(f)$ over H.*

Compositional inversion and compositional conjugation in Hardy fields. Let H be a Hardy field, and let $g \in \mathcal{C}^1$ be such that $g > \mathbb{R}$ and $g' \in H$. Then we have a Hardy field $H(g)$, and the compositional inverse $g^{\text{inv}} \in \mathcal{C}^1$ of g satisfies

$$g^{\text{inv}} > \mathbb{R}, \qquad (g^{\text{inv}})' \;=\; (1/g') \circ g^{\text{inv}} \in H \circ g^{\text{inv}}$$

and yields an ordered field isomorphism

$$h \mapsto h \circ g^{\text{inv}} \;:\; H \to H \circ g^{\text{inv}}$$

such that for all $h \in H$,

$$(h \circ g^{\text{inv}})' \;=\; (h' \circ g^{\text{inv}}) \cdot (g^{\text{inv}})' \;=\; (h'/g') \circ g^{\text{inv}} \in H \circ g^{\text{inv}}.$$

Thus $H \circ g^{\text{inv}}$ is again a Hardy field, and for $\phi = g'$ this yields an isomorphism

$$h \mapsto h \circ g^{\text{inv}} \;:\; H^\phi \to H \circ g^{\text{inv}}$$

of pre-H-fields. If $H \subseteq \mathcal{C}^\infty$ and $g \in \mathcal{C}^\infty$, then $H \circ g^{\text{inv}} \subseteq \mathcal{C}^\infty$; likewise with $\mathcal{C}^\omega$ instead of $\mathcal{C}^\infty$. For later use, a *$\mathcal{C}^\infty$-Hardy field* is a Hardy field $H \subseteq \mathcal{C}^\infty$, and a *$\mathcal{C}^\omega$-Hardy field* (also called an *analytic Hardy field*) is a Hardy field $H \subseteq \mathcal{C}^\omega$.

3. Extending Hardy Fields to ω-free Hardy Fields

In this section we assume familiarity with [2, Sections 5.2, 11.5–11.8]. Here we summarize some of this material, and then use this to prove Theorem 3.1 below. In the *Notations and terminology* at the end of the introduction we defined for any differential ring R functions $\omega\colon R \to R$ and $\sigma\colon R^\times \to R$. We define likewise

$$\omega \;:\; \mathcal{C}^1_a \to \mathcal{C}^0_a, \qquad \sigma \;:\; (\mathcal{C}^1_a)^\times \to \mathcal{C}^0_a$$

by

$$\omega(z) \;=\; -2z' - z^2 \quad \text{ and } \quad \sigma(y) \;=\; \omega(z) + y^2 \text{ for } z := -y^\dagger.$$

To clarify this role of ω and σ in connection with second-order linear differential equations, let $f \in \mathcal{C}^0_a$ and consider the differential equation

$$4Y'' + fY \;=\; 0.$$

Suppose $y \in \mathcal{C}^2_a$ is a *non-oscillating* solution, that is, a solution with $y(t) \neq 0$ for all sufficiently large t, say for all $t \geqslant b$, where $b \geqslant a$. Then $z \in \mathcal{C}^1_b$ given by $z(t) = 2y'(t)/y(t)$ satisfies the first-order differential equation $-2z' - z^2 = f$

on $[b,\infty)$. Thus if the germ of f at $+\infty$ belongs to a Hardy field H, then by Proposition 2.14 the germ of z at $+\infty$ (also denoted by z) generates a Hardy field $H(z)$ with $\omega(z)=f$, which in turn yields a Hardy field $H(z,y)$ with y now denoting its germ at $+\infty$ so that $y\in(\mathcal{C}^{<\infty})^\times$ and $2y^\dagger=z$ in $\mathcal{C}^{<\infty}$. Thus $y_1:=y$ lies in a Hardy field extension of H. The germ y_2 of the function $[b,+\infty)\to\mathbb{R}$ given by $t\mapsto y(t)\int_b^t\frac{1}{y(s)^2}\,ds$ also satisfies $4y_2''+fy_2=0$, and y_1, y_2 are $\mathbb{R}$-linearly independent [4, Chapter 6, Lemma 3]. By Proposition 2.13(iv), y_2 lies in a Hardy field extension of $H\langle y_1\rangle=H(y,z)$; see also [13, Theorem 2, Corollary 2].

There might not exist a non-oscillating solution y, but we do have $\mathbb{R}$-linearly independent solutions $y_1,y_2\in\mathcal{C}^2_a$. We saw before that $w:=y_1y_2'-y_1'y_2\in\mathbb{R}^\times$. Set $y:=y_1+y_2i$. Then $4y''+fy=0$ and $y(t)\neq0$ for all $t\geqslant a$, and for $z\in\mathcal{C}^1_a[\mathrm{i}]$ given by $z(t)=2y'(t)/y(t)$ we have $-2z'-z^2=f$. Now

$$z\ =\ \frac{2y_1'+2iy_2'}{y_1+iy_2}\ =\ \frac{2y_1'y_1+2y_2'y_2-2i(y_1'y_2-y_1y_2')}{y_1^2+y_2^2}\ =\ \frac{2(y_1'y_1+y_2'y_2)+2iw}{y_1^2+y_2^2},$$

$$\text{so}\quad \operatorname{Re}(z)\ =\ \frac{2(y_1'y_1+y_2'y_2)}{y_1^2+y_2^2}\in\mathcal{C}^1_a,\qquad \operatorname{Im}(z)\ =\ \frac{2w}{y_1^2+y_2^2}\in\mathcal{C}^2_a.$$

Thus $\operatorname{Im}(z)\in(\mathcal{C}^2_a)^\times$ and $\operatorname{Im}(z)^\dagger=-\operatorname{Re}(z)$ and $\sigma\big(\operatorname{Im}(z)\big)=\omega(z)=f$ in $\mathcal{C}^1_a$. Replacing y_1 by $-y_1$ changes w to $-w$; in this way we can arrange that $w>0$.

The property of ω-freeness. Let $H\supseteq\mathbb{R}$ be a Liouville closed Hardy field. Note that then $x\in H$ and $\log f\in H$ for all $f\in H^>$. To express the property of ω-freeness for H we introduce the "iterated logarithms" ℓ_ρ; more precisely, transfinite recursion yields a sequence (ℓ_ρ) in $H^{>\mathbb{R}}$ indexed by the ordinals ρ less than some infinite limit ordinal κ as follows: $\ell_0=x$, and $\ell_{\rho+1}:=\log\ell_\rho$; if λ is an infinite limit ordinal such that all ℓ_ρ with $\rho<\lambda$ have already been chosen, then we pick ℓ_λ to be any element in $H^{>\mathbb{R}}$ such that $\ell_\lambda\prec\ell_\rho$ for all $\rho<\lambda$, if there is such an ℓ_λ, while if there is no such ℓ_λ, we put $\kappa:=\lambda$. From (ℓ_ρ) we obtain the sequences (γ_ρ) in $H^>$ and (λ_ρ) in H as follows:

$$\gamma_\rho\ :=\ \ell_\rho^\dagger,\qquad \lambda_\rho\ :=\ -\gamma_\rho^\dagger\ =\ -\ell_\rho^{\dagger\dagger}\ :=\ -(\ell_\rho^{\dagger\dagger}).$$

Then $\lambda_{\rho+1}=\lambda_\rho+\gamma_{\rho+1}$ and we have

$$\begin{aligned}\gamma_0\ &=\ \ell_0^{-1}, & \gamma_1\ &=\ (\ell_0\ell_1)^{-1}, & \gamma_2\ &=\ (\ell_0\ell_1\ell_2)^{-1},\\ \lambda_0\ &=\ \ell_0^{-1}, & \lambda_1\ &=\ \ell_0^{-1}+(\ell_0\ell_1)^{-1}, & \lambda_2\ &=\ \ell_0^{-1}+(\ell_0\ell_1)^{-1}+(\ell_0\ell_1\ell_2)^{-1},\end{aligned}$$

and so on. Indeed, $v(\gamma_\rho)$ is strictly increasing as a function of ρ and is cofinal in $\Psi_H=\big\{v(f^\dagger):f\in H,\ 0\neq f\not\asymp1\big\}$; we refer to [2, Section 11.5] for this and some of what follows. Also, (λ_ρ) is a strictly increasing pc-sequence which is cofinal in $\Lambda(H)$; see [2, Section 11.8] for the definition of the set $\Lambda(H)\subseteq H$, which is downward closed since H is Liouville closed. The latter also gives that H is λ-free as defined in [2, Section 11.6], equivalently, (λ_ρ) has no pseudolimit in H. The function $\omega\colon H\to H$ is strictly increasing on $\Lambda(H)$ and setting $\omega_\rho:=\omega(\lambda_\rho)$ we obtain a strictly increasing pc-sequence (ω_ρ) which is cofinal in $\omega\big(\Lambda(H)\big)=\omega(H)$:

$$\omega_0\ =\ \ell_0^{-2},\qquad \omega_1\ =\ \ell_0^{-2}+(\ell_0\ell_1)^{-2},\qquad \omega_2\ =\ \ell_0^{-2}+(\ell_0\ell_1)^{-2}+(\ell_0\ell_1\ell_2)^{-2},$$

and so on; see [2, Sections 11.7, 11.8] for this and some of what follows. Now H being ω-free is equivalent to (ω_ρ) having no pseudolimit in H. By [2, Corollary 11.8.30] the pseudolimits of (ω_ρ) in H are exactly the $\omega \in H$ such that $\omega(H) < \omega < \sigma\big(\Gamma(H)\big)$. Here the upward closed subset $\Gamma(H)$ of H is given by

$$\Gamma(H) \ = \ \big\{a^\dagger:\ a \in H,\ a \succ 1\big\} \ = \ \{a \in H:\ a > \gamma_\rho \text{ for some } \rho\},$$

and σ is strictly increasing on $\Gamma(H)$. Thus H is not ω-free if and only if there exists an $\omega \in H$ such that $\omega(H) < \omega < \sigma\big(\Gamma(H)\big)$.

We are now ready to prove the following:

Theorem 3.1. *Every Hardy field has an* ω*-free Hardy field extension.*

Proof. It is enough to show that every maximal Hardy field is ω-free. That reduces to showing that every non-ω-free Liouville closed Hardy field containing $\mathbb{R}$ has a proper Hardy field extension. So assume $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and H is not ω-free. We shall construct a proper Hardy field extension of H. We have $\omega \in H$ such that

$$\omega(H) \ < \ \omega \ < \ \sigma\big(\Gamma(H)\big).$$

Take $a \in \mathbb{R}$ such that ω is the germ of a function in $\mathcal{C}^2_a$, this function also to be denoted by ω. With ω in the role of f in the discussion preceding the statement of the theorem, we have $\mathbb{R}$-linearly independent solutions $y_1, y_2 \in \mathcal{C}^2_a$ of the differential equation $4Y'' + \omega Y = 0$ whose germs at $+\infty$ (also denoted by y_1 and y_2) lie in $\mathcal{C}^{<\infty}$. Then the complex solution $y = y_1 + y_2 i$ is a unit of $\mathcal{C}^2_a[i]$, and so we have $z := 2y^\dagger \in \mathcal{C}^1_a[i]$. The germs of y and z at $+\infty$ are also denoted by y and z and lie in $\mathcal{C}^{<\infty}[i]$. We shall prove that the elements $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ of $\mathcal{C}^{<\infty}$ generate a Hardy field extension $K = H\big(\mathrm{Re}(z), \mathrm{Im}(z)\big)$ of H with $\omega = \sigma\big(\mathrm{Im}(z)\big) \in \sigma(K^\times)$. We can assume that $w := y_1 y_2' - y_1' y_2 \in \mathbb{R}^>$, so $\mathrm{Im}(z)(t) > 0$ for all $t \geqslant a$.

We have $\omega_\rho \rightsquigarrow \omega$, with $\omega - \omega_\rho \sim \gamma^2_{\rho+1}$ by [2, Lemma 11.7.1]. We set $g_\rho := \gamma_\rho^{-1/2}$, so $2g_\rho^\dagger = \lambda_\rho = -\gamma_\rho^\dagger$. For $h \in H^\times$ we also have $\omega(2h^\dagger) = -4h''/h$, hence $P := 4Y'' + \omega Y \in H\{Y\}$ gives

$$P(g_\rho) \ = \ g_\rho(\omega - \omega_\rho) \ \sim \ g_\rho \gamma^2_{\rho+1},$$

and so with an eye towards using Lemma 2.12:

$$g_\rho^3 P(g_\rho) \ \sim \ g_\rho^4 \gamma^2_{\rho+1} \ \sim \ \gamma^2_{\rho+1}/\gamma^2_\rho \ \asymp \ 1/\ell^2_{\rho+1}.$$

Thus with $g := g_\rho = \gamma_\rho^{-1/2}$, $\phi = g^{-2} = \gamma_\rho$ we have $A_\rho \in \mathbb{R}^>$ such that

$$g^3 P^\phi_{\times g}(Y) \ = \ 4Y'' + g^3 P(g) Y, \quad |g^3 P(g)(t)| \ \leqslant \ A_\rho/\ell_{\rho+1}(t)^2, \text{ eventually.} \tag{3.1}$$

From $P(y) = 0$ we get $P^\phi_{\times g}(y/g) = 0$, that is, $y/g \in \mathcal{C}^{<\infty}[i]^\phi$ is a solution of $4Y'' + g^3 P(g) Y = 0$, with $g^3 P(g) \in H \subseteq \mathcal{C}^{<\infty}$. Now $\ell'_{\rho+1} = \ell^\dagger_\rho = \phi$, so the end of the previous section yields the isomorphism $H^\phi \to H \circ \ell^{\mathrm{inv}}_{\rho+1}$ of H-fields, where $\ell^{\mathrm{inv}}_{\rho+1}$ is the compositional inverse of $\ell_{\rho+1}$. Under this isomorphism the equation $4Y'' + g^3 P(g) Y = 0$ corresponds to the equation

$$4Y'' + f_\rho Y \ = \ 0, \qquad f_\rho \ := \ g^3 P(g) \circ \ell^{\mathrm{inv}}_{\rho+1} \in H \circ \ell^{\mathrm{inv}}_{\rho+1} \ \subseteq \ \mathcal{C}^{<\infty}.$$

The equation $4Y'' + f_\rho Y = 0$ has the "real" solutions

$$y_{i,\rho} \ := \ (y_i/g) \circ \ell^{\mathrm{inv}}_{\rho+1} \in \mathcal{C}^{<\infty} \circ \ell^{\mathrm{inv}}_{\rho+1} \ = \ \mathcal{C}^{<\infty} \qquad (i = 1, 2),$$

and the "complex" solution

$$y_\rho := y_{1,\rho} + y_{2,\rho} i = (y/g) \circ \ell^{\mathrm{inv}}_{\rho+1},$$

which is a unit of the ring $\mathcal{C}^{<\infty}[\mathrm{i}]$. We set $z_\rho := 2y_\rho^\dagger \in \mathcal{C}^{<\infty}[\mathrm{i}]$. The bound in (3.1) gives

$$|f_\rho(t)| \ \leqslant\ A_\rho/t^2, \text{ eventually,}$$

which by Corollary 2.11 yields positive constants B_ρ, c_ρ such that

$$|z_\rho(t)| \ \leqslant\ B_\rho t^{c_\rho}, \qquad \text{eventually.}$$

Using $(\ell_{\rho+1}^{\mathrm{inv}})' = (1/\ell'_{\rho+1}) \circ \ell_{\rho+1}^{\mathrm{inv}}$ we obtain

$$z_\rho \ = \ 2\big((y/g)^\dagger \circ \ell_{\rho+1}^{\mathrm{inv}}\big)\cdot(\ell_{\rho+1}^{\mathrm{inv}})' \ = \ 2\big((y/g)^\dagger/\ell'_{\rho+1}\big)\circ \ell_{\rho+1}^{\mathrm{inv}} \ = \ \big((z-2g^\dagger)/\ell'_{\rho+1}\big)\circ \ell_{\rho+1}^{\mathrm{inv}}.$$

In combination with the eventual bound on $|z_\rho(t)|$ this yields

$$\begin{aligned}
\left|\frac{z(t)-2g^\dagger(t)}{\ell'_{\rho+1}(t)}\right| \ &\leqslant\ B_\rho\,\ell_{\rho+1}(t)^{c_\rho} \ \text{ eventually, hence}\\
|z(t)-\lambda_\rho(t)| \ &\leqslant\ B_\rho\,\ell_{\rho+1}(t)^{c_\rho}\,\ell'_{\rho+1}(t) \ = \ B_\rho\,\ell_{\rho+1}(t)^{c_\rho}\,\gamma_\rho(t), \text{ eventually, so}\\
z(t) \ &= \ \lambda_\rho(t) + R_\rho(t), \quad |R_\rho(t)| \leqslant B_\rho\,\ell_{\rho+1}(t)^{c_\rho}\,\gamma_\rho(t), \text{ eventually.}
\end{aligned}$$

We now use this last estimate with $\rho+1$ instead of ρ, together with

$$\lambda_{\rho+1} \ = \ \lambda_\rho + \gamma_{\rho+1}, \quad \ell_{\rho+1}\gamma_{\rho+1} \ = \ \gamma_\rho.$$

This yields

$$\begin{aligned}
z(t) \ &= \ \lambda_\rho(t) + \gamma_{\rho+1}(t) + R_{\rho+1}(t) \ \text{ eventually, with}\\
|R_{\rho+1}(t)| \ &\leqslant\ B_{\rho+1}\,\ell_{\rho+2}(t)^{c_{\rho+1}}\,\gamma_{\rho+1}(t)\\
&= \ B_{\rho+1}\big(\ell_{\rho+2}(t)^{c_{\rho+1}}/\ell_{\rho+1}(t)\big)\,\gamma_\rho(t) \ \text{ eventually,}\\
\text{so } z(t) \ &= \ \lambda_\rho(t) + o\big(\gamma_\rho(t)\big) \ \text{ as } \ t\to\infty, \ \text{ and thus}\\
\mathrm{Re}(z)(t) \ &= \ \lambda_\rho(t) + o\big(\gamma_\rho(t)\big), \quad \mathrm{Im}(z)(t) \ = \ o\big(\gamma_\rho(t)\big), \ \text{as } \ t\to\infty.
\end{aligned}$$

Recall that (λ_ρ) is a strictly increasing divergent pc-sequence (λ_ρ) in H which is cofinal in $\Lambda(H)$. By the above, $\lambda := \mathrm{Re}(z) \in \mathcal{C}^{<\infty}$ satisfies $\Lambda(H) < \lambda < \Delta(H)$. This yields an ordered subfield $H(\lambda)$ of $\mathcal{C}^{<\infty}$, which by Lemma 2.6 is an immediate valued field extension of H with $\lambda_\rho \rightsquigarrow \lambda$.

Pick functions in $\mathcal{C}_a^0$ whose germs at $+\infty$ are the elements ℓ_ρ, γ_ρ, λ_ρ of H; we denote these functions also by ℓ_ρ, λ_ρ, γ_ρ. From $\ell_\rho^\dagger = \gamma_\rho$ and $\gamma_\rho^\dagger = -\lambda_\rho$ in H we obtain constants $c_\rho, d_\rho \in \mathbb{R}^>$ such that for all $t \geqslant a$,

$$\ell_\rho(t) \ = \ c_\rho \exp\left[\int_a^t \gamma(s)\,ds\right], \quad \gamma_\rho(t) \ = \ d_\rho \exp\left[-\int_a^t \lambda_\rho(s)\,ds\right].$$

Set $\gamma := \mathrm{Im}(z)$, so $\gamma^\dagger = -\lambda$, and both γ and λ are already given as elements of $\mathcal{C}_a^0$. Since $\gamma(t) > 0$ for all $t \geqslant a$ we have a constant $d \in \mathbb{R}^>$ such that for all $t \geqslant a$,

$$\gamma(t) \ = \ d\exp\left[-\int_a^t \lambda(s)\,ds\right].$$

The above estimate for $\lambda = \mathrm{Re}(z)$ gives

$$\lambda_\rho(t) \ < \ \lambda(t) \ < \ \lambda_\rho(t) + \gamma_\rho(t), \text{ eventually,}$$

so we have constants $a_\rho, b_\rho \in \mathbb{R}$ such that

$$\int_a^t \lambda_\rho(s)\,ds \ < \ a_\rho + \int_a^t \lambda(s)\,ds \ < \ b_\rho + \int_a^t \lambda_\rho(s)\,ds + \int_a^t \gamma_\rho(s)\,ds, \quad \text{eventually,}$$

which by applying $\exp(-*)$ yields

$$\frac{1}{d_\rho}\gamma_\rho(t) \ > \ \frac{1}{\mathrm{e}^{a_\rho}\, d}\gamma(t) \ > \ \frac{c_\rho}{\mathrm{e}^{b_\rho}\, d_\rho}\gamma_\rho(t)/\ell_\rho(t), \quad \text{eventually.}$$

Here the positive constant factors don't matter, since the valuation of γ_ρ is strictly increasing and that of $\gamma_\rho/\ell_\rho = (1/\ell_\rho)'$ is strictly decreasing with ρ. Thus for all ρ we have $\gamma_\rho > \gamma > (1/\ell_\rho)'$, in $\mathcal{C}^{<\infty}$. In view of Lemma 2.7 applied to $H(\lambda)$, γ in the role of K, f this yields an ordered subfield $H(\lambda,\gamma)$ of $\mathcal{C}^{<\infty}$. Moreover, γ is transcendental over $H(\lambda)$ with $\gamma^\dagger = -\lambda$, and γ satisfies the second-order differential equation $2yy'' - 3(y')^2 + y^4 - \omega y^2 = 0$ over H (obtained from the relation $\sigma(\gamma) = \omega$ by multiplication with γ^2). It follows that $H(\lambda,\gamma)$ is closed under the derivation of $\mathcal{C}^{<\infty}$, and hence $H(\lambda,\gamma) = H\langle\lambda\rangle$ is a Hardy field. □

The proof also shows that every $\mathcal{C}^\infty$-Hardy field has an ω-free $\mathcal{C}^\infty$-Hardy field extension, and the same with $\mathcal{C}^\omega$ instead of $\mathcal{C}^\infty$.

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WHEN DOES NIP TRANSFER FROM FIELDS TO HENSELIAN EXPANSIONS?

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Abstract. Let K be an NIP field and let v be a henselian valuation on K. We ask whether (K, v) is NIP as a valued field. By a result of Shelah, we know that if v is externally definable, then (K, v) is NIP. Using the definability of the canonical p-henselian valuation, we show that whenever the residue field of v is not separably closed, then v is externally definable. We also give a weaker statement for the case of separably closed residue fields.

1. Introduction and Motivation

There are many open questions connecting NIP and henselianity, most prominently

Question 1.1. (1) *Is any valued NIP field (K, v) henselian?*
(2) *Let K be an NIP field, neither separably closed nor real closed. Does K admit a definable non-trivial henselian valuation?*

Both of these questions have been recently answered positively in the special case where 'NIP' is replaced with 'dp-minimal' (cf. Johnson's results in [Joh15]).

The question discussed here is the following:

Question 1.2. *Let K be an NIP field and v a henselian valuation on K. Is (K, v) NIP?*

Note that this question neither implies nor is implied by any of the above questions, it does however follow along the same lines aiming to find out how close the bond between NIP and henselianity really is.

The first aim of this article is to show that the answer to Question 1.2 is 'yes' if Kv is not separably closed:

Theorem A. *Let (K, v) be henselian and such that Kv is not separably closed. Then v is definable in the Shelah expansion K^{Sh}.*

See section 2.1 for the definition of K^{Sh}. The theorem follows immediately from combining Propositions 2.4 and 2.5. If v is definable in K^{Sh}, then one can add a symbol for the valuation ring $\mathcal{O}$ to any language $\mathcal{L}$ extending $\mathcal{L}_{\mathrm{ring}}$ and obtain that if K is NIP as an $\mathcal{L}$-structure, then (K, v) is NIP as an $\mathcal{L} \cup \{\mathcal{O}\}$-structure. Theorem A is proven using the definability of the canonical p-henselian valuation. We make a case distinction between when Kv is neither separably closed nor real closed (Proposition 2.4) and when Kv is real closed (Proposition 2.5).

On the other hand, if Kv is separably closed, then - by a result of Johnson (see also Example 3.2) - we cannot hope for a result in the same generality: it is well-known that any algebraically closed valued field is NIP in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$, however, any algebraically closed field with two independent valuations has IP ([Joh13, Theorem 6.1]). In this case, we can still consider the question in the language of rings: Given an NIP field K and a henselian valuation v on K, is (K, v) NIP in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$? The result we show here is weaker than what one might hope for, in that we can only answer Question 1.2 completely (and positively) if the valuation has no coarsenings with non-perfect residue field. We give however a partial answer in the general case, showing that a henselian valuation on an NIP field can always be decomposed into two NIP valuations:

Theorem B. *Let K be NIP, v henselian on K.*

(1) *There is some (possibly trivial) $\mathcal{L}_{\text{ring}}$-definable coarsening w of v such that (K, w) and $(Kw, \bar{v})$ are NIP as pure valued fields. In particular, v can be decomposed into a composition of two NIP valuations.*
(2) *Moreover, if all proper coarsenings of v (including the trivial valuation) have perfect residue field, then (K, v) is NIP as a pure valued field.*

Theorem B is proven as Theorem 3.10 in section 3. The proof of the theorem uses a NIP transfer theorem recently proven in [JS16]. A transfer theorem gives criteria under which dependence of the residue field implies dependence of the (pure) valued field. Delon proved a transfer theorem for henselian valued fields of equicharacteristic 0 (see [Del81]), and Bélair proved a version for equicharacteristic Kaplansky fields which are algebraically maximal (see [Bél99]). The transfer theorem proven in [JS16, Theorem 3.3] generalizes these known results to separably algebraically maximal Kaplansky fields of finite degree of imperfection, in particular, it also works in mixed characteristic. See section 3 for definitions and more details. Combining this transfer theorem with an idea of Scanlon and some standard trickery concerning definable valuations yields the first statement of Theorem B. However, all known transfer theorems only work for perfect residue fields. Moreover, the question whether the composition of two henselian NIP valuations is again NIP seems to be open. For the case when the residue field of the coarser valuation is stably embedded, this follows from [JS16, Proposition 2.5]. Together with more facts about separably algebraically maximal Kaplansky fields, this allows us to prove the second part of Theorem B.

The paper is organized as follows: In section 2, we first recall the necessary background concerning the Shelah expansion. We then discuss the definition and definability of the canonical p-henselian valuation. In the final part, we use these two ingredients to prove Theorem A. In particular, we conclude that for any henselian NIP field the residue field is NIP as a pure field. We also obtain as a consequence that if a field admits a non-trivial henselian valuation and is NIP in some $\mathcal{L} \supseteq \mathcal{L}_{\text{ring}}$, then there is *some* non-trivial valuation v on K such that (K, v) is NIP in $\mathcal{L} \cup \{O\}$ (Corollary 2.8).

In the third section, we treat the case of separably closed residue fields. We first give Johnson's example which shows that we have to restrict Question 1.2 to the language of pure valued fields. We then briefly review different ingredients, starting with the transfer theorem for separably algebraically maximal Kaplansky fields and use it to give an answer to Question 1.2 in the (perfect) equicharacteristc setting (Proposition 3.5). After quoting a result by Delon and Hong, we state and prove a Proposition by Scanlon (Proposition 3.7) which implies that on an NIP field, any valuation with non-perfect residue field is $\mathcal{L}_{\text{ring}}$-definable. We then recall some facts about stable embeddedness and show that in some finitely ramified henselian fields, the residue field is stably embedded (which follows from the well-known corresponding result for unramified henselian fields). In the final subsection, we prove Theorem B and state some immediate consequences (Corollaries 3.12 and 3.13).

Finally, in section 4, we treat the much simpler case of convex valuation rings on an ordered field (K, v). As any convex valuation ring is definable in $(K, <)^{\mathrm{Sh}}$, we conclude that if $(K, <)$ is an ordered NIP field in some language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}} \cup \{<\}$ and v is a convex valuation on K, then (K, v) is NIP in $\mathcal{L} \cup \{\mathcal{O}\}$ (Corollary 4.3).

Throughout the paper, we use the following notation: for a valued field (K, v), we write vK for the value group, Kv for the residue field and $\mathcal{O}_v$ for the valuation ring of v.

2. Non-separably closed residue fields

2.1. **Externally definable sets.** Throughout the subsection, let M be a structure in some language $\mathcal{L}$.

Definition. *Let $N \succ M$ be an $|M|^+$-saturated elementary extension. A subset $A \subseteq M$ is called* externally definable *if it is of the form*

$$\{a \in M^{|\bar{x}|} \mid N \models \varphi(a, b)\}$$

for some $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ and some $b \in N^{|\bar{y}|}$.

The notion of externally definable sets does not depend on the choice of N. See [Sim15, Chapter 3] for more details on externally definable sets.

Definition. *The* Shelah expansion M^{Sh} *is the expansion of M by predicates for all externally definable sets.*

Note that the Shelah expansion behaves well when it comes to NIP:

Proposition 2.1 (Shelah, [Sim15, Corollary 3.14]). *If M is NIP then so is M^{Sh}.*

The way the Shelah expansion is used in this paper is to show that any coarsening of a definable valuation on an NIP field is an NIP valuation. Thus, the following example is crucial:

Example 2.2. *Let (K, w) be a valued field and v be a coarsening of w, i.e., a valuation on K with $\mathcal{O}_v \supseteq \mathcal{O}_w$. Then, there is a convex subgroup $\Delta \leq wK$ such that we have $vK \cong wK/\Delta$. As Δ is externally definable in the ordered abelian group wK, the valuation ring $\mathcal{O}_v$ is definable in $(K, w)^{\mathrm{Sh}}$.*

2.2. **p-henselian valuations.** Throughout this subsection, let K be a field and p a prime. We recall the main properties of the canonical p-henselian valuation on K. We define $K(p)$ to be the compositum of all Galois extensions of K of p-power degree (in a fixed algebraic closure). Note that we have

- $K \neq K(p)$ iff K admits a Galois extension of degree p and
- if $[K(p) : K] < \infty$ then $K = K(p)$ or $p = 2$ and $K(2) = K(\sqrt{-1})$ (see [EP05, Theorem 4.3.5]).

A field K which admits exactly one Galois extension of 2-power degree is called *Euclidean.* Any Euclidean field is uniquely ordered, the positive elements being exactly the squares (see [EP05, Proposition 4.3.4 and Theorem 4.3.5]). In particular, the ordering on a Euclidean field is $\mathcal{L}_{\mathrm{ring}}$-definable.

Definition. *A valuation v on a field K is called* p-henselian *if v extends uniquely to $K(p)$. We call K* p-henselian *if K admits a non-trivial p-henselian valuation.*

In particular, every henselian valuation is p-henselian for all primes p. Assume $K \neq K(p)$. Then, there is a canonical p-henselian valuation on K: We divide the class of p-henselian valuations on K into two subclasses,

$$H_1^p(K) = \{v \text{ } p\text{-henselian on } K \mid Kv \neq Kv(p)\}$$

and

$$H_2^p(K) = \{v \text{ } p\text{-henselian on } K \mid Kv = Kv(p)\}.$$

One can show that any valuation $v_2 \in H_2^p(K)$ is *finer* than any $v_1 \in H_1^p(K)$, i.e. $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1^p(K)$ are comparable. Furthermore, if $H_2^p(K)$ is non-empty, then there exists a unique coarsest valuation v_K^p in $H_2^p(K)$; otherwise there exists a unique finest valuation $v_K^p \in H_1^p(K)$. In either case, v_K^p is called the *canonical p-henselian valuation* (see [Koe95] for more details).

The following properties of the canonical p-henselian valuation follow immediately from the definition:

- If K is p-henselian then v_K^p is non-trivial.
- Any p-henselian valuation on K is comparable to v_K^p.
- If v is a p-henselian valuation on K with $Kv \neq Kv(p)$, then v coarsens v_K^p.
- If $p = 2$ and Kv_K^2 is Euclidean, then there is a (unique) 2-henselian valuation v_K^{2*} such that v_K^{2*} is the coarsest 2-henselian valuation with Euclidean residue field.

Theorem 2.3 ([JK15b, Corollary 3.3])**.** *Let p be a prime and consider the (elementary) class of fields*

$$\mathcal{K} = \{K \mid K \text{ } p\text{-henselian, with } \zeta_p \in K \text{ in case } \mathrm{char}(K) \neq p\}$$

There is a parameter-free $\mathcal{L}_{ring}$-formula $\psi_p(x)$ such that

(1) *if $p \neq 2$ or Kv_K^2 is not Euclidean, then $\psi_p(x)$ defines the valuation ring of the canonical p-henselian valuation v_K^p, and*
(2) *if $p = 2$ and Kv_K^2 is Euclidean, then $\psi_p(x)$ defines the valuation ring of the coarsest 2-henselian valuation v_K^{2*} such that Kv_K^{2*} is Euclidean.*

2.3. **External definability of henselian valuations.** In this subsection, we apply the results from the previous two subsections to prove Theorem A from the introduction.

Proposition 2.4. *Let (K, v) be henselian such that Kv is neither separably closed nor real closed. Then v is definable in K^{Sh}.*

Proof. Assume Kv is neither separably closed nor real closed. For any finite separable extension F of K, we use u to denote the (by henselianity unique) extension of v to F. Choose any prime p such that Kv has a finite Galois extension k of degree divisible by p^2. Consider a finite Galois extension $N \supseteq K$ such that $Nu = k$. Note that such an N exists by [EP05, Corollary 4.1.6]. Now, let P be a p-Sylow of $\mathrm{Gal}(Nu/Kv)$. Recall that the canonical restriction map

$$\mathrm{res} : \mathrm{Gal}(N/K) \to \mathrm{Gal}(Nu/Kv)$$

is a surjective homomorphism ([EP05, Lemma 5.2.6]). Let $G \leq \mathrm{Gal}(N/K)$ be the preimage of P under this map, and let $L := \mathrm{Fix}(G)$ be the intermediate field fixed by G. In particular, L is a finite separable extension of K. By construction, the extension $Lu \subseteq Nu$ is a Galois extension of degree p^n for some $n \geq 2$, in particular, we have $Lu \neq Lu(p)$.

Hence, we have constructed some finite separable extension L of K with $Lu \neq Lu(p)$. Moreover, we may assume that L contains a primitive pth root of unity in case $p \neq 2$ and $\mathrm{char}(K) \neq p$: The field $L' := L(\zeta_p)$ is again a finite separable extension of K and its residue field is a finite extension of Lu. Thus, by [EP05, Theorem 4.3.5], we get $L'u \neq L'u(p)$. Similarly, in case $p = 2$ and $\mathrm{char}(K) = 0$, we may assume that L contains a square root of -1: By construction, Lu has a Galois extension of degree p^n for some $n \geq 2$. Consider $L' := L(\sqrt{-1})$, then $L'u$ is not 2-closed and not orderable. In this case, no 2-henselian valuation on L' has Euclidean residue field (see [EP05, Lemma 4.3.6]).

Finally, v_L^p is definable on L by a parameter-free $\mathcal{L}$-formula $\varphi_p(x)$. It follows from the defining properties of v_L^p that $\mathcal{O}_{v_L^p} \subseteq \mathcal{O}_u$ holds. As L/K is finite, L is interpretable in K. Hence, $\mathcal{O}_w := \mathcal{O}_{v_L^p} \cap K$ is an $\mathcal{L}_{\mathrm{ring}}$-definable valuation ring of K with $\mathcal{O}_w \subseteq \mathcal{O}_v$. By Example 2.2, v is definable in K^{Sh}. □

Proposition 2.5. *Let (K, v) be henselian such that Kv is real closed. Then v is definable in K^{Sh}.*

Proof. Assume that (K, v) is henselian and Kv is real closed. Then K is orderable. We first reduce to the case that K is Euclidean: Note that v is a 2-henselian valuation with Euclidean residue field. Let v_K^{2*} be the coarsest 2-henselian valuation on K with Euclidean residue field, which is $\emptyset$-definable on K in $\mathcal{L}_{\mathrm{ring}}$ by Theorem 2.3. Now, if the induced valuation $\overline{v}$ on Kv_K^{2*} is definable in $(Kv_K^{2*})^{\mathrm{Sh}}$, then the valuation ring of v, which is the composition of v_K^{2*} and $\overline{v}$, is also definable in K^{Sh}.

Thus, we may assume that K is Euclidean. In this case, K is uniquely ordered and the ordering on K is $\mathcal{L}_{\mathrm{ring}}$-definable. Let $\mathcal{O}_w \subseteq K$ be the convex hull of $\mathbb{Z}$ in K. Then, $\mathcal{O}_w$ is definable in K^{Sh}. By [EP05, Theorem 4.3.7], (K, w) is a 2-henselian valuation ring on K with Euclidean residue field. As w has no proper refinements, w is the canonical 2-henselian valuation on K. Thus, we get $\mathcal{O}_w \subseteq \mathcal{O}_v$ and hence $\mathcal{O}_v$ is also definable in K^{Sh} by Example 2.2. □

Note that combining Propositions 2.4 and 2.5 immediately yields Theorem A from the introduction. Applying Proposition 2.1, we conclude:

Corollary 2.6. *Let K be a field and v a henselian valuation on K. Assume that $\mathrm{Th}(K)$ is NIP in some language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}}$. If Kv is not separably closed, then (K, v) is NIP in the language $\mathcal{L} \cup \{\mathcal{O}_v\}$.*

As separably closed fields are always NIP in $\mathcal{L}_{\mathrm{ring}}$, we note that the residue field of a henselian valuation on an NIP field is NIP as a pure field.

Corollary 2.7. *Let K be a field and v henselian on K. Assume that $\mathrm{Th}(K)$ is NIP in some language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}}$. Then Kv is NIP as a pure field.*

Recall that a field K is called *henselian* if it admits some non-trivial henselian valuation.

Corollary 2.8. *Let K be a henselian field such that $\mathrm{Th}(K)$ is NIP in some language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}}$. Assume that K is neither separably closed nor real closed. Then K admits some non-trivial externally definable henselian valuation v. In particular, (K, v) is NIP in the language $\mathcal{L} \cup \{\mathcal{O}_v\}$.*

Proof. If K admits some non-trivial henselian valuation v such that Kv is not separably closed, the result follows immediately by Propositions 2.4 and 2.5. Otherwise, K admits a non-trivial $\mathcal{L}_{\text{ring}}$-definable henselian valuation by [JK15a, Theorem 3.8]. □

The question of what happens in case Kv is separably closed is addressed in the next section.

3. Separably closed residue fields

In this section, we give a partial answer to Question 1.2 in case the residue field is separably closed. Recall that when (K, v) is henselian and the residue field is not separably closed, we may add a symbol for $\mathcal{O}_v$ to *any* NIP field structure on K and obtain an NIP structure. First, we note that we cannot expect the same when it comes to separably closed residue fields:

Theorem 3.1 ([Joh13, Theorem 6.1])**.** *Let K be an algebraically closed field and v_1 and v_2 two independent valuations on K. Then (K, v_1, v_2) has IP in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{O}_1\} \cup \{\mathcal{O}_2\}$.*

There are of course many examples of algebraically closed fields with independent valuations:

Example 3.2. *Let $\mathbb{Q}^{\text{alg}}$ be an algebraic closure of $\mathbb{Q}$ and let $p \neq l$ be prime. Consider a prolongation v_p (respectively v_l) of the p-adic (respectively l-adic) valuation on $\mathbb{Q}$ to $\mathbb{Q}^{\text{alg}}$. Then v_p and v_l are independent, thus the bi-valued field $(\mathbb{Q}, v_p, v_l)$ has IP.*

As any algebraically closed valued field has NIP in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$ and any valuation is henselian on an algebraically closed field, we cannot expect an analogue of Corollary 2.6 to hold for separably closed residue fields. We will instead focus on the following version of Question 1.2:

Question 3.3. *Let K be NIP as a pure field and v a henselian valuation on K with Kv separably closed. Is (K, v) NIP in $\mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$?*

3.1. Separably algebraically maximal Kaplansky fields and the equicharacteristic case.

The equicharacteristic case follows from the existing transfer theorems for NIP valued fields. First, we recall the relevant definitions:

Definition. *Let (K, v) be a valued field and $p = \text{char}(Kv)$.*

1. *We say that (K, v) is* (separably) algebraically maximal *if (K, v) has no immediate (separable) algebraic extensions.*
2. *We say that (K, v) is* Kaplansky *if the value group vK is p-divisible and the residue field Kv is perfect and admits no Galois extensions of degree divisible by p.*

Note that separable algebraic maximality always implies henselianity. See [Kuh13] for more details on (separably) algebraically maximal Kaplansky fields. As mentioned in the introduction, there is a transfer theorem which works for separably algebraically maximal Kaplansky fields:

Theorem 3.4 ([JS16, Theorem 3.3])**.** *Any complete theory of separably algebraically maximal Kaplansky fields of finite degree of imperfection is NIP if and only if corresponding theories of the residue field and value group are NIP.*

Using this transfer theorem, we can now prove that in equicharacteristic, the answer to Question 3.3 is mostly 'yes':

Proposition 3.5. *Let K be NIP, v henselian on K with* $\mathrm{char}(K) = \mathrm{char}(Kv)$*. In case* $\mathrm{char}(K) > 0$*, assume further that K has finite degree of imperfection. Then, (K, v) is NIP as a pure valued field.*

Proof. In case Kv is non-separably closed, the statement follows from Corollary 2.6. Now assume that Kv is separably closed, in particular, Kv is NIP as a pure field. If $\mathrm{char}(Kv) = 0$, the statement follows immediately from Delon's classical result ([Del81]) - or by the fact that any equicharacteristic 0 henselian valued field is separably algebraically maximal Kaplansky. On the other hand, if $\mathrm{char}(K) = p > 0$, then K admits no Galois extensions of degree divisible by p by [KSW11, Corollary 4.4]. Thus, vK is p-divisible and Kv is perfect (for an argument for the latter, see the proof of [JS16, Proposition 4.1]). As Kv is separably closed, we conclude that (K, v) is Kaplansky. Moreover, any immediate separable extension of K has degree divisible by p by the lemma of Ostrowski ([Kuh11, see (3) on p. 280 for the statement and p. 300 for the proof]). Thus, (K, v) is also separably algebraically maximal. By Theorem 3.4, (K, v) is NIP. □

3.2. **More helpful results.** The case of SCVF has also previously been answered (independently) by Delon and Hong, though Delon's proof remains unpublished and Hong's proof only works for finite degree of imperfection:

Theorem 3.6 (Delon, Hong, see [Hon13, Corollary 5.2.13])**.** *Let K be separably closed and let v be a valuation on K. Then (K, v) has NIP as a pure valued field.*

Using an argument by Scanlon, we can also reduce Question 1.2 to the case of algebraically closed residue fields.

Proposition 3.7 (Scanlon)**.** *Let (K, v) be a henselian valued field with* $\mathrm{char}(Kv) =$ *p, such that Kv is not perfect and has no separable extensions of degree divisible by p. Then $\mathcal{O}_v$ is definable in $\mathcal{L}_{\mathrm{ring}}$.*

Proof. Choose $t \in \mathcal{O}_v$ such that we have $\bar{t} \in Kv \setminus Kv^p$. Consider the $\mathcal{L}_{\mathrm{ring}}$-definable subset of K given by

$$S := \{a \in K \mid \exists\, L \supseteq K \text{ with } [L : K] < p \text{ and } \exists y \in L : y^p - ay = t\}.$$

We claim that $S = \{a \in K \mid v(a) \leq 0\}$ holds. We first show the inclusion $S \subseteq \{a \in K \mid v(a) \leq 0\}$. Assume for a contradiction that there is some $a \in S$ with $v(a) > 0$. Take $L \supseteq K$ and $y \in L$ witnessing $a \in S$, i.e., we have $[L : K] < p$ and $y^p - ay = t$. Let w denote the unique prolongation of v to L. Note that, as $w(t) \geq 0$ and $w(a) > 0$, we have $w(y) \geq 0$. Hence, we get $\bar{y}^p = \bar{t} \in Lw$. However, as $[Lw : Kv] \leq [L : K] < p$, this gives the desired contradiction.
For the other inclusion, suppose that we have $v(a) \leq 0$. Choose any $b \in K^{\mathrm{alg}}$ with $b^{p-1} = a$ and set $L := K(b)$. In particular, we have $[L : K] \leq p - 1 < p$. Let w denote the unique extension of v to L. Consider the equation

$$baZ^p - Zba - t = (bZ)^p - a(bZ) - t = 0$$

over L. As we have $w(ba) \leq 0$, this equation has a solution in L if and only if the equation

$$Z^p - Z - \frac{t}{ba} = 0$$

over $\mathcal{O}_w$ has a solution in $\mathcal{O}_w$. As (L, w) is henselian and Lw, a separable extension of Kv, also has no separable extensions of degree divisible by p, there is some $z \in \mathcal{O}_w$ with $z^p - z = \frac{t}{ba}$. For $y = zb$, we conclude $y^p - ay = t$ as desired.
It now follows immediately from the claim that $\mathcal{O}_v$ is also definable. □

3.3. **Stable embeddedness.** In the proof of Theorem 3.10, we decompose the valuation v on K into several pieces: a (definable) coarsening u of v and a valuation $\bar{v}$ on Ku such that v is the composition of $\bar{v}$ and u. However, in general, it is not clear whether showing that each of these is NIP is sufficient to show that v is NIP. The situation is simpler if the residue field Ku of u is stably embedded.

Definition. *Let M be a structure in some language $\mathcal{L}$ and $\mathcal{N} \succ M$ sufficiently saturated. A definable set D is said to be* stably embedded *if for every formula $\phi(x; y)$, y a finite tuple of variables from the same sort as D, there is a formula $d\phi(z; y)$ such that for any $a \in \mathcal{N}^{|x|}$, there is a tuple $b \in D^{|z|}$, such that $\phi(a; D) = d\phi(b; D)$.*

See [Sim15, Chapter 3] for more on stable embeddedness. Note that [JS16, Proposition 2.5] proves that we can add NIP structure on a stably embedded set and stay NIP. Moreover, the residue field in any separably tame algebraically maximal Kaplansky field is stably embedded (this is a special case of [JS16, Lemma 3.1]). These are not the only examples of henselian fields with stably embedded residue fields.

Definition. *Let (K, v) be a valued field of characteristic* $(\mathrm{char}(K), \mathrm{char}(Kv)) = (0, p)$ *for some prime $p > 0$. We say that (K, v) is*

(1) unramified *if $v(p)$ is the smallest positive element of vK.*
(2) finitely ramified *if the interval $[0, v(p)] \subseteq vK$ is finite.*

For unramified henselian fields with perfect residue field, it is well-known that the residue field is stably embedded.

Proposition 3.8 ([vdD14, Theorem 7.3])**.** *Let (K, v) be an unramified henselian valued field with Kv perfect. Then Kv is stably embedded.*

We require an analogue of the above proposition for finitely ramified henselian fields. One way one could show it would be by going through the proofs in [vdD14] and checking that - if one adds a constant symbol for a uniformizer - everything also works in the finitely ramified case. The route we choose here is significantly shorter though arguably less elegant. Recall that a valued field is called *complete* if every Cauchy sequence converges in K (see [EP05, Section 2.4] for more details).

Corollary 3.9. *Let (K, v) be a complete henselian valued field with Kv perfect and $vK \cong \mathbb{Z}$. Then Kv is stably embedded.*

Proof. Let (K, v) be a complete henselian valued field with Kv perfect and $vK \cong \mathbb{Z}$. By [War93, Theorem 22.7], there is a subfield $K_0 \subseteq K$ such that K is a finite algebraic extension of K_0 and the restriction v_0 of v to K_0 is unramified and satisfies $K_0 v_0 = Kv$ (in [War93], such fields are called Cohen subfields). Note that both v_0 and v are definable in $\mathcal{L}_{\text{ring}}$ on K_0 and K respectively (see [Koe04, Lemma 3.6]). Now, assume that some set $U \subseteq (Kv)^n$ is definable in (K, v). Then, as (K, v) is interpretable in (K_0, v_0), U is also a definable subset of $(K_0 v_0)^n$ in (K_0, v_0). Thus, by Proposition 3.8, U is already definable in $K_0 v_0 = Kv$. □

3.4. **A partial answer for separably closed residue fields.** In this subsection, we prove our second main result which was mentioned as Theorem B in the introduction.

Theorem 3.10. *Let K be NIP, v henselian on K.*

(1) *There is some (possibly trivial) $\mathcal{L}_{\text{ring}}$-definable coarsening w of v such that (K, w) and $(Kw, \bar{v})$ are NIP as pure valued fields. In particular, v can be decomposed into a composition of two NIP valuations.*
(2) *Moreover, if all proper coarsenings of v (including the trivial valuation) have perfect residue field, then (K, v) is NIP as a pure valued field.*

Proof. If Kv is not separably closed, the statement follows from 2.6. In the case when Kv is separably closed and non-perfect, the theorem holds by Proposition 3.7. Thus, we may assume that Kv is algebraically closed. Moreover, by Theorem 3.6, we may assume that K is not separably closed. The equicharacteristic 0 case follows from Proposition 3.5. We divide the proof into several cases.

Case 1: Assume $\text{char}(K) = p > 0$, $K \neq K^{\text{sep}}$ and $Kv = Kv^{\text{alg}}$. Then, G_K is pro-soluble, so there is some prime q such that K has a Galois extension of degree q (for more details, see the proof of [JK15a, Theorem 3.10]). Now, the restriction u of $v^q_{K(\zeta_q)}$ to K is a $\emptyset$-definable henselian valuation which coarsens v. If Ku is perfect, we get that $(Ku, \bar{v})$ is NIP by Proposition 3.5. If Ku is non-perfect but separably closed, we conclude that $(Ku, \bar{v})$ is NIP by Theorem 3.6. Finally, assume that Ku is non-perfect and non-separably closed. Then, there is again some prime l such that Ku has a Galois extension of degree l, so the restriction ν of $v^l_{Ku(\zeta_l)}$ to Ku is a non-trivial $\emptyset$-definable henselian valuation which coarsens $\bar{v}$. Note that its residue field $(Ku)\nu$ is perfect since Ku is an NIP field and thus admits no separable extensions of degree divisible by p (see also the proof of [JS16, Proposition 4.1]). Define w to be the composition of ν and u, this is a $\emptyset$-definable henselian valuation on K which coarsens v. Now, using Proposition 3.5 again, $(Kw, \bar{v})$ is NIP.
Note that in case all coarsenings of v have perfect residue field, K is perfect, so (K, v) is NIP by Proposition 3.5.

Case 2: Assume $\text{char}(K) = 0$ and $\text{char}(Kv) = p > 0$.

Case 2.1: Assume that there is some coarsening u of v such that Ku is not perfect. Then u is definable by Proposition 3.7. We now proceed as in Case 1: if Ku is separably closed, then $(Ku, \bar{v})$ is NIP by Theorem 3.6. If Ku is not separably closed, there is some non-trivial coarsening ν of $\bar{v}$ on Ku which is $\emptyset$-definable. The composition w of ν and u is a definable non-trivial henselian valuation on K with perfect residue field that coarsens v. Once more, Proposition 3.5 implies that $(Kw, \bar{v})$ is NIP.

Case 2.2: Assume that for all coarsening u of v the residue field Ku is perfect.

Claim: For any $(K, v) \prec (K', v')$ we also have that for all coarsenings u' of v' the residue field $K'u'$ is perfect.
Proof of claim: Let $(K, v) \prec (K', v')$ and assume for a contradiction that u' is a coarsening of v' such that $K'u'$ is not perfect. Then, by the same arguments as before, $K'u'$ is either separably closed or (K', u') is NIP. In either case, $K'u'$ is Artin-Schreier closed. By Proposition 3.7, there is some $t \in \mathcal{O}_{u'}$ and an $\mathcal{L}_{\text{ring}}$-formula $\phi(x, y)$ such that $\phi(x, t)$ defines $\mathcal{O}_{u'}$ in K'. Thus, the $\mathcal{L}_{\text{val}}$-sentence

$$\exists t[\phi(x, t) \text{ is a valuation ring } \mathcal{O} \text{ with non-perfect residue field and } \mathcal{O}_{v'} \subsetneq \mathcal{O} \subsetneq K']$$

holds in (K', v') and thus also in (K, v). Hence, there is some (proper) coarsening of v on K with non-perfect residue field. This proves the claim.

Thus, we may assume that (K, v) is $\aleph_1$-saturated. Following the proof of [Joh15, Lemma 6.8], we consider a decomposition of v (writing $\Gamma := vK$). Let $\Delta_0 \leq \Gamma$ be the biggest convex subgroup not containing $v(p)$ and let $\Delta \leq \Gamma$ be the smallest convex subgroup containing $v(p)$. We get the following decomposition of the place $\varphi_v : K \to Kv$ corresponding to v:

$$K = K_0 \xrightarrow{\Gamma/\Delta} K_1 \xrightarrow{\Delta/\Delta_0} K_2 \xrightarrow{\Delta_0} K_3 = Kv$$

where every arrow is labelled with the corresponding value group. Note that $\mathrm{char}(K) = \mathrm{char}(K_1) = 0$ and $\mathrm{char}(K_2) = \mathrm{char}(Kv) = p$. Let v_i denote the valuation on K_i corresponding to the place $K_i \to K_{i+1}$.

As the composition of v_1 and v_0 is a henselian valuation on an NIP field, Corollary 2.7 implies that K_2 is NIP as a pure field. Note that, as K_2 is perfect, (K_2, v_2) is NIP by Proposition 3.5. In particular, K_2 admits no separable extensions of degree divisible by p ([KSW11, Corollary 4.4]) and hence Δ_0 is p-divisible. By [AK16, Theorem 1.13], the value group v_1K_1 of (K_1, v_1) is either isomorphic to $\mathbb{Z}$ or $\mathbb{R}$. Moreover, by saturation (and since Δ/Δ_0 has rank 1), (K_1, v_1) is spherically complete and thus algebraically maximal (compare also again the proof of [Joh15, Lemma 6.8]).

In case v_1K_1 is isomorphic to $\mathbb{Z}$, the composition u of v_1 and v_0 is finitely ramified and thus definable on K ([Hon14, Theorem 4]). In particular, (K, u) is NIP and thus so is (K_1, v_1) by Proposition 2.1. As (K_1, v_1) is spherically complete, it is in particular complete. Here, we use our assumption that all coarsenings of v (and hence in particular v_1) have perfect residue field: Corollary 3.9 implies that K_2 is stably embedded in (K_1, v_1). Moreover, K_1 is stably embedded in (K_0, v_0) by [vdD14, Corollary 5.25]. Now, applying [JS16, Proposition 2.5] twice, we conclude that that (K, v) is NIP.

On the other hand, in case (K_1, v_1) has divisible value group, we show that (K_1, v_1) is Kaplansky. Note we have already shown that K_2 is NIP as a pure field, thus admits no separable extensions of degree divisible by p. Moreover, by assumption, K_2 is perfect. Hence, we have shown that (K_1, v_1) is separably algebraically maximal Kaplansky. As $K_1v_1 = K_2$ is NIP, we can apply Theorem 3.4 to conclude that (K_1, v_1) is NIP. Now (K_2, v_2) and (K_0, v_0) are also NIP by Proposition 3.5. Finally, $K_0v_0 = K_1$ is stably embedded in (K_0, v_0) and $K_1v_1 = K_2$ is stably embedded in (K_1, v_1) by [vdD14, Corollary 5.25] and [JS16, Lemma 3.1] respectively. Thus, applying [JS16, Proposition 2.5] twice again, we conclude that (K, v) is NIP. □

It would be interesting to know an answer to the following

Question 3.11. *Let (K, v) be henselian such that the residue field Kv is not perfect and NIP. Is (K, v) NIP? Are there any examples of NIP valued fields with imperfect residue field?*

Finally, we discuss a criterion which ensures that all proper coarsenings of v have perfect residue field.

Corollary 3.12. *Let (K, v) be a henselian valued field of mixed characteristic $(0, p)$ and assume that $|K^\times/(K^\times)^p|$ is finite. If K is NIP in $\mathcal{L}_{\mathrm{ring}}$, then (K, v) is NIP in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}_v\}$.*

Proof. Let (K, v) be a henselian valued field of mixed characteristic $(0, p)$ and assume that $|K^\times/(K^\times)^p|$ is finite. For ease of notation, we use Γ to denote vK. As in the proof of Theorem 3.10, consider the decomposition of v into three parts: let $\Delta_0 \leq \Gamma$ be the biggest convex subgroup not containing $v(p)$ and let $\Delta \leq \Gamma$ be the smallest convex subgroup containing $v(p)$. We get the following decomposition of the place $\varphi_v : K \to Kv$ corresponding to v:

$$K = K_0 \xrightarrow{\Gamma/\Delta} K_1 \xrightarrow{\Delta/\Delta_0} K_2 \xrightarrow{\Delta_0} K_3 = Kv$$

where every arrow is labelled with the corresponding value group. Note that $\mathrm{char}(K) = \mathrm{char}(K_1) = 0$ and $\mathrm{char}(K_2) = \mathrm{char}(Kv) = p$. Let v_i denote the valuation on K_i corresponding to the place $K_i \to K_{i+1}$.

Then (K_1, v_1) is a rank-1 valuation and we have $|K_1^\times/(K_1^\times)^p| < \infty$ (the latter is straightforward, for a proof see [JS16, Lemma 4.4]). By [Koe04, Proposition 3.2], its residue field $K_2 = K_1v_1$ is perfect. As any proper coarsening of the composition u of v_1 and v_0 has residue characteristic 0, all of them have perfect residue field. Any valuation w on K with $\mathcal{O}_v \subseteq \mathcal{O}_w \subsetneq \mathcal{O}_u$ induces a non-trivial henselian valuation on the perfect field Ku with residue field Kw. As $\mathrm{char}(Ku) > 0$, this implies that also Kw is perfect. □

The previous corollary allows us to give a rather less complicated set of assumptions under which the answer to Question 3.3 is 'yes'. Recall that a field K has *small absolute Galois group* if K has finitely many Galois extensions of degree n for each $n \in \mathbb{N}$. Note that if G_K is small, then $|K^\times/(K^\times)^p|$ is finite for all $p \neq \mathrm{char}(K)$. Thus, we obtain the following:

Corollary 3.13. *Let (K, v) be a henselian field with small absolute Galois group. In case* $\mathrm{char}(K) > 0$ *assume that K has finite degree of imperfection. If K is NIP in $\mathcal{L}_{\mathrm{ring}}$, then (K, v) is NIP in $\mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}_v\}$.*

Proof. The equicharacteristic case follows immediately from Proposition 3.5. The mixed characteristic case is a consequence of Corollary 3.12. □

4. Ordered fields

In this section, we use the same technique as in the proof of Proposition 2.5 to study convex valuation rings on an ordered field. We show that any convex valuation ring $\mathcal{O}_v$ on K is definable in $(K, <)^{\mathrm{Sh}}$. The idea to consider convex valuation rings on ordered fields was suggested by Salma Kuhlmann.

Definition. *Let $(K, <)$ be an ordered field and $R \subseteq K$ a subring.*

1. *The* $<$-convex hull of R in K *is defined as*
$$\mathcal{O}_R(<) := \{x \in K \,:\, x, -x < a \textit{ for some } a \in R\}.$$
2. *We say that R is* $<$-convex *if $\mathcal{O}_R(<) = R$.*

The following facts about convex valuation rings are well-known.

Fact 4.1 ([EP05, p. 36])**.** *Let $(K, <)$ be an ordered valued field.*

1. *Any convex subring of K containing 1 is a valuation ring.*
2. *A subring $R \subseteq K$ is $<$-convex if and only if R is a convex subgroup of the additive group of K. Thus, any two valuations v, w on K which are convex with respect to $<$ are comparable.*

(3) *There is a (unique) finest valuation* v_0 *on* K *which is convex with respect to* $<$. *It is called the* natural valuation *of* $(K, <)$. *The valuation ring* $\mathcal{O}_{v_0}$ *is the convex hull of the integers in* $(K, <)$.

It is now an easy consequence of the properties of the natural valuation that convex valuation rings are definable in the Shelah expansion:

Proposition 4.2. *Let* $(K, <)$ *be an ordered field and* $\mathcal{O}_v$ *a convex valuation ring on* K. *Then* $\mathcal{O}_v$ *on* K *is definable in* $(K, <)^{\mathrm{Sh}}$.

Proof. As the valuation ring of the natural valuation v_0 is exactly the convex closure of $\mathbb{Z}$ in K, it is definable in $(K, <)^{\mathrm{Sh}}$. As any convex valuation v on K is a coarsening of v_0, the valuation ring of v is also definable in $(K, <)^{\mathrm{Sh}}$. □

Applying Proposition 2.1, this yields the following

Corollary 4.3. *Let* K *be an ordered field such that* $\mathrm{Th}(K)$ *is NIP in some language* $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{of}}$ *and let* v *be a convex valuation on* K. *Then,* (K, v) *is NIP in* $\mathcal{L} \cup \{\mathcal{O}_v\}$.

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A New Dynamic Method to Find Roots of Polynomials with Coefficients in Ultranormed Commutative Rings*

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Dédié à Danielle Gondard. Le temps n'efface rien. Tu te souviens que tu as été mon élève au cours que j'ai ministré en 1969 à Paris, à l'amphithéâtre "Hermite" sur l'Arithmétique des Corps. Ton intérêt au sujet été majeur et pour cette raison nous avons collaboré en une recherche sur la Géométrie Algébrique Réelle et des sujets corrélates.

Pour la première fois les méthodes de théorie des modèles ont été appliquées avec grand succès pour obtenir nos résultats. C'était très original et aussi élégant. Encore très jeune ce succès a eté continué par toi en approfondissant ta recherche. Dans cette ligne tu est devenue l'autorité.

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Cet article est une extension écrite de mon exposé, où je voulu indiquer les lignes maîtresses de ma recherche à venir sur les racines de polynômes à coefficients dans des anneaux ultranormés – travail conjoint avec Sibylla Priess-Crampe.

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Très amicalement,

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Abstract

This paper is an extract from two chapters in the book entitled "Theory of Ultrametric Spaces" by Sibylla Priess-Crampe and Paulo Ribenboim.

We consider polynomials with coefficients in an ultranormed commutative ring. Our purpose is to determine their roots and this will be achieved using a new dynamic method.

There will be two types of ultranormed rings. The first one is related to the classical p-adic numbers. The second type of ultranormed rings involves functions and as such, it is essentially distinct from the first type. Ultranormed rings of functions are important in Functional Analysis, for example, but this aspect is not explored in the present article.

There will be good guesses for the roots, associated dynamic mappings, fixed points and common points theorems, which produce roots or common roots to polynomials.

The required notions are explained in the text and may be seeing in all detail in the links indicated.

*The writing of this paper began on March 14, 2019 and the paper was submitted on March 21, 2019.

• The determination of roots

In the determination of roots there will be existence theorems and approximation algorithms.

Existence theorems. After making a "good guess" for the eventual roots of the given polynomial, the Fixed Point Theorem and Common Point Theorem are applied, under appropriate assumptions, to show the actual existence of the roots, or, the existence of stable balls.

The seminal theorem in this context is the well-known and important Hensel's Lemma for the ring of p-adic integers.

Approximation algorithms. The knowledge of the existence of the roots needs to be supplemented by the determination of their location and by the indication of an algorithm to reach the roots or to provide an asymptotic approximation. The Approximation Theorem will play the front role.

We shall define successively the notion of a good guess, the dynamic mapping associated to the good guess. We will need the concept of balls which play the role of neighbourhoods in the case of topological space. We will describe types of balls including those which are stable under the actions of mappings.

Before coming to the all important Fixed Point Theorem and Common Point Theorem we will require concepts related to completeness of the ultranormed ring, which play in the context a role analogous to completeness of uniform structures or even of compactness.

A Preliminaries

I. The ultranormed ring (R, w, Γ)

R: R is a commutative and associative ring with unit element denoted by 1 and $1 \neq 0$ (the zero element of R).
$(\Gamma, \leq)$: Γ is an ordered set with a smallest element denoted by 0. Moreover Γ is a sup-lattice. The set Γ is endowed with a binary commutative and associative operation of multiplication. Γ has a neutral element with respect to multiplication, denoted by 1. The following relations hold:

$0\gamma = 0, \quad 1\gamma = \gamma$ for all $\gamma \in \Gamma$.
If $\gamma_1, \gamma_2, \delta \in \Gamma$ and $\gamma_1, \leq \gamma_2$ then $\gamma_1\delta \leq \gamma_2\delta$.
$\sup\{\gamma_1, \gamma_2\}\delta = \sup\{\gamma_1\delta, \gamma_2\delta\}$ for all $\gamma_1, \gamma_2, \delta \in \Gamma$.

Unless it is stated explicitly, we do not assume that the order on Γ is a total order. We also will have to consider the following two properties of the order.

Strict order: The order on Γ is said to be *strict*, when the following condition is satisfied:

(S) If $\gamma_1 < \gamma_2$ and $\delta \neq 0$ then $\gamma_1\delta < \gamma_2\delta$.

Weakly strict order: The order on Γ is said to be *weakly strict* when the following condition is satisfied:

(WS) If $\gamma < 1$ and $\delta \neq 0$ then $\gamma\delta < \delta$.

We observe that if $\Gamma^\bullet = \Gamma \backslash \{0\}$ is a group then Γ is strictly ordered.

The *mapping* $w \colon R \to \Gamma$: The following properties hold for all $x, y \in R$ and $\gamma \in \Gamma$:

(w1) $w(x) = 0$ if and only if $x = 0$.

(w2) $w(-x) = w(x)$.

(w3) $w(x+y) \leq \sup\{w(x), w(y)\}$.

(w4) $w(1) = 1$.

(w5) $w(xy) \leq w(x)w(y)$.

w is an ultranorm and (R, w, Γ) is an ultranormed ring. The ultranorm w on R is called multiplicative if $w(xy) = w(x)w(y)$ for all $x, y \in R$.

- **The associated ultrametric ring**

Let (R, w, Γ) be an ultranormed ring. Let $d(x, y) = w(x - y)$ for all $x, y \in R$. The mapping $d\colon R \times R \to \Gamma$ is an ultrametric distance, which means that the following conditions are valid:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in R$;

(iii) If $\gamma \in \Gamma$ satisfies $\gamma \geq d(x, y)$ and $\gamma \geq d(y, z)$ than $\gamma \geq d(x, z)$.

(R, d, Γ) is the ultrametric ring associated to (R, w, Γ). Conversely if (R, d, Γ) is an ultrametric ring, define $w\colon R \to \Gamma$ by $w(x) = d(x, 0)$ for all $x \in R$. Then (R, w, Γ) is an ultranormed ring which is said to be associated to (R, d, Γ).We shall use the notation $(R, w, \Gamma) \rightleftharpoons (R, d, \Gamma)$. This relationship allows to make statements either for ultranormed rings or for the associated ultrametric rings as it is fit for convenience.

- **Special subsets of R**

$A = \{a \in R \mid w(a) \leq 1\}$
$T = \{a \in R \mid w(a) = 1\}$

$M = \{a \in R \mid w(a) < 1\}$

U denotes the set of elements of A which are invertible in A. Also called the units of A. It is true that if $a \in U$ then $w(a) = 1$ that is, $U \subset T$. However there are examples when $U \neq T$. If $U = T$ then $U \cap M = \emptyset$.

II. Theoretical support

To find the roots we need to consider balls. Like neighbourhood in topological spaces they have the role of studying approximation.

- **Balls**

For each $\gamma \in \Gamma^\bullet$ and $x \in X$ the set $B_\gamma(x) = \{y \in X \mid d(x, y) \leq \gamma\}$ is called a ball with center x and radius γ. There are several types of balls:

Principal balls: The ball $B_\gamma(x)$ such that $\gamma = d(t, x)$ for some $t \in X$ is called a principal ball. The notation $B(x, t) = B_{d(t,x)}(x)$ is often used. If (X, d, Γ) is solid (that is, for each $\gamma \in \Gamma^\bullet$ and $x \in X$ there exists $y \in X$ such that $d(x, y) = \gamma$), then every ball is a principal ball.

Some facts about balls:

1) If $B_\gamma(x) \cap B_\delta(y) \neq \emptyset$ and $\gamma \leq \delta$ then $B_\gamma(x) \subseteq B_\delta(y)$.

2) If $x \neq z$ and x or z belong to $B_\delta(y)$, then $B(x, z) \subseteq B_\delta(y)$ if and only if $d(x, z) \leq \delta$. If $B(x, z) \subset B_\delta(y)$ then $d(x, z) < \delta$. If $y \neq u$ and $B(x, z) = B(y, u)$ then $d(x, z) = d(y, u)$.

3) If Γ is totally ordered, $\gamma \neq 0$ and $B_\gamma(x) \subset B_\delta(y)$ then $\gamma < \delta$.

Stable balls: Let (X, d, Γ) be an ultrametric space. Let $\varphi\colon X \to X$ be a mapping. Denote by B_x the ball of X with radius $d(\varphi(x), x)$ and centre x. The *ball* B_x is said to be *stable* by φ when for every $y \in B_x$ we have $B_y = B_x$.

Chains of balls: A set of balls which is totally ordered by inclusion is called a chain of balls. The empty set of balls is also called a chain of balls.

Cauchy chains of balls: A Cauchy chain of balls is a chain $\{B_i \mid i \in I\}$ of balls $B_i = B_{\gamma_i}(x_i)$ such that $\inf \gamma_i = 0$.

• Completeness concepts

The successive improvement of approximation to a root of a polynomial involves decreasing chains of balls. This leads to appropriate concepts related to completeness.

Spherically complete ultrametric spaces: (X, d, Γ) is said to be spherically complete when every chain of balls in X has a non-empty intersection.

Principally complete ultrametric spaces: (X, d, Γ) is said to be principally complete when every chain of principal balls in X has a non-empty intersection. Clearly, every spherically complete ultrametric space is principally complete.

Complete: (X, d, Γ) is called complete when every Cauchy chain of balls in X has a non-empty intersection. If Γ is totally ordered and $\Gamma^\bullet$ does not have a smallest element, this concept of completeness coincides with that given by the uniformity on X which is induced by the ultrametric distance d. Obviously, every spherically complete ultrametric space is complete.

• Types of mappings

The determination of the roots and stable balls depends on the dynamic mapping associated to the good guess as will be soon defined. At this point we classify the possible types of mappings.

Contracting: We call φ contracting if for all $x, y \in X$, $d(\varphi(x), \varphi(y)) \leq d(x, y)$.

Strictly contracting: The mapping φ is said to be strictly contracting, if for all $x, y \in X$, with $x \neq y$, $d(\varphi(x), \varphi(y)) < d(x, y)$.

Strictly contracting on orbits: φ is called strictly contracting on orbits, if for all $x \in X$, with $x \neq \varphi(x)$, $d(\varphi^2(x), \varphi(x)) < d(\varphi(x), x)$.

Lemma A.1. *Assume that $\varphi\colon L \to L$ is contracting and strictly contracting on orbits, then L does not contain any ball stable by φ.*

The next theorem may be called the Main Theorem for Fixed Points and Stable Balls.

• The Fixed Point Theorem

Let (X, d, Γ) be an ultrametric space, let $\varphi : X \to X$. An element $z \in X$ is called a fixed point of φ if $\varphi(z) = z$.

Theorem A.2. *Assume that X is principally complete.*

1) *If φ is contracting, then X contains a fixed point of φ or X contains a ball stable by φ.*
2) *If φ is contracting and strictly contracting on orbits, then X does not contain any ball stable by φ, so X contains a fixed point.*
3) *If φ is strictly contracting, then X contains a unique fixed point, but no stable ball.*

III. Examples

We describe two types of examples of ultranormed rings. In the first one the set Γ is totally ordered, while in the second type of examples, Γ is not totally ordered.

Example A.3. *For every integer $n \in \mathbb{Z}$ and every prime number p, we write $p^{v_p(n)}$ when this number divides n but $p^{v_p(n)+1}$ does not divide n.*

The Fundamental Theorem of Arithmetic states that

$$|n| = \prod_p p^{-v_p(n)} .$$

The p-adic ultranorm is defined as follows: $w_p(n) = p^{-v_p(n)}$.

$(\mathbb{Z}, w_p, \mathbb{Q})$ is called the p-adic ultranormed ring.

This is the seminal example from which a large arithmetic theory has developed. In this example the set of values of the ultranormed is totally ordered.

Example A.4. *Let X be a non-empty set, let Y be an ultranormed commutative ring with unit element. Let F be the set of mappings f from X to Y. For every $f \in F$ we define $w(f) = \{x \in X \mid f(x) \neq 0\}$. It is easy to verify that $(F, w, \mathcal{P}(X))$ is an ultranormed commutative ring.*

It is an interesting exercise to prove that F is spherically complete. Moreover, except when X has only one element, the order in $\mathcal{P}(X)$ is not total.

Examples of this kind have many applications in the theory of ultrametric spaces, specially in the Ultrametric Functional Analysis.

IV. The search for the roots

We describe the main steps in the dynamic methods for the determination of the roots of the polynomial with coefficients in ultranormed rings.

• The concept of a good guess

We say that a is a good guess for a root of $f \in \mathbf{F}$ in $L \in \mathbf{L}$ when $f(a) \in L$.

The set of good guesses defined above is denoted by $GG(f, L)$.

Lemma A.5. *The set $GG(f, \boldsymbol{L})$ is empty or it is a union of concepts of A modulo the additive subgroup of L.*

If $GG(f, L)$ is empty the dynamic method to be described is not applicable.

• The dynamic method defined by a good guess

In order to define the dynamic method some estimates have to be obtained. For this purpose the Taylor expansion of the polynomial f will play an important role.

• The Taylor expansion of $f \in \mathbf{F}$

Let Y be an indeterminate, let $f = a_0 + a_1X + a_2X^2 + \cdots + a_kX^k$.

Lemma A.6. *Let $f \in \mathbf{F}$, with degree of $f = k \geq 1$. There exist uniquely defined polynomials $p_0, p_1, \ldots, p_k \in A[Y]$ such that $f(Y + X) = p_0 + p_1X + p_2X^2 + \cdots + p_kX^k$.*

Moreover, if A is a domain with field of quotients of characteristic 0 or greater than k and $a \in A$ then $f(a + X) = f(a) + f'(a)X + \cdots + \frac{f^{(k)}(a)}{k!}X^k$.

We obtain the following corollary:

Corollary A.7. *Let $x \in L$.*

1) $f(a+x) = f(a) + dx + qx^2$, *where* $d = f'(a)$, $q = \sum_{j=2}^{k} p_j(a)x^{j-2}$.
2) $f(a+x) \in f(a) + L$ *and if* $f(a) \in L$ *then* $f(a+x) \in L$.

V. Interlude

In this Interlude we shall consider good guesses for roots of polynomials and their associated dynamic mappings.

We describe conditions for the existence of fixed points for the dynamic mapping, which, under appropriate conditions will assure the existence of roots for the polynomials.

There are three types of good guesses:

a) The general good guess
b) The non-singular good guess
c) The singular good guess which is classified by levels.

The types (b) and (c) are special cases of the general good guess.

VI. The general good guess

Notations:

(R, w, Γ) is an ultranormed ring;

A = set of elements $x \in R$ such that $w(x) \leq 1$.

$\mathbf{F}$ = is a non-empty set of polynomials $f \in A[X]$, such that $\deg f \geq 1$.

$\mathbf{L}$ = the set of ideals L such that $L \neq A$.

An element $a \in A$ is a *general good guess* for a root of a polynomial $f \in \mathbf{F}$ in $\mathbf{L}$ when $f(a) \in L$.

The set of good guesses defined above is denoted by: $GG(f, L)$.

Lemma A.8. *The set* $GG(f, L)$ *is empty or it is a union of additive cosets of* A *modulo* L.

If the set $GG(f, L)$ is empty, then the dynamic method to be described cannot be used.

• The dynamic mapping

Let $a \in GG(f, L)$. Let $e \in A$, $e \neq 0$ and let $x \in L$.

By Corollary (A.7) $f(a+x) \in L$. We define $\varphi_{f,a,e}(x) = x - ef(a+x)$. The parameter e will be necessary in the study of non-singular good guesses.

As stated above $\varphi_{f,a,e}(x) \in L$. The mapping $\varphi_{f,a,e}$ is called *the dynamic mapping associated to* $a \in GG(f, L)$ (with respect to the parameter e).

• Properties of the dynamic mapping. Due to the relevance of the result in the next Lemma its proof will be included.

Lemma A.9. *With the above notations, we have:*

1) $\varphi_{f,a,e}$ *is a contracting mapping;*

2) *$\varphi_{f,a,e}$ is strictly contracting on orbits if and only if the following condition is satisfied for every $x \in L$ such that $\varphi_{f,a,e}(x) \neq x$: $w(\varphi_{f,a,e}(x) - x) > w(e(a + x - e(a + x)))$ However, L does not contain any ball stable by $\varphi_{f,a,e}$.*

Proof

1) Let $x, y \in L$, $x \neq y$ then

$$\varphi_{f,a,e}(x) = x - ef(a + x)$$
$$\varphi_{f,a,e}(y) = y - ef(a + y)$$

then $\varphi_{f,a,e}(x) - \varphi_{f,a,e}(y) = x - y - e(f(a + x) - f(a + y))$.
By Lemma (A.6) there exist polynomials $p_2, p_3, \ldots, b_k \in A[X]$ such that

$$f(a + x) - f(a + y) = (x - y) - f'(a)(x - y) - \sum_{j=2}^{k} p_j(a) \frac{x^j - y^j}{x - y}(x - y).$$

Let $r = \sum_{j=2}^{k} p_j(a) \frac{x^j - y^j}{x - y} \in L$ because for $j \geq 2 \frac{x^j - y^j}{x - y} \in L$.
Putting together $w(\varphi_{f,a,e}(x) - \varphi_{f,a,e}(y)) = w(x - y)(ef'(a) + er)) \leq w(x - y)$ because $1 + e(f'(a) + e(r) \in a)$.

2) The proof is straightforward and therefore it is left to the reader. ☐

In the next Lemma we give a condition which implies the existence of a root of $f \in L$.

Lemma A.10. *With the above notations we assume that $\varphi_{f,a,e}$ is strictly contracting on orbits and that e is not a zero divisor in A. If (L, w, Γ) is principally complete, then there exists an element $b \in L$ such that $f(a + b) = 0$.*

Proof. By Lemma (A.1) L does not contain any ball stable by $\varphi_{f,a,e}$. By Lemma (A.9) $\varphi_{f,a,e}$ is a contracting mapping.

By the Main Theorem $\varphi_{f,a,e}$ has a fixed point in L, that is, there exists $b \in L$ such that $b = \varphi_{f,a,e}(b)$, thus $ef(a + b) = 0$ and finally $f(a + b) = 0$. ☐

VII. The special case of a non-singular good guess

Let $a \in A$, let f, L be as before. We write $d = f'a$. We say that a is a *non-singular good guess* for a root of $f \in L$ when the following two conditions are satisfied:

$NSGG1)$ $f(a) \in L$.

$NSGG2)$ $\overline{d} \in U(\overline{A})$.

The above condition is equivalent to:

$NSGG2')$ There exists an element $e \in A$ such that $ed \in 1 + L$.

We denote by $NSGG(f, L)$ the set of elements defined above.

Lemma A.11. *The set $NSGG(f, L)$ is empty or it is a union of cosets of A modulo the additive subgroup L.*

Proof Assume that $NSGG(f, L) \neq \emptyset$. We show that $a + x \in NSGG(f, L)$ for each good guess a and every $x \in L$. For this purpose we verify that the required conditions are satisfied.

$(NSGG1)$ Since $f(a) \in L$ and $x \in L$ by (A.8) $f(a + x) \in L$.

$(NSGG2)$ By assumption there exists $e \in A$ such that $ef'(a) \in 1 + L$. Since $f' \in A[X]$, by Corollary (A.7) $f'(a + x) = f'(a) + y$, with $y \in L$. Hence $ef'(a + x) = ef'(a) + ey \in 1 + L$. ☐

We give an example where $NSGG(f, L) = \emptyset$.

Example A.12. Let L be a non-zero prime ideal of A, let $f = X^2$. If $a \in NSGG(f, L)$ then $f(a) = a^2 \in L$, thus $a \in L$. We have $f'(a) = 2a \in L$, hence $\overline{f'(a)} = \overline{0} \in U(\overline{A})$. This is a contradiction, so $NSGG(f, L) = \emptyset$.

If $NSGG(f, L)$ is empty, the dynamic method described below is not applicable.

• The dynamic mapping

Let f, a, e be as before. For every $x \in L$, let $\varphi_{f,a,e}(x) = x - ef(a + x)$. By Corollary (A.7) $\varphi_{f,a,e}(L) \subseteq L$. $\varphi_{f,a,e}$ is called the *dynamic mapping associated to the good guess* $a \in NSGG(f, L)$ *and the parameter* e. For simplicity we shall often write φ_e or φ instead of $\varphi_{f,a,e}$.

Lemma A.13.

1) *φ is strictly contracting on orbits if and only if for every $x \in L$ such that $ef(a + x) \neq 0$ the following inequality holds:*

$$w(ef(a + x - ef(a + x))) < w(ef(a + x)).$$

2) *φ is a contracting mapping.*

3) *If $L \subseteq M$ and Γ satisfies the condition (WS) then φ is strictly contracting.*

Proof.

1) The proof is left to the reader.

2,3) Let $x, y \in L$, $x \neq y$. Then by Corollary (A.7) $\varphi(x) - \varphi(y) = (x - ef(a + x)) - (y - ef(a + y)) = x - y - e(f(a) + dx + q(x)x^2) + e(f(a) + dy + q(y)y^2) = (x - y)(1 - ed) - e(q(x)x^2 - q(y)y^2)$, where $q(x) = \sum_{j=2}^{k} p_j(a)\, x^{j-2}$.

Hence if $s = \sum_{j=2}^{k} p_j(a)\, \frac{x^j - y^j}{x - y}$ we have $\varphi(x) - \varphi(y) = (x - y)(1 - ed - es)$, where $s \in L$, thus $\varphi(x) - \varphi(y) = c(x - y)$ with $c = 1 - ed - es \in L$.

It follows that $w(\varphi(x) - \varphi(y)) = w(c(x - y)) \leq w(c)w(x - y) \leq w(x - y)$, hence φ is contracting.

If moreover, $L \subseteq M$ and Γ satisfies (WS) then $w(c) < 1$ and therefore by (WS), $w(\varphi(x) - \varphi(y)) = w(c(x - y)) \leq w(c)w(x - y) < w(x - y)$. So in this case, φ is strictly contracting.

So φ is strictly contracting on orbits if and only if the inequality indicated is satisfied. □

The next lemma will be used in the proof of the Main Theorem.

Lemma A.14. *Assume that $L \subseteq M$ and that Γ satisfies the condition (WS). Then we have:*

1) *For every $l \in L$, $1 - l$ is not a zero-divisor in L.*

2) *If $a \in NSGG(f, L)$ then $d = f'(a)$ is not a zero-divisor in A.*

Proof.

1) Let $l \in L$ and assume that there exists $t \in A$, $t \neq 0$, such that $(1 - l)t = 0$. Then $t = lt$. Since $l \in L \subseteq M$ then $w(l) < 1$. From $t \neq 0$ by the condition (WS), $w(t) = w(lt) \leq w(l)w(t) < w(t)$, which is a contradiction.

2) Since $a \in NSGG(f, L)$, there exists an element $e \in A$ such that $ed = 1 - l$, where $l \in L$. Now let $t \in A$ be such that $dt = 0$. Then $(1 - l)t = edt = 0$. By (1) $t = 0$.

□

Now we state and prove the Main Theorem.

Theorem A.15. *We assume that* (L, w, Γ) *is principally complete, that* $L \subseteq M$ *and that* Γ *satisfies the condition (WS). Let* $a \in NSGG(f, L)$. *There exists a unique element* $b \in L$ *such that* $f(a+b) = 0$. L *does not contain any ball stable by* $\varphi_{f,a,e}$, *for any* $e \in A$ *such that* $de \in 1+L$.

Proof. We shall apply the Fixed Point Theorem (Theorem A.2). Let $e \in A$ be such that $de \in 1 + L$; let $\varphi_{f,a,e}$ be the associated dynamic mapping. By Lemma (A.13) $\varphi_{f,a,e}$ is strictly contracting. Since L is principally complete, then $\varphi_{f,a,e}$ has a unique fixed point $b \in L$ and that L does not contain any ball stable by $\varphi_{f,a,e}$. We stress that this holds for any $e \in A$ with $de \in 1 + L$. From $\varphi_{f,a,e}(b) = b$ we deduce that $ef(a + b) = 0$. Since $de = 1 - l$, with $l \in L$, it follows that $(1 - l)f(a + b) = \text{def}(a + b) = 0$. So by the (A.14) $f(a + b) = 0$.

We show the uniqueness of $b \in L$ such that $f(a + b) = 0$. Let $b, b' \in L$ be such that $f(a + b) = f(a + b') = 0$. Choose any $e' \in A$ such that $e'd \in 1 + L$, so $e'f(a + b) = 0$ and $e'f(a + b') = 0$. Then $\varphi_{f,a,e'}(b) = b$, $\varphi_{f,a,e'}(b') = b'$. Since $\varphi_{f,a,e'}$ has only one fixed point, then $b = b'$. □

Corollary A.16. *Let* $h \in L[X]$, *let* $g \in A[X]$, *with* $\deg g \geq 1$. *If* $a \in NSGG(g, L)$ *and* $\deg(g - h) \geq 1$ *then there exists a unique element* $b \in L$ *such that* $g(a + b) = h(a + b)$. *In particular, if* $c \in L$ *there exists a unique element* $b \in L$ *such that* $g(a + b) = c$.

Proof. Let $f = g - h$, so deg $f \geq 1$. We have $f(a) = g(a) - h(a) \in L$. We also have $\overline{f'(a)} = \overline{g'(a)} - \overline{h'(a)} \in U(\overline{A})$, because $a \in NSGG(g, L)$ and $h \in L[X]$. Thus $a \in NSGG(f, L)$. By (A.16) there exists a unique element $b \in L$ such that $f(a+b) = 0$, that is $g(a+b) = h(a+b)$. In particular, taking $h(x) = c \in L$, there exists a unique element $b \in L$ such that $f(a+b) = c$. □

• Henselian pairs

We recall the following notations:

$\mathbf{F} =$ is a non-empty set of polynomials $f \in A[X]$ such that $\deg f \geq 1$.

$\mathbf{L} =$ the set of all proper ideals L of A, so $L \neq A$.

We say that $(A, \mathbf{L})$ is a Henselian Pair when the following condition is satisfied: For every $a \in A$ and for each $f \in \mathbf{F}$ there exists $L \in \mathbf{L}$ such that there exists a unique element $b \in L$ satisfying $f(a + b) = 0$.

VIII. The special case of a singular good guess classified by levels

• Definitions

Let A be an associative, commutative ring with unit element 1, such that $1 \neq 0$. Let $f \in A[X]$, with deg $f = k \geq 1$, so $f \neq 0$. Let L be a non-trivial ideal of A, thus $L \neq A$ and $L \neq 0$. For every integer $q \geq 1$ let $G_q(f, L) = \{a \in A \mid f(a) \in f'(a)^{q+1}L\}$. The elements of $G_q(f, L)$are said to have *level* q. We shall often write $d = f'(a)$ and $G_q = G_q(f, L)$.

It is easy to verify that $G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{q-1} \supseteq G_q \supseteq \cdots$, because $d \in A$ and $AL \subseteq L$.

• The polynomial h

Let $a \in G_q$, then $f(a) \in d^{q+1}L$. We choose an element $c \in L$ such that $f(a) = d^{q+1}c$. We write the Taylor expansion of $f(a + d^qX)$. By (A.6) there exist uniquely determined polynomials $p_j \in A[X]$, for $j = 2, \ldots, k$, such that $f(a + d^qX) = d^{q+1}(c + X + X^2r(X))$, where $r(X) = \sum_{j=2}^{k} p_j(a)d^{qj-(q+1)}, X^{j-2}$ if $k \geq 2$ and $r(X) = 0$ if $k = 1$.

By definition $h(X) = c + X + r(X)X^2$, so $f(a + d^qX) = d^{q+1}h(X)$.

• **Properties of $h(X)$:** $h(0) = c \in L$, $h'(0) = 1$. The image of $h'(0)$ modulo L is $1 \in U(A/L)$, thus $0 \in NSGG(h, L)$. Let $\psi = \psi_{h,0,1}$ be defined by $\psi(x) = x - 1h(0+x) = x - h(x)$. So $\psi(L) \subseteq L$ and ψ is the dynamic mapping of h in L.

Lemma A.17.

1) *ψ is a contracting mapping.*
2) *ψ is strictly contracting on orbits if and only if the following inequality*

$$w(h(x - h(x))) < w(h(x))$$

holds for all $x \in L$ such that $h(x) \neq 0$.

3) *If $L \subseteq M$ and Γ satisfies the condition (WS) then ψ is a strictly contracting mapping.*

Proof. The lemma is a special case of (A.13) for the polynomial h. □

The next proposition is the analogue for h of Theorem (A.15).

Proposition A.18. *Assume that L is principally complete, $L \subseteq M$ and that Γ satisfies the condition (WS). Let $f \in A[X]$, with $\deg f \geq 1$, let $q \geq 1$ and $a \in G_q(f, L)$. Then there exists an element $b \in L$ such that $f(a + d^q b) = 0$ and L does not contain any ball which is stable by ψ. If d is not a zero-divisor in A then the element b with $f(a + d^q b) = 0$ is uniquely determined.*

Proof. The analogue for h of Corollary (A.16) tells: There exists a unique element $b \in L$ such that $h(b) = 0$ and L does not contain any ball stable by ψ. It follows that $f(a + d^q b) = d^{q+1}\, h(b) = 0$. We show that $b \in L$ is unique with this property if d is not a zero-divisor in A. Let $b' \in L$ be such that $0 = f(a + d^q\, b') = d^{q+1}\, h(b')$. Since d is not a zero-divisor in A then $h(b') = 0$ and therefore $b' = b$. □

B Some Applications

We shall discuss three types of applications:

I) Γ is the set Σ_F of finitely generated ideals of A.

II) The L-adic case.

III) The case of valued fields.

I. The Case $\Gamma = \Sigma_F$

Let A be a commutative ring with unit element 1. Let $\Gamma = \Sigma_F$ be the set of finitely generated ideals of A.

Let $w\colon A \to \Sigma_F$ be the mapping defined by $w(x) = Ax$, where Ax is the principal ideal of A generated by x. The image of A by w is the set Σ_P of principal ideals of A.

• **Properties of Σ_F:** The set Σ_F is ordered by inclusion, it has the smallest element $\{0\}$ and the largest element $A = A1$. If $H_1, H_2, H_3 \in \Sigma_F$ then $H_1 + H_2 \in \Sigma_F$ and $H_1 H_2 \in \Sigma_F$, $(H_1 + H_2)H_3 = H_1 H_3 + H_2 H_3$, and both operations, the addition and multiplication, are commutative and associative. Σ_F is a sup-lattice, with $\sup\{H_1, H_2\} = H_1 + H_2$. If $H_1 \subseteq H_2$ then $H_1 H_3 \subseteq H_2 H_3$ for every ideal $H_3 \in \Sigma_F$.

The mapping w: We have for all $x, y \in A$:

$w(x) = \{0\}$ if and only if $x = 0$,

$w(x) = Ax = w(-x)$,
$w(x + y) = A(x + y) \subseteq Ax + Ay = \sup\{w(x), w(y)\}$,
$w(1) = A$,
$w(xy) = A(xy) = AxAy = w(x)w(y)$.

In view of the properties, (A, w, Σ_F) is a multiplicative ultranormed ring. We denote the unit element of $\Gamma = \Sigma_F$ by 1 and the smallest element by 0.

The ultranormed ring (A, w, Σ_F)*:* We have $A = \{a \in A \mid w(a) \leq 1\}$, $M = \{a \in A \mid w(a) < 1\}$, $T = \{a \in A \mid w(a) = 1\}$, so $A = M \cup T$, $M \cap T = \emptyset$. Let U be the set of elements of A which are invertible.

The following lemma holds:

Lemma B.1.

1) *$U = T$.*
2) *If L is a non-trivial ideal of A then $L \subseteq M$.*
3) *If M is an ideal of A (for example, this happens when Σ_F is totally ordered) then M is the unique maximal ideal of A.*
4) *If M is an ideal of A and (A, w, Σ_F) is principally complete then (M, w, Σ_F) is principally complete.*

Now we will give a condition on the ring A which implies that Σ_F has the property (WS).

Prüfer domains: A fractional ideal H of an integral domain A is said to be invertible if there exists a fractional ideal J of A such that $HJ = A$. It is well known and easy to prove, that every invertible ideal is finitely generated. The integral domain A is said to be a *Prüfer domain* when every finitely generated non-zero fractional ideal is invertible. Prüfer domains have the following property: If A is a Prüfer domain, if H is any maximal ideal of A then the localization A_H is a valuation domain.

In the next lemma we give conditions on A which imply that Σ_F satisfies the condition (WS).

Lemma B.2. *If A is a Prüfer domain then Σ_F satisfies the condition (WS).*

Proof. Let $P, Q \in \Sigma_F$ be such that $P \neq A$ and $Q \neq \{0\}$. Then $PQ \subseteq AQ = Q$. We assume that $PQ = Q$. There exists a fractional ideal B such that $BQ = A$. Then $P = AP = BQP = BQ = A$, which is absurd. Hence $BQ \subset Q$, showing that Σ_F satisfies the condition (WS). □

Proposition B.3. *Let A be a Prüfer domain. Assume that (L, w, Σ_F) is principally complete. Then (A, L) is Henselian.*

Proof. By (B.2) Σ_F satisfies the condition (WS). As stated in (B.1), $L \subseteq M$. Since (L, w, Σ_F) is principally complete, then by (A.15) (A, L) is Henselian. □

II. The L-adic Case

Let A be a commutative ring, let L be a non-trivial ideal of A. We recall that for all $n \geq 1$, L^n is the ideal of A generated by $\{x_1 \times x_2 \times \cdots \times x_n$, where each $x_i \in L\}$. So $L^1 = L$; we also define $L^0 = A$.

We assume that $\bigcap_{n\geq 1} L^n = \{0\}$. Then $\{L^n \mid n \geq 1\}$ is a basis of neighborhoods of 0 for a Hausdorff topology on A, which is known as the *L-adic topology.*

Let $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, let

$$\Gamma = \{0\} \cup \{\alpha^n \mid n \geq 1\} \subset \mathbb{R},$$

so Γ is a totally ordered abelian multiplicative monoid which satisfies the condition (WS).

Let $w\colon A \to \Gamma$ be defined by $w(0) = 0$ and if $a \neq 0$ let $w(a) = \alpha^n$ where $n \geq 0$ is such that $a \in L^n \backslash L^{n+1}$. It is easy to verify that w is an ultranorm on the ring A. Moreover, $M = \{a \in A \mid w(a) < 1\} = L$. Since Γ is totally ordered, (A, w, Γ) is spherically complete if and only if it is principally complete.

Proposition B.4.

1) *The following statements are equivalent:*
   a) (A, w, Γ) *is principally complete.*
   b) (L, w, Γ) *is principally complete.*
   c) *Endowed with the L-adic topology, A is complete.*
2) *In the above situation, (A, L) is Henselian.*

Corollary B.5. *Let A be a local noetherian ring with maximal ideal M. If A is complete in the M-adic topology then (A, M) is Henselian.*

Proof. By Krull's Theorem, $\bigcap\limits_{n\geq 1} M^n = \{0\}$. We may apply the preceding proposition to obtain the corollary. □

The following special case is of interest in Algebraic Geometry.

Let K be a field, $n \geq 1$ and $A = K[X_1, \dots, X_n]$, let M be a maximal ideal of A. We denote by $A' = A_M$ the localization of A at M, and we write $M' = A_M M$ for the unique maximal ideal of A'. The ring A is noetherian, hence A' is noetherian, so $\bigcap\limits_{n\geq 1} (M')^n = \{0\}$. It follows that the M'-adic topology on A' is a Hausdorff topology. We denote by $\widehat{A'}$ the completion (relative to the M'-adic topology) of A', and let $\widehat{M'}$ be the topological closure of M' in $\widehat{A'}$. So $\widehat{A'}$ is a local noetherian ring, $\widehat{M'}$ is its unique maximal ideal. With these notations we have:

Corollary B.6. $(\widehat{A'}, \widehat{M'})$ *is Henselian.*

Proof. This is a particular case of the preceding corollary. □

III. The Case of Valued Fields

The results to follow are about valued fields (K, v, Δ), or correspondingly, about the associated ultranormed fields (K, w, Γ). As usual, A is the ring of the valuation and M is its maximal ideal.

Proposition B.7. *If the valued field (K, v, Δ) is maximal then (A, M) is Henselian.*

Proof. (K, w, Γ) is principally complete, it follows easily that (A, w, Γ) is also principally complete. Thus by (B.1) (M, w, Γ) is principally complete. $\Gamma^\bullet$ is a totally ordered abelian multiplicative group, so it satisfies the condition (WS). By (A.15) (A, M) is Henselian. □

Corollary B.8.

1) *Let Δ be a totally ordered abelian additive group, let R be a field and let $H = R((\Delta))$ be the Hahn field of generalized series with coefficients in R, exponents in Δ, and let v be its canonical valuation. Then (A, M) is Henselian.*
2) *Let $H = \mathbb{R}((\mathbb{R}))$. Then (A, M) is Henselian.*

Proof. 1) $R((\Delta))$ is spherically complete. The valued field (H, v, Δ) is maximal. Thus by the preceding proposition, (A, M) is Henselian.
2) This is a particular case of (1). □

Corollary B.9. *Let v be a discrete valuation of the field K and assume that (K, v) is complete. Then (A, M) is Henselian.*

Proof. As it is known, the valued field (K, v) is maximal. By (B.7), (A, M) is Henselian. $\square$

The next special case is the Original Hensel's Lemma.

Corollary B.10. *Let p be any prime number. Then $(\mathbb{Z}_p, \mathbb{Z}_p\, p)$ is Henselian.*

Proof. The p-adic valuation is discrete and the field of p-adic numbers is complete. So the statement is a special case of (B.9). $\square$

The Original Hensel's Lemma concerns non-singular good guesses. We state it explicitly:

Theorem B.11 (Hensel's Lemma)**.** *Let p be a prime number, let $f \in \mathbb{Z}_p[X]$, with $\deg f \geq 1$, let $a \in \mathbb{Z}_p$ be such that $f(a) \in \mathbb{Z}_p\, p$ and $f'(a) \notin \mathbb{Z}_p\, p$. Then there exists a unique element $b \in \mathbb{Z}_p\, p$ such that $f(a + b) = 0$.*

The analogous lemma for good guesses of level $q \geq 1$ asserts:

Lemma B.12. *Let $f \in \mathbb{Z}_p[X]$, with $\deg f \geq 1$, $a \in \mathbb{Z}_p$, such that $d = f'(a) = p^e$ where $e \geq 1$. Let $q \geq 1$, and assume that $f(a) \in \mathbb{Z}_p\, p^{e(q+1)+1}$. Then there exists a unique element $b \in \mathbb{Z}_p\, p$ such that $f(a + p^{eq}\, b) = 0$.*

Proof. It suffices to remark that $a \in GG_q(f, \mathbb{Z}_p\, p)$, so the lemma follows from (A.18). $\square$

A noteworthy application is the proof of existence of non-trivial solutions in p-adic integers for Fermat's equation.

Proposition B.13. *Let p and ℓ be prime numbers. Then the equation $X^\ell + Y^\ell = Z^\ell$ has non-trivial solutions in p-adic integers.*

Proof. If $\ell = 2$ the Pythagorean triples are non-trivial solutions in $\mathbb{Z}$, hence also in $\mathbb{Z}_p$.
Now let $\ell \neq 2$.

First case: $\ell \neq p$.
Let $f = X^\ell + p^\ell - 1$. We have $f(1) = p^\ell \in \mathbb{Z}_p\, p$ and $f'(1) = \ell$, which is a unit in $\mathbb{Z}_p$, so $1 \in NSGG(f, \mathbb{Z}_p p)$. By the Original Hensel's Lemma (B.11) there exists a unique element $b \in \mathbb{Z}_p\, p$ such that $(1 + b)^\ell + p^\ell - 1 = 0$ and clearly $1 + b \neq 0$.

Second case: $\ell = p$.
Let $f = X^p + p^p - 1$. We have $f(1) = p^p$ and $f'(1) = p$. The ideal $M = \mathbb{Z}_p\, p$ is the unique maximal ideal of $\mathbb{Z}_p$ and $f(1) \in p^{p-1}M$, so $1 \in G_{p-2}(f, M)$. We note that $p - 2 \geq 1$ because $p \neq 2$. Hence it follows by the preceding lemma that there exists a unique element $b \in \mathbb{Z}_p\, p$ such that $f(1 + pb) = 0$, that is $(1 + pb)^p + p^p - 1 = 0$, and clearly $1 + pb \neq 0$. $\square$

C The Case when (L, w, Γ) is Step-Complete

In this subsection we shall assume that Γ is totally ordered.

1º) Preliminaries

I. Recapitulation

We gather here some concepts and facts from § 2.

Let (X, d, Γ) be an ultrametric space.

• Cauchy Families and Pseudo-Convergent Families

Let μ be a limit ordinal, let $\xi = (x_\iota)_{\iota<\mu}$ be a family of elements of X. We say that ξ is a *Cauchy family* if for every $\gamma \in \Gamma^\bullet$ there exists $\iota_0 = \iota_0(\gamma, \xi) < \mu$ such that if $\iota_0 \leq \iota < \kappa < \mu$ then $d(x_\iota, x_\kappa) < \mu$.

The family $\xi = (x_\iota)_{\iota<\mu}$ is said to be *pseudo-convergent* if there exists $\iota_0 = \iota_0(\xi) < \mu$ such that if $\iota_0 \leq \iota < \kappa < \nu < \mu$ then $d(x_\kappa, x_\nu) < d(x_\iota, x_\kappa)$. We note that if $\xi = (x_\iota)_{\iota<\mu}$ is pseudo-convergent, the elements x_ι, for $\iota_0(\xi) \leq \iota < \mu$, are all distinct and if $\iota_0(\xi) \leq \iota < \kappa < \nu < \mu$ then $d(x_\iota, x_\kappa) = d(x_\iota, x_\nu)$; this element is denoted by ξ_ι. Hence if $\iota_0 \leq \iota < \kappa < \mu$ then $\xi_\iota > \xi_\kappa$.

The element $y \in X$ is a *limit* of the Cauchy family ξ if for every $\gamma \in \Gamma^\bullet$ there exists $\iota_1 = \iota_1(\gamma) < \mu$, such that if $\iota_1 \leq \iota < \mu$ then $d(y, x_\iota) < \gamma$. A Cauchy family has at most one limit. Indeed, if y, z are limits, then $d(y, z) < \gamma$ for all $\gamma \in \Gamma^\bullet$, so $y = z$.

The element $y \in X$ is a *pseudo-limit* of the pseudo-convergent family $\xi = (x_\iota)_{\iota<\mu}$ if there exists $\iota_1 = \iota_1(\xi, y)$, $\iota_0(\xi) \leq \iota_1 < \mu$, such that if $\iota_1 \leq \iota < \mu$ then $d(y, x_\iota) \leq \xi_\iota$. If y is a pseudo-limit of ξ then $z \in X$ is a pseudo-limit of ξ if and only if $d(y, z) < \xi_\iota$ for all ι such that $\iota_1 \leq \iota < \mu$.

We recall that X is said to be spherically complete if every chain of balls has a non-empty intersection. A chain of balls $B_{\gamma_i}(a_i)$, $i \in I$, with $\inf \gamma_i = 0$, is called a Cauchy chain. X is said to be complete if every Cauchy chain of X has a non-empty intersection.

Lemma C.1. 1) *X is complete if and only if every Cauchy family of X has a limit in X.*

2) *X is spherically complete if and only if every pseudo-convergent family of X has a pseudo-limit in X.*

• Distinguished pseudo-convergent families

We shall now discuss another kind of completeness, which may be considered to be intermediate between completeness and spherical completeness.

We assume that Γ is endowed with an associative and commutative multiplication with unit element 1. We also assume that $0\gamma = \gamma 0 = 0$ and that if $\gamma \leq \gamma'$ then $\gamma\delta \leq \gamma'\delta$ for all $\gamma, \gamma', \delta \in \Gamma$. Let $(0, 1] = \{\gamma \in \Gamma \mid 0 < \gamma \leq 1\}$ and $(0, 1) = \{\gamma \in \Gamma \mid 0 < \gamma < 1\}$. We shall consider lower classes of $(0, 1)$. The lower class Λ is said to be a *prime lower class* when the following holds: if $\pi_1, \pi_2 \in (0, 1]$ and $\pi_1, \pi_2 \notin \Lambda$ then $\pi_1 \cdot \pi_2 \notin \Lambda$. The empty set is a prime lower class.

Let (X, d, Γ) be an ultrametric space where Γ satisfies the conditions indicated. Let μ be a limit ordinal, let $\xi = (x_\iota)_{\iota<\mu}$ be a pseudo-convergent family in X. Let $\Lambda(\xi) = \{\gamma \in (0, 1) \mid \gamma < \xi_\iota$ for all ι such that $\iota_0 \leq \iota < \mu\}$. So $\Lambda(\xi)$ is a lower class of $(0, 1)$. It is not excluded that $\Lambda(\xi) = \emptyset$. We say that ξ is a *distinguished pseudo-convergent* family if $\Lambda(\xi)$ is a prime lower class and $\Lambda(\xi) \subset (0, 1)$. The ultrametric space (X, d, Γ) is said to be *step-complete* when every distinguished pseudo-convergent family has a pseudo-limit in X.

Proposition C.2. *Let (X, d, Γ) be an ultrametric space for which Γ has the properties indicated above.*

1) *If X is step-complete then it is complete.*
2) *Assume moreover that $\Gamma^\bullet$ is an archimedean totally ordered group. Then if X is complete it is step-complete.*

There are examples of a step-complete ultrametric space which is not spherically complete and of a complete ultrametric space which is not step-complete. These ultrametric spaces are associated to valued fields.

II. Step-Complete Ultranormed Rings

Let (R, w, Γ) be an ultranormed ring as it was defined in Part (A), but recall that now Γ is assumed to be totally ordered. Let $A = \{x \in R \mid w(x) \leq 1\}$, $M = \{x \in R \mid w(x) < 1\}$, and let L be a non-trivial ideal of A.

We say that an ultranormed ring is step-complete if the associated ultrametric ring is step-complete. The ideal L or the set M is called step-complete if it is step-complete as an ultrametric subspace of (A, d, Γ).

2º) The Henselian Property for Non-Singular Good Guesses when (L, w, Γ) is Step-Complete

As before, let $A = \{x \in R \mid w(x) \leq 1\}$. Let L be a non-trivial ideal of A and U the set of units of A. Let $f \in A[X]$, with $deg\, f = k \geq 1$, $a \in NSGG(f, L)$, $d = f'(a)$ and let $e \in A$ be such that $ed \in 1 + L$.

Lemma C.3. *Assume that L is step complete. If $U + L = U$ then the subset $\Xi = \{w(f(a+x)) \mid x \in L\}$ of Γ has a smallest element.*

Proof. Since $ed \in 1 + L \subseteq U + L = U$, the element d is a unit of A.

We assume that Ξ does not have a smallest element and we shall derive a contradiction. By assumption, there exists a limit ordinal $\mu > 0$ and a strictly decreasing family $(w(f(a+x_\iota)))_{\iota<\mu}$ in Ξ which is coinitial in Ξ. We shall prove:

i) $\xi = (x_\iota)_{\iota<\mu}$ is a pseudo-convergent family in L.

ii) ξ is a distinguished pseudo-convergent family.

iii) We derive a contradiction.

Proof of (i): For every $x, y \in L$ we obtain by (A.7)

$$f(a + x) = f(a) + dx + \textstyle\sum_{j=2}^{k} p_j(a)x^j, \text{ and } \quad f(a + y) = f(a) + dy + \textstyle\sum_{j=2}^{k} p_j(a)y^j,$$

where $p_2, \ldots, p_k \in A[X]$. Then,

$$f(a + x) - f(a + y) = (x - y)(d + s),$$

where

$$s = \sum_{j=2}^{k} p_j(a)\, \frac{x^j - y^j}{x - y} \in L.$$

Since $d \in U$ then $d + s \in U + L = U$, hence $w(f(a + x) - f(a + y)) = w(x - y)$.

We apply this calculation. If $\iota < \kappa < \nu < \mu$ then $w(x_\iota - x_\kappa) = w(f(a + x_\iota) - f(a + x_\kappa)) = w(f(a+x_\iota)) > w(f(a+x_\kappa)) = w(x_\kappa - x_\nu)$. This proves that $\xi = (x_\iota)_{\iota<\mu}$ is a pseudo-convergent family in L.

We write $\xi_\iota = w(x_\iota - x_\kappa)$ for all κ such that $\iota < \kappa < \mu$. So $\xi_\iota = w(f(a + x_\iota))$ for all $\iota < \mu$.

Proof of (ii): We consider the lower class $\Lambda(\xi) = \{\gamma \in \Gamma \mid 0 < \gamma < 1 \text{ and } \gamma < \xi_\iota \text{ for all } \iota < \lambda\}$. We shall prove that $\Lambda(\xi)$ is a prime lower class of Γ. This is true if $\Lambda(\xi) = \emptyset$, thus we assume that $\Lambda(\xi) \neq \emptyset$. Let $\pi_1, \pi_2 \in \Gamma\backslash\Lambda(\xi)$, so there exist $\iota_1, \iota_2 < \lambda$ such that $\xi_{\iota_1} \leq \pi_1$ and $\xi_{\iota_2} \leq \pi_2$. Say $\xi_{\iota_1} \leq \xi_{\iota_2}$, then $\xi_{\iota_1}^2 \leq \xi_{\iota_1}\, \xi_{\iota_2} \leq \pi_1\, \pi_2$.

Let $q = a + x_{\iota_1}$, so $f(q) \in L$. By (A.7) there exists $y \in L$ such that $f'(q) = f'(a) + y \in U + L = U$. Let $u = -f'(q)^{-1} f(q) \in L$. Then by (A.7) there exists $c \in A$ such that $f(q + u) = f(q) + f'(q)u + cu^2 = cu^2$. Thus $w(f(a + x_{\iota_1} + u)) = w(f(q + u)) = w(cu^2) \leq w(u^2) \leq (w(u))^2 = (w(-f'(q)f(q)))^2 = (w(f(q)))^2 = (w(f(a + x_{\iota_1})))^2 = \xi_{\iota_1}^2 \leq \pi_1\, \pi_2$. Since $u \in L$ then $w(f(a + x_{\iota_1} + u)) \in \Xi$, thus $\pi_1\, \pi_2 \notin \Lambda(\xi)$.

This proves that $\Lambda(\xi)$ is a prime lower class of Γ, so ξ is a distinguished pseudo-convergent family in L.

Proof of (iii): We shall derive a contradiction. By assumption, L is step-complete, so ξ has a pseudo-limit $z \in L$. Thus there exists $\iota_0 < \mu$ such that if $\iota_0 < \iota < \mu$ then $w(z - x_\iota) = \xi_\iota = w(f(a + x_\iota)) \in \Xi$. By assumption, Ξ does not have a smallest element. Since $w(f(a + z)) \in \Xi$ there exists $\iota' < \mu$, which may be taken such that $\iota_0 < \iota'$, satisfying $w(z - x_{\iota'}) = w(f(a + x_{\iota'})) < w(f(a + z)) = \max\{w(f(a + z)),\, w(f(a + x_{\iota'}))\} = w(f(a + z) - f(a + x_{\iota'})) = w(z - x_{\iota'})$, as it was calculated in (i). This is a contradiction and concludes the proof of the lemma. $\square$

Let $\varphi = \varphi_{f,a,e}$ be the dynamic mapping in L, which is associated to f, a, e and defined by $\varphi(x) = x - ef(a + x)$.

Proposition C.4. *Assume that L is step-complete and $U + L = U$. Then we have:*

1) *There exists an element $b \in L$ such that $f(a + b) = 0$, or L contains a ball stable by φ. Moreover, if $w(f(a + x - ef(a + x))) < w(f(a + x))$ for all $x \in L$ such that $f(a + x) \neq 0$, then L does not contain any ball stable by φ.*
2) *If $L \subseteq M$ and Γ satisfies the condition (WS) then there exists a unique element $b \in L$ such that $f(a + b) = 0$ and L does not contain any ball stable by φ.*

Proof. 1) By the preceding lemma, there exists $b \in L$ such that$w(f(a + b))$ is the smallest element of $\Xi = \{w(f(a + x)) \mid x \in L\}$.

If $f(a + b) \neq 0$ we consider the ball in L, $B = B_{w(f(a+b))}(b) \cap L$ and we show that B is stable by φ. Let $x \in L$ be such that $w(x - b) \leq w(f(a + b))$. By (A.13) the mapping $\varphi\colon L \to L$ is contracting. Hence $w(\varphi(x) - \varphi(b)) = w(x - ef(a + x) - b + ef(a + b)) \leq w(x - b)$. So $w(e(f(a + x) - f(a + b))) = w(f(a + x) - f(a + b)) \leq w(x - b) \leq w(f(a + b))$ and therefore $w(f(a + x)) \leq w(f(a + b))$. Since $w(f(a + x)) \in \Xi$ then $w(f(a + x)) = w(f(a + b))$, so $w(x - \varphi(x)) = w(ef(a + x)) = w(f(a + b))$, showing that B is a ball stable by φ.

As it was seen in (A.13), φ is strictly contracting on orbits if and only if the inequality of the statement holds. In this case, by (A.1) L does not contain any ball stable by φ.

2) Since $L \subseteq M$ and Γ satisfies the condition (WS), by (A.13) φ is a strictly contracting mapping, so it is strictly contracting on orbits. By (A.1) L does not contain any ball which is stable by φ. By (1), $f(a + b) = 0$, where $w(f(a + b))$ is the smallest element of Ξ. We show that b is unique with the above property. Let $b_1 \in L$ with $b_1 \neq b$ and $f(a + b_1) = 0$, so $\varphi(b) = b$ and $\varphi(b_1) = b_1$. Since φ is strictly contracting, then $w(b - b_1) = w(\varphi(b) - \varphi(b_1)) < w(b - b_1)$, which is a contradiction. $\square$

Remark C.5. *1) The above proposition is valid under the assumptions that (L, w, Γ) is step-complete and $U = \{x \in A \mid w(x) = 1\}$. Indeed, since L is a non-trivial ideal and Γ is totally ordered, then $L \subseteq M$.*

If $u \in U$ and $x \in L$ then $w(u+x) = \max\{w(u), w(x)\} = 1$, thus $u+x \in U$. So the assertions of the proposition are true.

2) We note that under the assumption that Γ is totally ordered, $U = \{x \in A \mid w(x) = 1\}$ and $a \in NSGG(f, L)$,
eqrefthm:A15 follows from the above proposition. Indeed, since (L, w, Γ) is principally complete and Γ is totally ordered, (L, w, Γ) is spherically complete, so it is step-complete, showing that (A.15) *follows from* (C.4)*.*

Theorem C.6. *Assume that (L, w, Γ) is step-complete and that $U + L = U$. If $L \subseteq M$ and Γ satisfies the condition (WS) then (A, L) is Henselian for non-singular good guesses.*

Proof. The theorem is a rephrasement of part (2) of the preceding proposition. $\square$

Corollary C.7. *Assume that $\Gamma^\bullet$ is an archimedean group, L is complete and $U + L = U$. Let $f \in A[X]$, with $\deg f \geq 1$ and $a \in NSGG(f, L)$. Then the assertions (1) and (2) of* (C.4) *are true. In particular, if $L \subseteq M$ then (A, L) is Henselian for non-singular good guesses.*

Proof. Since L is complete and $\Gamma^\bullet$ is an archimedean group, then by (C.2) L is step-complete.

From $U + L = U$ it follows that the assertions of (C.4) are true. In particular, noting that Γ satisfies the condition (WS), if $L \subseteq M$ then (A, L) is Henselian for non-singular good guesses. □

Corollary C.8. *Let (K, w, Γ) be an ultranormed field which is associated to a valued field and assume that (A, w, Γ) is step-complete. Then:*

1) *The subspace M is step-complete.*
2) *(A, M) is Henselian for non-singular good guesses.*

Proof. 1) It suffices to show that if λ is a limit ordinal, if $\xi = (x_\iota)_{\iota<\lambda}$ is a distinguished pseudo-convergent family in M then ξ has a pseudo-limit in M. For every $\iota < \lambda$, $\xi_\iota = w(x_\iota - x_\kappa)$ for any κ such that $\iota < \kappa < \lambda$, so $\xi_\iota < 1$. Let $\Lambda_A(\xi) = \{\gamma \in w(A) \mid 0 < \gamma < 1,\ \gamma < \xi_\iota$ for all $\iota < \lambda\}$ and let $\Lambda_M(\xi) = \{\gamma \in w(M) \mid 0 < \gamma < \xi_\iota$ for all $\iota < \lambda\}$, so $\Lambda_A(\xi) = \Lambda_M(\xi)$. If $\pi_1, \pi_2 \in w(A)\backslash\Lambda_A(\xi)$ and $\pi_1 = 1$ or $\pi_2 = 1$ then $\pi_1\pi_2 \notin \Lambda_A(\xi)$. If $\pi_1 \neq 1$ and $\pi_2 \neq 1$ then $\pi_1, \pi_2 \in w(M)\backslash\Lambda_M(\xi)$; since $\Lambda_M(\xi)$ is a prime lower class then $\pi_1\pi_2 \notin \Lambda_M(\xi) = \Lambda_A(\xi)$. This proves that $\Lambda_A(\xi)$ is a prime lower class. By assumption A is step-complete, so there exists a pseudo-limit $z \in A$ of ξ, thus $w(z - x_\iota) = \xi_\iota < 1$ (for $\iota_0 \leq \iota < \lambda$). From $x_\iota \in M$ and $z - x_\iota \in M$ then $z \in M$. This proves that M is step-complete.

2) Since (A, w, Γ) is associated to a valued field then $U = \{x \in A \mid w(x) = 1\}$ and Γ satisfies the condition (WS). By (1) M is step-complete, so by (C.6) (A, M) is Henselian for non-singular good guesses. □

3º) Elements of Level *q*

As before we keep the same notations. We recall (see Part (A),3º),II) that if $f \in A[X]$ with $\deg f = k \geq 1$, if $q \geq 1$ and $a \in G_q(f, L)$ there exists a polynomial $h(X) \in A[X]$, which depends on f, a, q, L, such that $f(a + d^q X) = d^{q+1} h(X)$ and $0 \in NSGG(h, L)$; let $\psi = \varphi_{h,0,1}$ be the dynamic mapping on L, which is associated to $h, 0, 1$ and defined by $\psi(x) = x - h(x)$.

Proposition C.9. *Assume that L is step-complete and $U + L = U$. Let $f \in A[X]$ with $\deg f \geq 1$, let $q \geq 1$, $a \in G_q(f, L)$. Then we have:*

1) *There exists $b \in L$ such that $f(a + d^q b) = 0$ or L contains a ball stable by ψ. Moreover, if the inequality $w(h(x - h(x))) < w(h(x))$ holds for all $x \in L$ such that $h(x) \neq 0$, then L does not contain any ball stable by ψ.*
2) *If $L \subseteq M$ and Γ satisfies the condition (WS), then there exists an element $b \in L$ such that $f(a + d^q b) = 0$ and L does not contain any ball stable by ψ. If d is not a zero-divisor in A then the element b with $f(a + d^q b) = 0$ is uniquely determined.*

Proof. 1) As it was just recalled $0 \in NSGG(h, L)$, $h'(0) = 1$ and $\psi = \varphi_{h,0,1}$. By (C.4) there exists $b \in L$ such that $h(b) = 0$ or L contains a ball stable by ψ. In the first case, $f(a + d^q b) = d^{q+1} h(b) = 0$. Moreover, if the inequality in the statement holds then L does not contain any ball stable by ψ.

2) It follows from (C.4) that there exists a unique element $b \in L$ such that $h(b) = 0$ and L does not contain any ball stable by ψ. We deduce that $f(a + d^q b) = d^{q+1} h(b) = 0$. If $b_1 \in L$ and $f(a + d^q b_1) = d^{q+1} h(b_1) = 0$, then since d is not a zero-divisor, $h(b_1) = 0$, hence $b = b_1$, as it was required to prove. □

Remark C.10. *1) The remark (1) in* (C.5) *is still valid for the above proposition.*
2) As in (C.5) *(2),* (A.15) *follows from* (C.9) *because* Γ *is totally ordered,* $U = \{x \in A \mid w(x) = 1\}$ *and* $a \in G_q(f, L)$.

Corollary C.11. *Assume that* $\Gamma^{\bullet}$ *is an archimedean group,* L *is complete and* $U + L = U$. *Let* $f \in A[X]$, *with* $\deg f \geq 1$, *let* $q \geq 1$ *and* $a \in G_q(f, L)$. *Then the assertions (1) and (2) of* (C.9) *are true.*

Proof. By (C.2) L is step-complete, because it is complete and $\Gamma^{\bullet}$ is an archimedean group. Since $U + L = U$ then the assertions of (C.9) are true. □

Let (K, w, Γ) be an ultranormed field which is associated to a valued field. Under the assumption that (A, w, Γ) is step-complete, we have shown in (C.8) that M is step-complete.

Corollary C.12. *Let* A *be the valuation ring of an ultranormed field. Let* $f \in A[X]$, *with* $\deg f \geq 1$, *let* $q \geq 1$ *and* $a \in G_q(f, M)$. *If* A *is step-complete then there exists a unique element* $b \in M$ *such that* $f(a + d^q b) = 0$.

Proof. Obviously, Γ satisfies (WS) and in analogy to (2) of (C.8), $U = A \backslash M$, so $U + M = U$. Since furthemore, as mentioned above, M is step-complete, it follows from (C.9) that there exists a unique element $b \in M$ such that $f(a + d^q b) = 0$. □

D The Approximation to Roots

We assume in this section that all ultrametric spaces have totally ordered value sets.

I. Recapitulation

We gather here some concepts and results which will be needed.

• Spherical completion and completion for ultrametric spaces

We recall that an ultrametric space (X, d, Γ) is said to be spherically complete when every chain of balls has a non-empty intersection. A chain of balls $B_{\gamma_i}(a_i)$ with $\inf \gamma_i = 0$ is called a Cauchy chain. X is said to be complete when every Cauchy chain of X has a non-empty intersection. Hence, if X is spherically complete, then it is also complete.

• Spherically complete and pseudo-convergent families, complete and Cauchy families

For the definition of pseudo-convergent families and pseudo-limits, and Cauchy families and limits, we refer to Part (C), I). From there we quote Lemma (C.1).

• Spherical completion and completion

Let (X, d, Γ) be a subspace of the ultrametric space (X', d, Γ) and assume that $d(X \times X) = d(X' \times X') = \Gamma$. If for every $x' \in X'$ and every $x \in X$ with $x \neq x'$ there exists $y \in X$ such that $d(y, x') < d(x, x')$, the extension $X' \succ X$ is said to be *immediate*, and this is denoted by $X' im \succ X$. If for every $x' \in X'$ and for every $0 < \gamma \in \Gamma$ there exists $x \in X$ such that $d(x, x') < \gamma$, the subspace X is said to be *dense in* X', this is denoted by $X de \prec X'$. Hence if $X de \prec X'$ then also $X' im \succ X$. The ultrametric space X' is a *completion* of X, if $X de \prec X'$ and X' is complete. We say that X' is a *spherical completion* of X, if $X' im \succ X$ and X' is spherically complete.

The following propositions hold:

Proposition D.1.

1) *X has a spherical completion.*
2) *X has a completion.*

Concerning the existence and uniqueness of spherical completions and completions of a subspace (X, d, Γ) in an extension (X', d, Γ'), we have the following result:

Proposition D.2. *Let $(X, d, \Gamma) \prec (X', d, \Gamma')$ and assume that $d(X \times X) = \Gamma$, $d(X' \times X') = \Gamma'$.*

1) *If X' is spherically complete then X has a spherical completion which is a subspace of X'.*
2) *If X' is complete and $\Gamma^{\bullet}$ is coinitial in $\Gamma'^{\bullet}$, then X has one and exactly one completion which is a subspace of X'. (This completion will be denoted by $\widehat{X}_{X'}$).*

Proposition D.3. *If X' is spherically complete then X has a completion which is a subspace of X'.*

By (D.2)(1), X has a spherical completion $\widetilde{X}$ in X'. Since $\widetilde{X}$ is spherically complete it is complete. Furthermore $d(X \times X) = d(\widetilde{X} \times \widetilde{X}) = \Gamma$. So by (D.2)(2), X has a completion $\widehat{X}_{\widetilde{X}}$ in $\widetilde{X}$, which obviously is a completion of X in X'.

The proofs of the above propositions require extensive theoretical support.

An ultrametric space (X, d, Γ) may have different spherical completions in a spherically complete ultrametric space (X', d, Γ'), hence X may also have different completions in (X', d, Γ').

• Spherical completion and completion for valued fields

The results to follow are about valued fields (K, v, Δ). They will be formulated for the associated ultranormed fields (K, w, Γ).

• Preliminaries

Let A be the valuation ring and M the valuation ideal of the ultranormed field (K, w, Γ). The ideal M is the unique maximal ideal of A, so $\overline{K}_w = A/M$ is a field, the residue field of K.

To (K, w, Γ) is canonically associated an ultrametric space (K, d, Γ), where $d : K \times K \to \Gamma$ is defined by $d(x, y) = w(x - y)$. Regarding K as an ultrametric space and A, M as subspaces of K, we have for them all the concepts and results which are well known. In particular, we have for them the notion of pseudo-convergence, pseudo-limit, immediate extension, spherical completion, Cauchy family and completion. Like for ultrametric spaces, we write $(K, w, \Gamma) \prec (K', w, \Gamma')$ if K is a subfield of K' and if the ultranorm on K is the restriction of the ultranorm on K'. In analogous way are defined the notations $(K', w, \Gamma') \succ (K, w, \Gamma)$, $K' im \succ K$ etc. We had proved that $K' im \succ K$ if and only if $w(K') = w(K)$ and $\overline{K}'_w = \overline{K}_w$. An ultranormed field is said to be *maximal* if it does not have any proper immediate field extension.

The following theorem was proved by Kaplansky:

Theorem D.4. *An ultranormed field is maximal if and only if it is spherically complete.*

II. Spherical completion and completion

For ultranormed fields, we have:

Proposition D.5. *Let $(K, w, \Gamma) \prec (K', w, \Gamma')$ and assume that $w(K) = \Gamma$, $w(K') = \Gamma'$. If K' is complete and $\Gamma^{\bullet}$ is coinitial in $\Gamma'^{\bullet}$ then K' contains as a subfield one and exactly one completion of K. This completion is denoted by $\widehat{K}_{K'}$.*

Corollary D.6. *Let $\widetilde{K}$ be a spherical completion of K. Then $\widetilde{K}$ contains as an ultranormed subfield exactly one completion of K.*

For the existence of spherical completions, respectively completions, we have:

Proposition D.7. *Let K be an ultranormed field.*

1) *K has a maximal immediate extension, hence a spherical completion.*
2) *K has a completion.*

The transfer of statement (1) to the case of ultranormed fields is not possible without further assumptions. A sufficient assumption is Kaplansky's "Hypothesis (A)" which runs as follows: If the residue field $\overline{K}_w$ of the ultranormed field (K, w, Γ), $\Gamma = w(K)$, has characteristic p, the following two conditions have to be satisfied:

(K1) Every equation of the form $X^{p^n} + a_1 X^{p^{n-1}} + \cdots + a_{n-1}X^p + a_n X + a_{n+1} = 0$, with coefficients in $\overline{K}_w$, has a root in $\overline{K}_w$.

(K2) The value group $\Gamma^\bullet$ satisfies $\Gamma^\bullet = (\Gamma^\bullet)^p$.

If the characteristic of $\overline{K}_w$ is 0, then "Hypothesis (A)" is meant to be vacuous. An ultranormed field satisfying Kaplansky's hypothesis (A), is said to be a *Kaplansky field.*

Using this hypothesis it is now possible to prove the following theorem:

Theorem D.8. *Let $(K, w, \Gamma) \prec (K', w, \Gamma')$, $w(K) = \Gamma$, $w(K') = \Gamma'$. Assume that K' is a spherically complete Kaplansky field. Then K has a spherical completion which is an ultranormed subfield of K'.*

As a corollary we obtain:

Corollary D.9. *Let $(K, w, \Gamma) \prec (K', w, \Gamma')$, $w(K) = \Gamma$, $w(K') = \Gamma'$. Assume that K' is a spherically complete Kaplansky field. Then K has a completion $\widehat{K}$, which is an ultranormed subfield of K'.*

Like for ultrametric spaces, we have for valued fields that a spherical completion of (K, w, Γ) in the spherically complete ultranormed field (K', w, Γ') is not uniquely determined, thus this also holds for a completion $\widehat{K}$ of K in K'.

In the next proposition and corollary, we show that every ultranormed field can be embedded into a maximal Kaplansky field.

Proposition D.10. *If an ultranormed field is algebraically closed it is a Kaplansky field.*

Corollary D.11. *Every ultranormed field (K, w, Γ) has an ultranormed field extension (K', w, Γ') which is a maximal Kaplansky field.*

The following remark will be useful.

Remark D.12. *Let K be an ultranormed field. Since $A = \{x \in K \mid w(x) \leq 1\}$ and $M = \{x \in K \mid w(x) < 1\}$, we have that $A = B_1(0)$ and $M = B_1^-(0)$. Hence if K is spherically complete (respectively complete), also A and M are spherically complete (respectively complete).*

• Approximants and the Main Theorem

We emphasize that in section (D) the value sets of the ultrametric distance mapping are always assumed to be totally ordered.

• Preliminary remarks

We shall consider the following situation: (X', d, Γ') is a spherically complete ultrametric space, (X, d, Γ) is a subspace of X' and $\varphi : X' \to X'$ is a strictly contracting mapping such that $\varphi(X) \subseteq X$. By the Fixed Point Theorem, φ has a unique fixed point $z \in X'$. We wish to approximate z by a family of elements of X.

• Approximants

Let λ be the smallest ordinal number such that $\operatorname{card}\lambda > \operatorname{card}\Gamma$. Let μ be an ordinal number with $\mu < \lambda$. The family $\xi = (x_\iota)_{\iota<\mu}$, with $x_\iota \in X$ for every $\iota < \mu$, is said to be an *approximant* in X to z (with respect to φ) when the following conditions are satisfied:

(A1) $\underline{\xi} = (\xi_\iota)_{\iota<\mu}$, $\xi_\iota = d(x_\iota, \varphi(x_\iota))$, is strictly decreasing.

(A2) If $\kappa + 1 < \mu$ then $x_{\kappa+1} = \varphi(x_\kappa)$.

Let $AP(X)$ denote the set of all approximants of X.

If $\xi = (x_\iota)_{\iota<\mu} \in AP(X)$ and if there exists $\kappa < \mu$ such that $x_\kappa = z$ we say that ξ *reaches* z. In this case, $\mu = \kappa + 1$, since if not, then $\kappa + 1 < \mu$, so by (A2) $x_{\kappa+1} = \varphi(x_\kappa) = z$ and therefore $\xi_{\kappa+1} = \xi_\kappa = 0$, contradicting (A1).

We define an order on $AP(X)$ by $\xi = (x_\iota)_{\iota<\mu} \le \xi' = (x'_\iota)_{\iota<\mu'}$ if $\mu \le \mu'$ and $x'_\iota = x_\iota$ for all $\iota < \mu$. An *approximant* ξ is said to be *maximal* if for any $\xi' \in AP(X)$, $\xi \le \xi'$always implies that $\xi = \xi'$.

The following lemma may be proved:

Lemma D.13. *Let $\eta \in AP(X)$. There exists an approximant $\xi \in AP(X)$ such that $\eta \le \xi$ and ξ is maximal. In particular, if $y \in X$ then there exists $\xi = (x_\iota)_{\iota<\mu} \in AP(X)$such that $x_0 = y$.*

Let $Q(X)$ be the set of maximal elements of $AP(X)$.

• Approximants and pseudo-convergent families

Approximants $\xi = (x_\iota)_{\iota<\mu}$ for which μ is a limit ordinal are closely related to pseudo-convergent families. (If a family ξ of elements of X is pseudo-convergent in X then it is also pseudo-convergent in X', and a pseudo-limit $y \in X$ of ξ is also a pseudo-limit of ξ in X'.)

We shall need the following lemma:

Lemma D.14. *Let $\xi = (x_\iota)_{\iota<\mu} \in AP(X)$, where μ is a limit ordinal. Then we have:*

1) *ξ is pseudo-convergent and z is a pseudo-limit of ξ.*
2) *If $\underline{\xi}$ is coinitial in $\Gamma^\bullet$ then ξ is a Cauchy family in X.*

Proof

1) First we show: If $x \in X$ then $d(\varphi(x), x) = d(z, x)$.
   This is clear if $x = z$. So we assume now that $x \ne z$. Then $d(\varphi(x), z) < d(z, x)$, thus $d(\varphi(x), x) = d(z, x)$. We conclude that for all $\iota < \mu$, $\xi_\iota = d(x_\iota, z)$.

   We prove now that ξ is pseudo-convergent. Let $\iota < \kappa < \sigma < \mu$. Then $\xi_\iota = d(x_\iota, z) > \xi_\kappa = d(x_\kappa, z)$ and therefore $\xi_\iota = d(x_\iota, x_\kappa)$. Similarly we obtain $\xi_\kappa = d(x_\kappa, x_\sigma)$. Since $\xi_\iota > \xi_\kappa$ thus $d(x_\iota, x_\kappa) > d(x_\kappa, x_\sigma)$. Hence ξ is pseudo-convergent.
   From $\xi_\iota = d(x_\iota, z)$ for all $\iota < \mu$ it follows that z is a pseudo-limit of ξ.
2) Let $0 < \gamma \in \Gamma$. Since $\underline{\xi}$ is coinitial in $\Gamma^\bullet$ there exists $\iota_0 < \mu$ such that $\xi_{\iota_0} < \gamma$. Hence for all ι, κ, $\iota_0 \le \iota < \kappa < \mu$, as shown in Part (1), $\xi_\iota = d(x_\iota, x_\kappa) \le \xi_{\iota_0} < \gamma$. This shows that ξ is a Cauchy family in X. $\square$

We now have all the preparations to bring a version of the Special Approximation Theorem, which is appropriate for this section.

- **The Main Theorem**

As mentioned above, we will consider a spherically complete ultrametric space (X', d, Γ'), a subspace (X, d, Γ) of X' and a strictly contracting mapping $\varphi : X' \to X'$ which maps X into itself. The element $z \in X'$ is the unique fixed point of φ. We know that X has a spherical completion $\widetilde{X}$ which is a subspace of X' and $\widetilde{X}$ contains exactly one completion $\widehat{X}$ of X. We shall now assume additionally that $\varphi(\widetilde{X}) \subseteq \widetilde{X}$. Then the restriction $\varphi|_{\widetilde{X}}$ of φ to $\widetilde{X}$ is a strictly contracting mapping from $\widetilde{X}$ into itself and $\widetilde{X}$ is spherically complete. So $\varphi|_{\widetilde{X}}$ has a unique fixed point $\widetilde{z} \in \widetilde{X}$. Then $\widetilde{z}$ is also a fixed point of $\varphi : X' \to X'$. Hence $\widetilde{z} = z$.

The following theorem is called the Main Theorem. We shall consider in this theorem limits in (X, d, Γ). If $\Gamma^\bullet$ is not coinitial in $\Gamma'^\bullet$ we have to distinguish them from limits in (X', d, Γ'). Therefore we denote limits in (X, d, Γ) by $\lim_\Gamma$.

Theorem D.15. *Let $X, X', \widetilde{X}, \widehat{X}$ and φ be as explained above. Let $\xi = (x_\iota)_{\iota<\mu} \in AP(X)$. There are the following mutually exclusive possibilities:*

1) *μ is not a limit ordinal and $\xi \in Q(X)$. Then z is reached by ξ, $z \in X$.*
2) *μ is not a limit ordinal and $\xi \notin Q(X)$. Then there exists $\xi' = (x'_\iota)_{\iota<\mu'} \in Q(X)$ such that $\xi' > \xi$. The ordinal μ' may or may not be a limit ordinal.*
3) *μ is a limit ordinal. Then ξ is pseudo-convergent and z is a pseudo-limit of ξ.*
   (a) *$\underline{\xi}$ is coinitial in $\Gamma^\bullet$. Then ξ is a Cauchy family in X and $z = \lim_\Gamma \xi \in \widehat{X}$.*
   (b) *$\underline{\xi}$ is not coinitial in $\Gamma^\bullet$. Then z is only a pseudo-limit and not a limit of ξ in $\widehat{X}$. If $\xi \in Q(X)$ then $z \in \widetilde{X} \backslash \widehat{X}$. If $\xi \notin Q(X)$ then there exists $\xi' = (x'_\iota)_{\iota<\mu'} \in Q(X)$ such that $\xi' > \xi$. The ordinal μ' may or may not be a limit ordinal.*

III. An Application to Polynomials

- **The Context**

(K', w, Γ') is a spherically complete ultranormed Kaplansky field, A' is the valuation ring and M' the valuation ideal of K'. Since K' is spherically complete, also A' and M' are spherically complete. Let $f = a_0 + a_1X + \cdots + a_{k-1}X^{k-1} + X^k \in A'[X]$ be a polynomial of degree k. Let $a \in NSGG(f, M')$ be a non-singular good guess for a root of f in respect to M'. So we have: $f(a) \in M'$ and $d = f'(a)$ is a unit in A', hence $e = (\,f'(a))^{-1} \in A'$.

The dynamic mapping $\varphi_{a,e,M'} : M' \to M'$ is given by $\varphi_{a,e,M'}(x) = x - ef(a + x)$. For simplicity, we write $\varphi = \varphi_{a,e,M'}$. We recall that φ is strictly contracting. Thus it has a unique fixed point $z \in M'$. We have $\varphi(z) = z$, that is $z = z - ef(a + z)$, if and only if $f(a + z) = 0$. Hence z is a fixed point of φ if and only if $a + z$ is a root of f.

Let F be the prime field of K' and let K be the subfield $F(a_0, a_1, \ldots, a_{k-1}, a)$ of K'. With the restriction of w to K, the field K is ultranormed. Let $w(K) = \Gamma$. By (14.46), K has a spherical completion $\widetilde{K}$ which is an ultranormed subfield of K', and by (14.44), K has a completion $\widehat{K} = \widehat{K}_{\widetilde{K}}$ in $\widetilde{K}$.

Let M denote the valuation ideal of K, $\widehat{M}$ that of $\widehat{K}$ and $\widetilde{M}$ the valuation ideal of $\widetilde{K}$. We have $M \subseteq \widehat{M} \subseteq \widetilde{M} \subseteq M'$, then $\widehat{M}$ is complete and $\widetilde{M}$ spherically complete.

We show that $\widetilde{M}\, im \succ M$. Let $x' \in \widetilde{M}$ and $x \in M$ with $x \neq x'$. Since $\widetilde{K}\, im \succ K$ there exists $y \in K$ such that $w(y - x') < w(x - x')$. From $w(x - x') < 1$ it follows that $w(y - x') < 1$, furthermore $w(x') < 1$. Hence $w(y) < 1$, thus $y \in M$. This proves that $\widetilde{M}\, im \succ M$. Since moreover, $\widetilde{M}$ is spherically complete, $\widetilde{M}$ is a spherical completion of M in M'.

Similarly, we show that $M de \preceq \widehat{M}$. Indeed, let $x' \in \widehat{M}$ and $0 < \gamma \in \Gamma$. We may assume $\gamma \leq 1$. Then there exists $x \in K$ such that $w(x - x') < \gamma \leq 1$, so $w(x) < 1$, thus $x \in M$, which proves that $M de \preceq \widehat{M}$. Since $\widehat{M}$ is complete, $\widehat{M}$ is a completion of M in $\widetilde{M}$.

Lemma D.16. *With $\varphi, M, \widehat{M}$ and $\widetilde{M}$ as described above, we have: $\varphi(M) \subseteq M, \varphi(\widehat{M}) \subseteq \widehat{M}$ and $\varphi(\widetilde{M}) \subseteq \widetilde{M}$.*

Proof. Applying the Taylor expansion to

$$f = a_0 + a_1 X + \cdots + a_{k-1} X^{k-1} + X^k \in A[X]$$

we obtain for $x \in M'$: $f(a+x) = f(a) + f'(a)x + \sum_{j=2}^{k} p_j(a)x^j$, where $p_2, \ldots, p_k \in A[X]$. Hence, $\varphi(x) = x - ef(a+x) = -ef(a) - e\sum_{j=2}^{k} p_j(a)x^j$.

The element a is a non-singular good guess with respect to M', so $f(a) \in M'$. Furthermore, $f \in A[X]$ and $a \in A' \cap K = A$, thus $f(a) \in A \cap M' = M$. From $f' \in A[X]$ and $a \in A$ it follows that $f'(a) \in A$. Moreover, $f'(a)$ is a unit in A', so $w(f'(a)) = 1$. Thus $f'(a)$ is a unit in A, therefore $e = (f'(a))^{-1} \in A$. Summarizing, we obtain

$$\varphi(x) = -ef(a) - x^2 \left(\sum_{j=2}^{k} ep_j(a)x^{j-2} \right),$$

where $ef(a) \in M$ and $ep_j(a) \in A$, $j = 2, \ldots, k$. Hence if $x \in M$ then $\varphi(x) \in M$, and if $x \in \widehat{M}$ then $\varphi(x) \in \widehat{M}$, and if $x \in \widetilde{M}$ then $\varphi(x) \in \widetilde{M}$. $\square$

IV. Explicit Approximation to Roots of *f*

As before, let $f = a_0 + a_1 X + \cdots + a_{k-1}X^{k-1} + X^k \in A[X]$, and let $a \in A$ be a non-singular good guess for f with respect to M.

We will now apply the Main Theorem to obtain or approximate a root of f. We shall take the ultrametric spaces $M, M', \widehat{M}$ and $\widetilde{M}$ at the place of $X, X', \widehat{X}$,and $\widetilde{X}$, respectively. Let $\varphi = \varphi_{a,e,M'}$. We have pointed out that $M, M', \widehat{M}$ and $\widetilde{M}$ have all the properties which are assumed for $X, X', \widehat{X}$, and $\widetilde{X}$. So we have in particular that the fixed point z of φ is an element of $\widetilde{M}$. Let $\xi = (x_\iota)_{\iota<\mu} \in AP(M)$, $\mu \leq \omega_0$, be an approximant to z with respect to φ. We may assume that $x_0 = 0$.

We obtain the following proposition:

Proposition D.17. *Let $\xi = (x_n)_{n<\mu} \in AP(M)$, $\mu \leq \omega_0$, $x_0 = 0$. We have the following excluding possibilities for ξ:*

1) *There exists $n_0 \geq 1$ such that $\varphi(x_{n_0}) = x_{n_0}$. Then $z = x_{n_0} \in M$ and $f(a + x_{n_0}) = 0$. So in this case, $\mu = n_0 + 1$ and $a + x_{n_0}$ is a root of f.*
2) *$x_n \neq \varphi(x_n)$ for all $n \geq 0$. Then $\mu = \omega_0$, the approximant ξ is pseudo-convergent and $z \in \widetilde{M}$ is a pseudo-limit of ξ.*
   - *(a) $\underline{\xi} = (w(x_n - x_{n+1}))_{n<\omega_0}$ is coinitial in $\Gamma^\bullet$. Then $z = \lim_\Gamma x_n \in \widehat{M}$ and $a + z$ is a root of f.*
   - *(b) $\underline{\xi}$ is not coinitial in $\Gamma^\bullet$. In this case, we have only a "measure" for the quality of the approximation to z which for every $n < \omega_0$ is given by $w(z - x_n) = w(x_{n+1} - x_n)$.*

In the following lemma, we give a sufficient condition, when $\underline{\xi}$ is coinitial in $\Gamma^\bullet$.

Lemma D.18. *Let $\xi = (x_n)_{n<\mu} \in AP(M)$, $\mu \leq \omega_0$, $x_0 = 0$, and assume that $x_n \neq \varphi(x_n)$ for all $n \geq 0$. If $(w(f(a))^n)_{n\geq 1}$ is coinitial in $\Gamma^\bullet$, then also $\underline{\xi} = (w(x_n - \varphi(x_n)))_{n<\omega_0}$ is coinitial in $\Gamma^\bullet$.*

Proof. We have $x_0 = 0$ and for all $n \geq 0$, $x_{n+1} = \varphi(x_n) = x_n - ef(a + x_n)$. So by the Taylor expansion,

($\star$) $x_{n+1} = -ef(a) - e\sum_{j=2}^{k} p_j(a)x_n^j$, where $p_2, \ldots, p_k \in A[X]$.

We show first by induction that $w(x_n) = w(f(a))$ for $n \geq 1$. This is clear, if $n = 1$. Using ($\star$), we obtain

$$w(x_{n+1}) = w\left(-ef(a) - ex_n\sum_{j=2}^{k} p_j(a)x_n^{j-1}\right) = w(f(a))$$

because $\sum_{j=2}^{k} p_j(a)x_n^{j-1} \in M$, so $w\left(ex_n\sum_{j=2}^{k} p_j(a)x_n^{j-1}\right) < w(x_n) = w(f(a))$.

Now we prove by induction that $w(x_{n+1} - x_n) \leq w(f(a))^{n+1}$ for all $n \geq 0$. Obviously, $w(x_1 - x_0) = w(x_1) = w(f(a))$. By ($\star$), $x_{n+2} - x_{n+1} = \varphi(x_{n+1}) - \varphi(x_n) = -e\sum_{j=2}^{k} p_j(a)(x_{n+1}^j - x_n^j) = -e(x_{n+1} - x_n)\sum_{j=2}^{k} p_j(a)(x_{n+1}^{j-1} + x_{n+1}^{j-2}x_n + \ldots + x_n^{j-1})$. From $w(x_{n+1}) = w(x_n) = w(f(a)) < 1$ for $n \geq 1$ and $x_0 = 0$ it follows that $w(x_{n+1}^{j-1} + x_{n+1}^{j-2}x_n + \cdots + x_n^{j-1}) \leq w(f(a))$ for every $j = 2, \ldots, k$ and $n \geq 0$. So in view of the induction hypothesis,

$$w(x_{n+2} - x_{n+1}) \leq w(f(a))^{n+1} \max\{w(p_j(a))w(f(a)) \mid j = 2, \ldots, k\} \leq w(f(a))^{n+2}.$$

Hence if $(w(f(a))^n)_{n\geq 1}$ is coinitial in $\Gamma^\bullet$ then also $\underline{\xi}$. □

We consider some examples.

V. Examples

Example D.19. Let (K', w, Γ') be a maximal ultranormed field with $\mathrm{char}(\overline{K}'_w) = 0$. Hence K' is a Kaplansky field. Let $f = X^k - (1 + c) \in A'[X]$, $0 \neq c \in M'$.

Since $\mathrm{char}(\overline{K}'_w) = 0$ we may identify the prime field F of K' with $\mathbb{Q}$. The element $1 \in A'$ is a non-singular good guess for f, because $f(1) = -c \in M'$ and $f' = kX^{k-1}$, so $f'(1) = k \in A' \setminus M'$ and $e = (f'(1))^{-1} = \frac{1}{k}$.

We obtain $K = \mathbb{Q}(c)$. Since $w(\mathbb{Q}^\bullet) = \{1\}$ and $w(c) < 1$, the element c is transcendental over $\mathbb{Q}$. Thus $K = \mathbb{Q}(c)$ is a simple transcendental extension of $\mathbb{Q}$ and the ultranorm w on K is discrete. For the group $w(K^\bullet) = \Gamma^\bullet$ we have (up to isomorphism) $\Gamma^\bullet = \{\beta^n \mid n \in \mathbb{Z}\}$, where β is a real number, $1 < \beta$, chosen arbitrarily, and $w(c) = \beta^{-1}$. It follows that $\widetilde{K} = \widehat{K}$ and that $\widetilde{K}$ is the field $\mathbb{Q}((c))$ of formal power series $\sum_{\nu=\nu_0}^{\infty} q_\nu c^\nu$, $q_\nu \in \mathbb{Q}$ (which is, up to isomorphism, the Hahn field $\mathbb{Q}[[\Gamma^\bullet]]$).

The dynamic mapping $\varphi = \varphi_{1,k^{-1},M'}$ is given by $\varphi(x) = x - \frac{1}{k}f(1 + x)$, hence $\varphi(\widetilde{M}) \subseteq \widetilde{M}$. Thus $z \in \widetilde{M}$.

For the approximant $\xi = (x_n)_{n<\mu} \in AP(M)$, $\mu \leq \omega_0$, $x_0 = 0$, we obtain that either there exists $n < \omega_0$ such that $x_n = z$ and then $1 + x_n \in \mathbb{Q}(c)$ is a root of f or that $\mu = \omega_0$ and then $\xi = (x_n)_{n<\omega_0}$ is a Cauchy family and $z = \lim_\Gamma x_n \in \mathbb{Q}((c))$. In this last case, $1 + z \in \mathbb{Q}((c))$ is a root of f.

We will now study example 1 more specifically for the following two cases (a) and (b), where K' is an ultranormed Hahn field.

We recall that the ultranormed Hahn field $H = E[[\Delta]]$ consists of all the mappings χ from the totally ordered abelian group Δ into the field E for which $\mathrm{supp}\chi = \{\delta \in \Delta \mid \chi(\delta) \neq 0\}$ is dually well-ordered. With an addition, defined by $(\chi + \theta)(\delta) = \chi(\delta) + \theta(\delta)$, $\delta \in \Delta$, and a multiplication, defined by $(\chi\theta)(\delta) = \sum_{\rho,\sigma\in\Delta,\rho\sigma=\delta} \chi(\rho)\theta(\sigma)$, $\delta \in \Delta$, H is a field. The mapping $w : H \to \Delta \cup \{0\}$, where $w(0) = 0$ and $w(\chi) = \max(\mathrm{supp}\chi)$ if $\chi \neq 0$, is an ultranorm on H, the canonical ultranorm.

For the cases (a) and (b), let $K' = \mathbb{Q}[[\Gamma'^{\bullet}]]$, where $\Gamma'^{\bullet}$ is the direct product $\alpha^{\mathbb{Z}} \times \beta^{\mathbb{Z}}$ of the groups $\alpha^{\mathbb{Z}} = \{\alpha^m \mid m \in \mathbb{Z}\}$ and $\beta^{\mathbb{Z}} = \{\beta^s \mid s \in \mathbb{Z}\}$, ordered lexicographically, that is $(\alpha^m, \beta^s) < (\alpha^{m'}, \beta^{s'})$ if $m < m'$ or if ($m = m'$ and $s < s'$). Let $w : K' \to \Gamma'$ be the canonical ultranorm on $\mathbb{Q}[[\Gamma'^{\bullet}]]$.

Case (a): Let $c \in K'$ be defined by

$$c(\gamma) = \begin{cases} 1, \gamma = (\alpha^{-1}, \beta^{s_0}) \\ 0, \gamma \neq (\alpha^{-1}, \beta^{s_0}) \end{cases} .$$

So $w(c) = (\alpha^{-1}, \beta^{s_0})$. Then $\Gamma^{\bullet} = \{(\alpha^{-n}, \beta^{ns_0}) \mid n \in \mathbb{Z}\}$. Hence $\Gamma^{\bullet}$ is coinitial in $\Gamma'^{\bullet}$. This implies that every Cauchy family of K is also a Cauchy family of K'. Thus in particular, if $z = \lim_{\Gamma} \xi$ in $\widehat{K}$ then $z = \lim \xi$ in $\widehat{K'}$.

Case (b): Let $c \in K'$ be defined by

$$c(\gamma) = \begin{cases} 1, \gamma = (1, \beta^{-1}) \\ 0, \gamma \neq (1, \beta^{-1}) \end{cases} .$$

Then $w(c) = (1, \beta^{-1})$ and $\Gamma^{\bullet} = \{(1, \beta^{-n}) \mid n \in \mathbb{Z}\}$. Thus $\Gamma^{\bullet}$ is not coinitial in $\Gamma'^{\bullet}$. It follows that if $z = \lim_{\Gamma} \xi$ then ξ is a Cauchy family in K, but not in K'. In this case, z is only a pseudo-limit of ξ in K'.

Example D.20. We consider the field $\mathbb{Q}$ and a p-adic valuation v_p on $\mathbb{Q}$. Let w be the ultranorm on $\mathbb{Q}$ associated to v_p. We extend w to the algebraic closure $AC(\mathbb{Q})$ of $\mathbb{Q}$ and denote this extension of w again by w. $AC(\mathbb{Q})$ is a Kaplansky field, hence also its maximal immediate extension K'. The value group $\Gamma'^{\bullet}$ of K' coincides with the value group of $AC(\mathbb{Q})$ and this last one is the divisible hull of $w(\mathbb{Q}^{\bullet})$. Thus $w(\mathbb{Q}^{\bullet})$ is coinitial in $\Gamma'^{\bullet}$, hence $\mathbb{Q}$ has exactly one completion $\widehat{\mathbb{Q}}$ in K'. Since $w(\mathbb{Q}^{\bullet})$ is discrete, $\widehat{\mathbb{Q}}$ is also a spherical completion of $\mathbb{Q}$. We identify $\widehat{\mathbb{Q}}$ with $\mathbb{Q}_p$, the field of the p-adics.

Let $f = a_0 + a_1X + \ldots + a_{k-1}X^{k-1} + X^k \in A'[X]$. We assume that there exists $a \in NSGG(f, M')$. So $c = f(a) \in M'$, $f'(a) \in A' \setminus M'$ and $e = (f'(a))^{-1}$. We assume that $c \neq 0$. The dynamic mapping $\varphi = \varphi_{a,e,M'} : M' \to M'$ is given by $\varphi(x) = x - ef(a + x)$. The prime field F of K' is the field $\mathbb{Q}$, thus $K = \mathbb{Q}(a_0, a_1, \ldots, a_{k-1}, a)$. Since K' is a maximal Kaplansky field, K has a spherical completion $\widetilde{K}$ which is an ultranormed subfield of K'. Let $\widehat{K}$ be the unique completion of K in $\widetilde{K}$. This is also the unique completion of K in K', because $\Gamma^{\bullet}$ is coinitial in $\Gamma'^{\bullet}$. $\widehat{\mathbb{Q}} = \mathbb{Q}_p$ is the unique completion of $\mathbb{Q}$ in K' and $\mathbb{Q}$ is a subfield of K. Hence $\mathbb{Q}_p$ is an ultranormed subfield of $\widehat{K}$.

Let $\xi = (x_n)_{n<\mu} \in AP(M)$, $\mu \leq \omega_0$, $x_0 = 0$. By (14.55), either there exists $n_0 > 0$ such that $x_{n_0} = z \in M$ and then $a + x_{n_0} \in K$ is a root of f, or $\xi = (x_n)_{n<\omega_0}$ and then ξ is pseudo-convergent and $z \in \widetilde{M}$ is a pseudo-limit of ξ. For the following, let $\xi = (x_n)_{n<\omega_0}$. Since $\Gamma^{\bullet}$, as a subgroup of $\Gamma'^{\bullet}$, is archimedean ordered, the subset $\{w(c)^n \mid n \geq 1\}$ is coinitial in $\Gamma^{\bullet}$. Hence $\underline{\xi} = (w(x_{n+1} - x_n))_{n \geq 0}$ is coinitial in $\Gamma^{\bullet}$. Thus ξ is a Cauchy family in M and $z = \lim_{\Gamma} \xi \in \widehat{M}$. Since $\Gamma^{\bullet}$ is coinitial in $\Gamma'^{\bullet}$, ξ is also a Cauchy family in M' and therefore $\lim_{\Gamma} \xi = \lim \xi$. Hence $z = \lim_{\Gamma} \xi = \lim \xi \in \widehat{M}$ and $a + z \in \widehat{K}$ is a root of f.

Example D.21. Let $E = \mathbb{F}_p(t)$ be the field of rational functions over the field $\mathbb{F}_p$ of p elements. Let w be the ultranorm associated to the valuation $v_{\deg}$. Let K' be a maximal immediate extension of the algebraic closure of E with respect to an extension of w. K' is a maximal Kaplansky field.

We consider the polynomial $f = a_0 + a_1X + \ldots + a_{k-1}X^{k-1} + X^k \in A'[X]$ and assume that there exists $a \in NSGG(f, M')$. So $c = f(a) \in M'$, $f'(a) \in A' \setminus M'$ and $e = (f'(a))^{-1}$. We assume that $c \neq 0$. The dynamic mapping $\varphi = \varphi_{a,e,M'} : M' \to M'$ is given by $\varphi(x) = x - ef(a + x)$. Since $\mathbb{F}_p$ is the prime field of K' we obtain $K = \mathbb{F}_p(a_0, a_1, \ldots, a_{k-1}, a)$. K' is a maximal Kaplansky field. Thus there exists a spherical completion $\widetilde{K}$ of K in K'. The value group $w(E^\bullet)$ is discrete and $\Gamma'^\bullet$ is the divisible hull of $w(E^\bullet)$, hence $\Gamma'^\bullet$ is isomorphic to the additive group of $\mathbb{Q}$. Thus, K has a unique completion in K' which therefore is equal to $\widehat{K}$. Similarly, we deduce that E has a unique completion $\widehat{E}$ in K'.

Let $\xi = (x_n)_{n<\mu} \in AP(M)$, $\mu \leq \omega_0$, $x_0 = 0$. Since $\{w(c)^n \mid n \geq 1\}$ is coinitial in $\Gamma'^\bullet$, so also in $\Gamma^\bullet$, it follows that either there exists $n_0 > 0$ such that $z = x_{n_0}$ or $\xi = (x_n)_{n<\omega_0}$ is a Cauchy family in M as well as in M'. In the first case, $a + x_{n_0} \in K$ is a root of f. In the other one, $z = \lim_\Gamma \xi = \lim \xi \in \widehat{M}$, thus then $a + z \in \widehat{K}$ is a root of f.

At the very end, we consider the special case that $a_0, a_1, \ldots, a_{k-1}, a$ are elements of the valuation ring of E (with respect to w). Then $K = \mathbb{F}_p(a_0, a_1, \ldots, a_{k-1}, a)$ is a subfield of E. Hence the unique completion $\widehat{K}$ of K in K' is a subfield of the unique completion $\widehat{E}$ of E inK'. Since K, as a subfield of E, has a discrete ultranorm, $\widehat{K}$ is spherically complete, hence $\widehat{K} = \widetilde{K}$.

Notes

The main focus of this paper is the existence and determination of roots of a given polynomial in one indeterminate with coefficients in an ultranormed ring.

The seminal result was due to Hensel [5], who showed how to obtain an exact factorization of a polynomial with p-adic coefficients knowing an approximate factorization. The so-called Original Hensel's Lemma in the text is another formulation of Hensel's result. It asserts the existence of a root of a polynomial with p-adic coefficients and gives an algorithm which begins with an approximate root (a "good guess") and leads to the root.

The Original Hensel's Lemma was extended to polynomials in wider classes of rings. Hensel had proved it for complete discrete valued fields, Kürschak [8], Ostrowski [15], [16] and Rychlik [22], [23] studied it for polynomials in a complete valued field, with a rank 1 valuation. Krull [7] considered the lemma for polynomials with coefficients in fields endowed with Krull valuations and proved that it remains valid for step-complete, as well as for maximal valued fields. We give statements which are equivalent to Hensel's Lemma for valued fields (see [19]). We refer to [21] for more details about the history of Henselian valuations of rank one.

The results described above concern Hensel's Lemma with non-singular good guesses. Already in 1924 Rychlik [23] proved Hensel's Lemma with good guesses which are, one could say, "weakly singular". In our terminology these are good guesses of level 1. In (A) we study the lemma with good guesses of any level q. Rychlik's Lemma is an important complement to Hensel's Lemma. For example, it allows to prove that Fermat's polynomial $X^p + Y^p + Z^p$ has non-trivial p-adic roots. A comprehensive study of Hensel's and Rychlik's Lemma including many properties which follow from these lemmas is given in [20].

In the multitude of papers on Hensel's Lemma we like to quote the ones by Brink [3], Khanduja and Saha [6], Perdry [17], which offer interesting insights. In view of applications to Algebraic Geometry, Lafon [10] considered a form of Hensel's Lemma for couples (A, L) where A is the coordinate ring of an affine algebraic variety and L is a non-trivial ideal of A.

In [1] Azumaya investigated conditions for a ring to have a valid Hensel's Lemma. These rings have been called Henselian rings. In his papers and the book "Local Rings" [11], [12], [13], [14] Nagata studied Henselian rings and Henselization of rings. Greco [4] added more results to the matter.

A very general and quite useful form of Hensel's Lemma was published by Bourbaki [2].

In [9] Kuhlmann proves for ultrametric spaces (with totally ordered set of distances) a kind of attractor theorem and gives with the help of this theorem a proof of Hensel's Lemma.

The dynamic method in this paper allows to obtain, in a unified and coherent manner, all the results in the literature, but also even more general ones, for example when the set Γ of distances of the ultranorm is not totally ordered. Existence and approximation of roots, which is the quintessence of the dynamic method, were given here a natural as well as elegant treatment. The presentation here is an improvement and continuation of the method and results which were given in our paper [18].

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Pell equations over polynomial rings *

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1 Pell-Abel equations

Teorema (Fermat) : *seja D um número inteiro positivo, que* não é um quadrado perfeito. *Então a equação de Pell-Fermat*

$$x^2 - Dy^2 = 1$$

possui uma solução em números inteiros (x, y), com $y \neq 0$, e portanto uma infinidade de tais soluções.

NB : (i) this implies that for $O = \mathbb{Z}[\sqrt{D}]$, the unit group O^* has rank 1.

(ii) the classical proof uses the box principle three times, and studies the *generalized Pell-Fermat equations* $x^2 - Dy^2 = n$.

(iii) non squarefree numbers are allowed. If $\Delta = f^2 D$, $[O : O_\Delta] = f$, then O^*_Δ contains $Ker\{O^* \to (O/fO)^*\}$, so has same rank as O^*.

Abel replaced $\mathbb{Z}$ by $\mathbb{C}[X]$ to study why $\int \frac{f(x)}{\sqrt{D(x)}} dx$ can sometimes be computed without abelian integrals. This leads to the following definition, where we restrict from now on to the base field $k = \overline{\mathbb{Q}}$: a non constant polynomial $D \in k[X]$ is *pellian* if there exists $A, B \in k[X]$, with $B \neq 0$, such that

$$A^2 - DB^2 = 1.$$

Then, $deg(D) = d$ is even, and D is not a square. Any such D with $d = 2$ is pellian, so from now on, $d \geq 4$, and we moreover assume that *D is square-free.* Consider the (hyper)elliptic curve $y^2 = D(x)$. The ring $k[x, y]$ is a Dedekind domain, the smooth complete model C of the curve has two additional places ∞^+, ∞^-, and genus $g(C) = (d/2) - 1$.

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Theorem (Abel) : • *D is pellian if and only if the class p of the divisor $(\infty^+) - (\infty^-)$ in the jacobian $Jac(C)$ of C is a torsion point.*

Let now $\Delta = (X-\rho)^2 D$, where $D(\rho) \neq 0$, and let $Jac_{\mathfrak{q}}(C)$ be the generalized jacobian for the modulus $\mathfrak{q} = \mathfrak{q}^+ + \mathfrak{q}^-$ (above ρ). Then,

• *Δ is pellian if and only if the class $\tilde{p}$ of $(\infty^+) - (\infty^-)$ in $Jac_{\mathfrak{q}}(C)$ is a torsion point.*

And if $\overline{\Delta} = (X-\rho)^2 D$, where $D(\rho) = 0$, so one point $\overline{\mathfrak{q}}$ above ρ :

• *$\overline{\Delta}$ is pellian if and only if the class $\overline{p}$ of $(\infty^+) - (\infty^-)$ in $Jac_{2\overline{\mathfrak{q}}}(C)$ is torsion.*

2 Unlikely intersections

How often does Abel's condition occur ?

This is the theme of "unlikely intersections", a topic initiated by Bombieri- Masser-Zannier for tori, then Zilber, then merging with conjectures of Pink which generalize Manin-Mumford, Mordell-Lang, André-Oort, ...

The general idea is as follows : *it is unlikely that a subvariety W of a special variety G meets the union of the special subvarieties of G of codimension $> dim(W)$ Zariski-densely, unless W lies in a proper special subvariety. Same principle for a family $\{G_\lambda, \lambda \in S\}$ of special varieties.*

Following Masser-Zannier, let the polynomial D_λ depend on a parameter λ varying on a *curve* $S/\overline{\mathbb{Q}}$ (but allow finite base change S'/S in next statements). Let

$$S_D = \{\lambda \in S(\overline{\mathbb{Q}}), D_\lambda \text{ is pellian}\}.$$

S_D can be empty, finite, full... Say it is *sparse* if it has finitely many elements of given degree over $\mathbb{Q}$; e.g. any subset of $S(\overline{\mathbb{Q}})$ of bounded height. Same notation for Δ_λ, $\overline{\Delta}_\lambda$.

Here are some examples.

[$d = 4$]
$(d = 4.i)$: $D_\lambda(x) = x^4 + x + \lambda \Rightarrow S_D$ is sparse, but infinite;
$(d = 4.ii)$: $D'_\lambda(x) = x^4 + (2\lambda + 1)x^3 + 3\lambda x^2 + \lambda \Rightarrow S_{D'}$ idem;
$(d = 4.iii)$: $D''_\lambda(x) = x(x^3 + x + \lambda) \Rightarrow S_{D''}$ idem.

[Masser-Zannier (2013), and (2015) for $d \geq 6$]
$(d = 6.i)$: *$\mathcal{D}_\lambda(x) = x^6 + x + \lambda \Rightarrow S_{\mathcal{D}}$ is finite;*
$(d = 6.ii)$: *$\mathcal{D}''_\lambda(x) = x^6 + x^2 + \lambda \Rightarrow S_{\mathcal{D}''}$ is sparse, but infinite !*

[B.-Masser-Pillay-Zannier & B.-Edixhoven (2016)]
$(d = 4 + 2.i)$: *$\Delta_\lambda = (x + \frac{1}{2})^2 D_\lambda(x) \Rightarrow S_\Delta$ is finite;*
$(d = 4 + 2.ii)$: *$\Delta'_\lambda(x) = (x + \frac{1}{2})^2 D'_\lambda(x) \Rightarrow S_{\Delta'} = S_{D'}$ is sparse, but infinite !!*

[H. Schmidt (2017)]
$(d = 3+3)$: $\overline{\Delta}''_\lambda = x^3(x^3+x+\lambda) = x^2 D''_\lambda(x) \Rightarrow S_{\overline{\Delta}''}$ *is finite.*

The next section sketches how these conclusions can be reached.

NB : families of Pell equations also occur in the classical case over $\mathbb{Z}$, cf. [R].

3 Relative Manin-Mumford

Theorem 1. [M-Z] *Let $G = A$ be an abelian scheme of relative dimension $g \geq 2$ over the curve $S/\overline{\mathbb{Q}}$, and let $p \in A(S)$ be a section. Then the set*

$$S_p := \{\lambda \in S(\overline{\mathbb{Q}}), p(\lambda) \in A_\lambda^{tor}\}$$

is infinite if and only if one of the following conditions holds :
a) *p is a torsion section;*
b) *there exists an elliptic subscheme E/S of A/S such that a multiple of p factors through E, and is not a constant section if E/S is isoconstant.*

In (**b**), we are in *" relative dimension 1"* : if p is a section of E/S which is not constant (and not torsion), it meets E^{tor} densely (but *with bounded height* (Silverman)). This explains the cases $(d = 4)$, as well as the unexpected case $(d = 6.ii)$, whereas $Jac(C)$ is a simple abelian scheme in the case $(d = 6.i)$

The cases $(d = 4{+}2)$ require the introduction of a semi-abelian scheme $G \in Ext_S(E, \mathbb{G}_m) \simeq \hat{E}(S) \simeq E(S)$, of relative dimension 2 .

Theorem 2. [B-M-P-Z, B-E] *Let $S/\overline{\mathbb{Q}}$, G/S, parametrized by $q \in E(S)$, and let $\tilde{p} \in G(S)$ be a section, with projection $p \in E(S)$. Then, the set*

$$S_{\tilde{p}} := \{\lambda \in S(\overline{\mathbb{Q}}), \tilde{p}(\lambda) \in G_\lambda^{tor}\}$$

is infinite if and only if one of the following conditions is satisfied :
a) *$\tilde{p}$ is a torsion section;*
b) *there exists an elliptic subscheme E'/S of G/S (equivalently, q is a torsion section) and a multiple of $\tilde{p}$ which factors through E', and is not constant if E'/S (equivalently E/S) is isoconstant;*
b') *a multiple of $\tilde{p}$ factors though $\mathbb{G}_{m/S}$ (equivalently, p is torsion), and is not constant;*
c) *$\tilde{p}$ is a "Ribet section" (in particular, $p(\lambda) \in E_\lambda^{tor} \Rightarrow \tilde{p}(\lambda) \in G_\lambda^{tor}$).*

Notice that for $G = Jac_{\mathfrak{q}}(C) \in Ext(E, \mathbb{G}_m)$, and on denoting by (.), resp. [.], the standard, resp. narrow, class of a divisor on C :

$$\mathfrak{q}^\pm = (\rho(\lambda), \pm\sqrt{D_\lambda(\rho(\lambda))}) \rightsquigarrow q = ((\mathfrak{q}^+) - (\mathfrak{q}^-)) \in \hat{E}(S) \rightsquigarrow G = G_q$$

$$G(S) \ni \tilde{p} = [(\infty_+) - (\infty_-)] \to p = ((\infty_+) - (\infty_-)) \in E(S).$$

Finally, the case $(d = 3+3)$ corresponds to an extension G of E by $\mathbb{G}_a$.

Now,

• in (4+2.i), E is not isoconstant, p is not torsion, most ρ's give a non-torsion $q \rightsquigarrow$ usual case of [BMPZ]. While if q is torsion, G is (iso)split, but $\tilde{p}$ projects to a non constant point of $\mathbb{G}_m$, barring its Case (**b**).

• in (4+2.ii), $E \simeq E' : y^2 = x^3 + \lambda(1-\lambda)x$, which has CM by $\mathbb{Z}[i]$, and the points $\infty^{\pm}, \mathfrak{q}^{\pm}$ are given on this model by $\mathfrak{p}'^{\pm}(\lambda) = (\lambda, \pm\lambda)$, $\mathfrak{q}'^{\pm}(\lambda) = (-\lambda, \pm i\lambda)$. So, $\mathfrak{q}'^{+} = [i]\mathfrak{p}'^{+}$, $\mathfrak{q}'^{-} = [i]\mathfrak{p}'^{-}$, and $q = [i]p$.

Then, $\tilde{p}' = [(\mathfrak{p}'^{+}) - (\mathfrak{p}'^{-})] \in G'(S) \simeq G(S) \ni \tilde{p} := s_R$ is a Ribet section, and by Weil's law of reciprocity :

$$[n]p(\lambda) = 0 \Rightarrow [2n^2]\tilde{p}(\lambda) = 0.$$

Combined with Principle (**b**), this shows that $\tilde{p}$ assumes torsion values infinitely often.

• as for the case $(d = 3+3)$: $G = Jac_{2\overline{\mathfrak{q}}}(C) \in Ext(E, \mathbb{G}_a)$ is never split.

4 Relative Mordell-Lang

This extension of the relative Manin-Mumford problem occurs in the study of the *generalized Pell-Abel equations*. These have recently been studied by Barroero-Capuano, after previous work of Masser-Zannier. Here is a slightly different presentation, in a special case.

Let D be separable as before, with $d \geq 4$, *fix* $\alpha \in k$ with $D(\alpha) \neq 0$, and consider the equation in unknowns $A, B \in k[X]$, $B \neq 0$ and $e \in \mathbb{Z}_{\geq 0}$:

$$A^2 - DB^2 = (X - \alpha)^e.$$

Let $\mathfrak{a}^{\pm} \in C(k)$ above α, and $s = ((\mathfrak{a}^+) - (\infty^-)) \in Jac(C)$. Recall that $p = ((\infty^+) - (\infty^-))$.

Proposition : *the equation has a solution if and only if the points s and p of $Jac(C)$ are linearly dependent over $\mathbb{Z}$.*

NB : p is torsion iff there is a solution with $e = 0$. Whereas $e > 0$ forces s to lie in the *divisible hull of $\mathbb{Z}.p$* (including torsion), as in the Mordell-Lang problem.

Now, how often (still over a parameter curve $S/\overline{\mathbb{Q}}$) can this condition occur ?

Theorem 3. [Ba-Ca](2018) *Let A/S be an abelian scheme of relative dimension $g \geq 2$, et let $P \in A(S)$ be a section. Let $A^{[2]}$ be the union of all the flat subgroup schemes of A/S of codimension ≥ 2. Then, the set*

$$S_P^{[2]} := \{\lambda \in S(\overline{\mathbb{Q}}), P(\lambda) \in (A^{[2]})_\lambda\}$$

is finite, unless P factors through a strict subgroup scheme of A/S.

In particular, let B/S be an abelian scheme, *whose generic fiber contains no elliptic curve*, and let $s, p \in B(S)$ be two sections. Assume that the set

$$S^{\ell d}_{s,p} = \{\lambda \in S(\overline{\mathbb{Q}}), s(\lambda) \text{ and } p(\lambda) \text{ are lin. dep. over } \mathbb{Z}\}$$

is infinite. Then, *s and p are linearly dependent over $\mathbb{Z}$.*

Similarly, replace B by $G \in Ext_S(E_0, \mathbb{G}_m)$, where E_0 has CM, and $G = G_q$ for a non constant $q \in E_0(S)$. In particular, $q \notin E_0^{tor}$, so G contains no elliptic curve. The Ribet sections form a group of rank 1 of $G(S)$, and we fix a non-torsion one, say $s_R = \tilde{p}_R$.

Fix another section $s \in G(S)$. Since $S_{s_R} = \{\lambda \in S(\overline{\mathbb{Q}}), s_R(\lambda) \in G^{tor}_\lambda\}$ is infinite (cf. Case 4+2.ii), we should focus on its complement in $S^{\ell d}_{s,s_R}$:

Theorem 4. [B-Sch](2018) *Let $s \in G(S)$, and assume that the set*

$$S_{R,s} = \{\lambda \in S(\overline{\mathbb{Q}}), s(\lambda) \text{ lies in the divisible hull of } \mathbb{Z}.s_R(\lambda) \text{ in } G_\lambda\}$$

is infinite. Then, $s = s' + s''$, where s' lies in the divisible hull of $\mathbb{Z}.s_R$ in G (i.e. is a Ribet section), and s'' factors through $\mathbb{G}_m$.

For the *proofs*, we refer to the bibliography below. They are based on the *o-minimal strategy* introduced by Pila-Zannier and Masser-Zannier in these topics, via the following steps :

- bounded height;
- height of relations controlled by degrees;
- *o*-minimal count on an incidence variety (Pila-Wilkie, Habegger-Pila);
- functional algebraic (in-)dependence $\rightsquigarrow$ possible obstructions.

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