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STRUCTURES ALGÉBRIQUES ORDONNÉES

SÉMINAIRE 2020-2022

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UNIVERSITÉS PARIS-CITÉ et SORBONNE UNIVERSITÉ

Projets Logique Mathématique et Théorie des Nombres

Institut de Mathématiques de Jussieu - Paris Rive Gauche – UMR 7586

SÉMINAIRE DE STRUCTURES ALGÉBRIQUES ORDONNÉES

Responsables: F. Delon, M. Dickmann, D. Gondard,

2021-2022

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SÉMINAIRE DE STRUCTURES ALGÉBRIQUES ORDONNÉES

Responsables: F. Delon, M. Dickmann, D. Gondard

2021 – 2022

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Hyperseries and surreal numbers

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Abstract

I give an informal presentation of the project of establishing a correspondence between certain types of germs, formal series and abstract numbers, in the framework of real asymptotic differential algebra.

1 Introduction

A standard method for studying well-behaved functions occurring in real geometry and analysis is to rely on formal counterparts of these functions: formal power series and generalized power series for analytic functions and quasianalytic classes, transseries for functions arising from o-minimal geometry…

One advantage of the formal method is to give a clear-cut setting in which to distinguish analytic or geometric properties from purely formal properties. The latter can sometimes be derived as consequences of universal identities, which may then be recovered in the geometric setting *via* various methods of evaluation or resummation (e.g. analytic or quasianalytic evaluation in almost convergent settings [35], Ecalle-Borel summation for certain classically divergent settings [15]).

One second advantage of formal objects over geometric ones is that by their nature, formal objects are uniquely determined by "what they look like", and do not suffer from ambiguities (for instance among solutions of a differential equation). However, an algebra of formal series is a useful tool in studying analytic-geometric problems only insofar as it is amenable to formal versions of the analytic operations involved: resolution of algebraic equations, derivation, integration, composition…

A particularly good example of such formal objects is Dahn-Göring [16] and Ecalle's [20] logarithmic-exponential transseries (or variants thereof). These were used for instance in order to study germs of Dulac maps [20, 28, 34] and to describe the differential algebra of germs at $+\infty$ of non-oscillating differentiable real-valued functions [27, 1]. In the latter case, transseries are sufficiently rich to fully describe the differential algebra of differential fields of such germs. In that sense, transseries can be understood as ideal notions of growth rates of non-oscillating functions at $+\infty$. However, they prove themselves insufficient, by a large margin, to describe the asymptotic behavior of non-oscillating solutions of more general functional equations. The main reason for this is that transseries are not closed under simple functional equations which have non-oscillating solutions in the realm of germs of real-valued functions. If one is to attempt to study tame ordered algebraic structures equipped with non-commutative operations (such as groups under composition of definable germs in o-minimal structures), it is necessary to construct larger formal models.

In this presentation, we describe these issues at the levels of germs and transseries, before showing how to construct fields of formal series called hyperseries which do not present the same shortcomings. Constructing such fields is technical and requires a few careful steps. We then explain how Conway's surreal numbers form, in a canonical way, a field of hyperseries which we believe is well suited to the project of studying functional equations over non-oscillating germs.

2 Germs

We first describe the type of terms we will be concerned with in the sequel.

2.1 Germs at $+\infty$ and Hardy fields

We identify two functions f, g in

$$\bigcap_{n \in \mathbb{N}} \bigcup_{r \in \mathbb{R}} \mathcal{C}^n((r, +\infty), \mathbb{R}),$$

if $f(r) = g(r)$ for all sufficiently large $r \in \mathbb{R}$ (written $r \gg 1$). Let $\mathcal{G}$ denote the differential ring of equivalence classes, called *germs at* $+\infty$. A *Hardy field* is a differential subfield of $\mathcal{G}$ containing $\mathbb{R}$. By the intermediate value theorem for continuous real valued functions (see [36, Introduction]), such a field field is linearly ordered by

$$\forall f, g \in \mathcal{H}, (f < g \Longleftrightarrow \forall r \gg 1, (f(r) < g(r))).$$

Example 2.1. Writing x for the germ of the identity, the fields $\mathbb{R}((x^r)_{r \in \mathbb{R}})$, $\mathbb{R}(x, \arctan x)$, $\mathbb{R}(x, \mathrm{e}^x)$, and $\mathbb{R}(\log x, \log \log x)$.

Example 2.2. Let $\mathcal{R} = (\mathbb{R}, +, \cdot, <, \ldots)$ be an o-minimal extension of the real ordered field. Write $\mathcal{H}(\mathcal{R})$ for the set of germs at $+\infty$ of functions $(a, +\infty) \longrightarrow \mathbb{R}, a \in \mathbb{R}$ that are definable in $\mathcal{R}$. Then $\mathcal{H}(\mathcal{R})$ is a Hardy field [31]. In particular, the field $\mathcal{H}_{\exp} = \mathcal{H}(\mathbb{R}, +, \cdot, \exp, <)$ is a Hardy field.

2.2 Maximal Hardy fields

A Hardy field $\mathcal{H}$ is said *maximal* if it has no proper superset which is a Hardy field. Maximal Hardy fields exist in virtue of Zorn's lemma, and are by nature very large. This can be illustrated in two ways.

First, it is known that maximal Hardy fields are closed under certain algebraic differential equations:

Theorem. [17] *If $\mathcal{H}$ is a maximal Hardy field and $f, h \in \mathcal{H}$, and $P(Y, Y') \in \mathcal{H}[Y, Y']$ satisfy*

$$P(f, f') < 0 < P(h, h'),$$

then there is $g \in \widetilde{(f, h)}$ with $P(g, g') = 0$.

It is conjectured [2, Conjecture A] that the above theorem can be extended to differential polynomials of arbitrary order. Secondly, maximality entails that the underlying linear order of a maximal Hardy field does not contain certain cuts. More precisely:

Theorem. [13, Theorem 1.1] *Let $\mathcal{H}$ be a Hardy field, and let $A \subseteq \mathcal{H}$ be a countable subset. There is an $\Omega \in \mathcal{G}$ such that*

$$\forall f \in A, f < \Omega,$$

and that $\mathcal{H}(\Omega, \Omega', \Omega'', \ldots)$ is a Hardy field.

It is conjectured that the underlying ordered set of a maximal Hardy field is an η_1-set, i.e.:

Conjecture 2.3. [2, Conjecture B] *Let $\mathcal{H}$ be a Hardy field, and let $A, B \subseteq \mathcal{H}$ be countable with $f < h$ for all $f \in A$ and $h \in B$. Then there is a $g \in \mathcal{G}$ such that*

$$\forall f \in A, \forall h \in B, f < g < h,$$

and that $\mathcal{H}(g, g', g'', \ldots)$ is a Hardy field.

In particular, applying Theorem 2.2 to the set $A = \{\exp, \exp \circ \exp, \exp \circ \exp \circ \exp, \ldots\}$ in $\mathcal{H}_{\exp}$, one has Hardy fields containing *transexponential* germs, i.e. germs which grow faster than any finite iterate $\exp_n := \exp \circ \cdots \circ \exp$ of the exponential. We will also write $\log_n = \log \circ \cdots \circ \log$ for $n \in \mathbb{N}$.

2.3 Transexponentials

Let us look a bit more closely to transexponential germs. Since germs of elementary functions arising in analysis (e.g. Hardy's logarithmico-exponential functions [23], or as solutions of ordinary differential equations) are not transexponential, it is necessary to consider different types of problems in order to encounter well-behaved . A natural is Abel's equation

$$\forall r \gg 1, E(r+1) = \exp(E(r)). \tag{2.1}$$

for the exponential function, whose solution E can be seen as a transfinite iterate of exp. It is known [29] that there is a strictly increasing analytic function $\exp_\omega : \mathbb{R}^{\geqslant} \longrightarrow \mathbb{R}$ which solves Abel's equation. The germ of $\exp_\omega$ is transexponential.

Theorem. [13] *The germ of E_ω generates a Hardy field, i.e. $\mathbb{R}(\exp_\omega, \exp_\omega', \exp_\omega'', \ldots)$ is a Hardy field.*

Recently [33], it was shown that there are Hardy fields containing the germ of $\exp_\omega$, of its eventual functional inverse $\log_\omega$, and which are closed under composition of germs.

3 Hahn series

We next give a brief description of the notion of field of generalized series, and introduce the main elements of structure on the field of transseries.

3.1 Ordered valued fields of Hahn series

Definition 3.1. [22] *Let $(\mathfrak{M}, \times, \prec)$ be a linearly ordered Abelian group (possibly class-sized). The* **Hahn series** *field $\mathbb{R}[[\mathfrak{M}]]$ is the class of functions $f : \mathfrak{M} \longrightarrow \mathbb{R}$ whose support*

$$\operatorname{supp} f := \{\mathfrak{m} \in \mathfrak{M} : f(\mathfrak{m}) \neq 0\} \subseteq \mathfrak{M}$$

is a well-ordered <u>*subset*</u> *of* $(\mathfrak{M}, \succ)$ *(where* $\succ$ *is the reverse ordering on* $\mathfrak{M}$*).*

The set $\mathbb{R}[[\mathfrak{M}]]$ has a natural structure of ordered valued field under:

- **pointwise sum:** $(f+g) = \mathfrak{m} \longmapsto f(\mathfrak{m}) + g(\mathfrak{m})$

  Cauchy product: $(fg) = \mathfrak{m} \mapsto \sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} f(\mathfrak{u})\, g(\mathfrak{v})$
- **linear ordering:** $0 < f \Longleftrightarrow 0 < f(\max \operatorname{supp} f)$

  valuation: $\forall f \in \mathbb{R}[[\mathfrak{M}]]^{\times}, \mathfrak{d}_f := \max \operatorname{supp} f \in \mathfrak{M}$

The function $\mathfrak{d}.\colon \mathbb{R}[[\mathfrak{M}]]^{\times} \longrightarrow \mathfrak{M}$ is a valuation with value group $(\mathfrak{M}, \cdot, \succ)$. We write $f \prec g$ if $f = 0$ and $g \neq 0$ or $f, g \in \mathbb{R}[[\mathfrak{M}]]^{\times}$ and $\mathfrak{d}_f \prec \mathfrak{d}_g$. This is a dominance relation as per [1, Definition 3.1.1].

We identify each $\mathfrak{m} \in \mathfrak{M}$ with the indicator function

$$\chi_{\{\mathfrak{m}\}}\colon \mathfrak{M} \longrightarrow \mathbb{R}$$

of the singleton $\{\mathfrak{m}\}$. Then $\mathfrak{M} \subseteq \mathbb{R}[[\mathfrak{M}]]^{\times}$ is a subgroup whose elements are called *monomials*.

3.2 Summability

Fix a Hahn series field $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$. There is a well-known notion of (formal) summability for certain infinite families in $\mathbb{S}$ called summable families. It is based on the straightforward idea of infinite pointwise sums of functions $\mathfrak{M} \longrightarrow \mathbb{R}$.

Definition 3.2. *Let* $\mathbf{I}$ *be a class, and let* $(s_i)_{i \in \mathbf{I}} \in \mathbb{S}^{\mathbf{I}}$ *be a family. Then* $(s_i)_{i \in \mathbf{I}}$ *is* **summable** *if*

a) For all $\mathfrak{m} \in \mathfrak{M}$*, the class* $\mathbf{I}_{\mathfrak{m}} := \{i \in \mathbf{I} : \mathfrak{m} \in \operatorname{supp} f_i\}$ *is finite.*

b) The class $\mathfrak{S} := \bigcup_{i \in \mathbf{I}} \operatorname{supp} f_i$ *is a well-ordered subset of* $(\mathfrak{M}, \succ)$*.*

Condition *a*) implies that the pointwise sum

$$\begin{aligned} \sum_{i \in \mathbf{I}} s_i \colon \mathfrak{M} &\longrightarrow \mathbb{R} \\ \mathfrak{m} &\longmapsto \sum_{i \in \mathbf{I}_{\mathfrak{m}}} s_i(\mathfrak{m}) \end{aligned}$$

is a well-defined map $\mathfrak{M} \longrightarrow \mathbb{R}$. Condition *b*) implies that the support $\operatorname{supp}\left(\sum_{i \in I} s_i\right) \subseteq \mathfrak{S}$ is well-ordered in $(\mathfrak{M}, \succ)$, whence $\sum_{i \in \mathbf{I}} s_i \in \mathbb{R}[[\mathfrak{M}]]$.

Remark 3.3. The family $\left(\left(\sum_{i \in I_{\mathfrak{m}}} s_i(\mathfrak{m})\right) \mathfrak{m}\right)_{\mathfrak{m} \in \mathfrak{M}}$ is summable, with

$$\sum_{i \in I} s_i = \sum_{\mathfrak{m} \in \mathfrak{S}} \left(\sum_{i \in I_{\mathfrak{m}}} s_i(\mathfrak{m}) \right).$$

Remark 3.4. For $s \in \mathbb{S}$, the family $(s(\mathfrak{m})\, \mathfrak{m})_{\mathfrak{m} \in \mathfrak{M}}$ is summable with sum

$$\sum_{\mathfrak{m} \in \mathfrak{M}} s(\mathfrak{m})\, \mathfrak{m} = s.$$

Hence elements in $\mathbb{R}[[\mathfrak{M}]]$ can be seen as formal series with coefficients in $\mathbb{R}$ and monomials in $\mathfrak{M}$.

Proposition 3.5. [32] *For* $(r_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ *and* $\varepsilon\in\mathbb{S}$ *with* $\varepsilon\prec 1$, *the family* $(r_n\,\varepsilon^n)_{n\in\mathbb{N}}$ *is summable.*

$$\textit{e.g.}\quad \frac{1}{1+\varepsilon}=\sum_{n\in\mathbb{N}}(-1)^n\,\varepsilon^n.$$

See [24, Appendix A] for more details on summability and [25] for generalizations of Neumann's summability result above.

3.3 Transseries

The field $\mathbb{T}_{\mathrm{LE}}$ of log-exp *transseries* is a field of Hahn series involving formal terms x, $\log x$ and e^x and combinations thereof.

$$\text{e.g.}\quad f:=\sum_{n=1}^{+\infty} n!\,\mathrm{e}^{x/n}+\underbrace{\log_2 x}_{=\log\log x}+7+x^{-1}\log x+\sum_{p=0}^{+\infty}\mathrm{e}^{-x^{p+1}}x^p\quad\text{is a log-exp transseries.}$$

The number of iterations of exp and log must be uniformly bounded. See [19] for a definition.

- $\mathbb{T}_{\mathrm{LE}}$ enjoys a *derivation* $\partial\colon\mathbb{T}_{\mathrm{LE}}\longrightarrow\mathbb{T}_{\mathrm{LE}}$ which acts termwise, e.g.

$$f\circ\log x=\sum_{n=1}^{+\infty} n!\,x^{1/n}+\log_3 x+7+(\log x)^{-1}\log_2 x+\sum_{p=0}^{+\infty}\mathrm{e}^{-(\log x)^{p+1}}(\log x)^p.$$

- $\mathbb{T}_{\mathrm{LE}}$ enjoys a *composition law* $\circ\colon\mathbb{T}_{\mathrm{LE}}\times\mathbb{T}_{\mathrm{LE}}^{>\mathbb{R}}\longrightarrow\mathbb{T}_{\mathrm{LE}}$ which acts termwise on the right:

$$f\circ\log x=\sum_{n=1}^{+\infty} n!\,x^{1/n}+\log_3 x+7+(\log x)^{-1}\log_2 x+\sum_{p=0}^{+\infty}\mathrm{e}^{-(\log x)^{p+1}}(\log x)^p.$$

The model theory of $(\mathbb{T}_{\mathrm{LE}},+,\cdot,\partial,<,\prec)$ is surprisingly tame

Theorem 3.6. [1, Chapter 16] $\mathrm{Th}(\mathbb{T}_{\mathrm{LE}},+,\times,\partial,\mathbb{R})$ *has QE in a natural language, is model-complete, decidable, and o-minimal at infinity.*

Good progress has been made on the following conjecture regarding the relation between $\mathbb{T}_{\mathrm{LE}}$ and maximal Hardy fields:

Conjecture 3.7. [2] $\mathrm{Th}(\mathbb{T}_{\mathrm{LE}},+,\times,\partial)$ *is the theory of maximal Hardy fields.*

As for the expansion of this ordered differential field by the composition law (for instance extended to $\mathbb{T}_{\mathrm{LE}}\times\mathbb{T}_{\mathrm{LE}}$ by setting $f\circ g:=0$ whenever $g\not>\mathbb{R}$), its first-order theory is much less tame. This can be illustrated by exhibiting simple functional equations (i.e. equational atomic formulas) without solution in $\mathbb{T}_{\mathrm{LE}}$ but which may have solutions in its extensions.

3.4 Cuts in transseries

We use the notion of cut of [26, Chapter 9] to illustrate the previous point.

Definition 3.8. *A* **cut** *in* $\mathbb{T}_{\mathrm{LE}}$ *is an* $\leqslant$*-initial subset* $\boldsymbol{c}\subseteq\mathbb{T}_{\mathrm{LE}}$ *without supremum in* $(\mathbb{T}_{\mathrm{LE}},\leqslant)$.

Certain monotonous operations $\boldsymbol{c}_1+\boldsymbol{c}_2, \boldsymbol{c}_1\times\boldsymbol{c}_2, \mathrm{e}^{\boldsymbol{c}_1}$ on $\mathbb{T}_{\mathrm{LE}}$ extend to the set of cuts $\boldsymbol{c}_1, \boldsymbol{c}_2$ in However, cut operations behave quite differently from standard operations: for instance the additive law of cuts is not cancellative. The good way to define operations on cuts is via surreal numbers (see Section 5).

In order to prove his intermediate value theorem for differential polynomials on the field $\mathbb{T}_g \subsetneq \mathbb{T}_{\mathrm{LE}}$ of grid-based transseries, van der Hoeven gave a classification of cuts in $\mathbb{T}_g$ in terms of transseries in $\mathbb{T}_{\mathrm{LE}}$. We can distinguish three types of cuts:

a) "Serial" cuts related to missing pseudo-limits:

$$\begin{aligned}
\mathbf{L} &:= \{f:\exists n\in\mathbb{N}, f<\log x+\log_2 x+\cdots+\log_n x\}\\
\boldsymbol{\Lambda} &:= \partial(\mathbf{L})=\left\{f:\exists n\in\mathbb{N}, f<\frac{1}{x}+\frac{1}{x\log x}+\cdots+\frac{1}{x\log x\cdots\log_n x}\right\}\\
\boldsymbol{\Gamma} &:= \mathrm{e}^{-\mathbf{L}}=\left\{f:\forall n\in\mathbb{N}, f<\frac{1}{x\log x\cdots\log_n x}\right\}.
\end{aligned} \tag{3.1}$$

Serial cuts are related to difference equations and extensions by adding Hahn series. For instance, the difference equation

$$f-f\circ\log x=\log x \qquad (\mathcal{E}_s)$$

has no solution in $\mathbb{T}_{\mathrm{LE}}$. Any solution f in extensions $\mathbb{T}\supsetneq\mathbb{T}_{\mathrm{LE}}$ of $\mathbb{T}_{\mathrm{LE}}$ fills $\mathbf{L}$.

Solution: $\log x+\log_2 x+\cdots$ in $\mathbb{R}[[\mathfrak{M}]]\supseteq\mathbb{T}_{\mathrm{LE}}$. We have $\partial(\mathbf{L})=\boldsymbol{\Lambda}$; we expect that

$$\partial(\log x+\log_2 x+\cdots)=\frac{1}{x}+\frac{1}{x\log x}+\cdots+\frac{1}{x\log x\cdots\log_n x}+\cdots$$

b) "Vertical" cuts

$$\begin{aligned}
\boldsymbol{\Omega} &:= \mathbb{T}_{\mathrm{LE}}=\left\{f:\exists n\in\mathbb{N}, f<\mathrm{e}^{\cdot^{\cdot^{\mathrm{e}^x}}}\ (n \text{ times})\right\}\\
\infty &:= \{f:\exists r\in\mathbb{R}, f<r\}=\{f:\forall n\in\mathbb{N}, f<\log_n x\}.
\end{aligned}$$

Vertical cuts are related to conjugacy equations and extensions by adding hyperexponentials or hyperlogarithms. For instance, the formal version of Abel's equations

$$f\circ(x+1)=\mathrm{e}^x\circ f \qquad (\mathcal{E}_v^+) \qquad \text{and} \qquad f\circ\log x=f-1 \qquad (\mathcal{E}_v^-)$$

have no solution in $\mathbb{T}_{\mathrm{LE}}$. Any solution of $(\mathcal{E}_v^+)$ (resp. $(\mathcal{E}_v^-)$) in extensions of $\mathbb{T}_{\mathrm{LE}}$ fills $\boldsymbol{\Omega}$ (resp. ∞).

Solutions: hyperexponential e_ω and hyperlogarithm ℓ_ω.

Note that $\partial(\infty)$ is the cut $\boldsymbol{\Gamma}$ of (3.1). Indeed we expect that

$$\partial(\ell_\omega)=\frac{1}{x\log x\cdots\log_n x\cdots}.$$

c) "Nested" cuts

$$\mathbf{N} := \left\{f:\exists n\in\mathbb{N}, f<\sqrt{x}+\mathrm{e}^{\sqrt{\log x}+\mathrm{e}^{\cdot^{\cdot^{\sqrt{\log_n x}}}}}\right\}. \tag{3.2}$$

Nested cuts are related to general functional equations and extensions by adding nested transseries. The equation

$$f=\sqrt{x}+\mathrm{e}^{f\circ\log x} \qquad f\sim\sqrt{x} \qquad (\mathcal{E}_n)$$

has no solution in $\mathbb{T}_{\mathrm{LE}}$. Any solution of $(\mathcal{E}_n)$ would fill $\mathbf{N}$.

Solutions: "infinitely nested" series

$$f_n = \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log_2 x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}}. \tag{3.3}$$

It is consistent [37, Section 2.5] to consider fields of transseries containing f_n.

4 Hyperseries

Let us next see how to construct fields of generalized transseries, called hyperseries, which account for each of this type of cuts, equations and extensions.

4.1 Logarithmic hyperseries

Schmeling defined [37, Sections 8–10] a Hahn series field $\mathbb{H}$ equipped with functions E_{ω^k}, $L_{\omega^k}: \mathbb{H}^{>\mathbb{R}} \longrightarrow \mathbb{H}^{>\mathbb{R}}$ for all $k \in \mathbb{N}$, where

$$\begin{aligned} E_{\omega^k} \circ L_{\omega^k} &= L_{\omega^k} \circ E_{\omega^k} = \mathrm{id}_{\mathbb{H}^{>\mathbb{R}}}, \quad \text{and} \\ E_{\omega^{k+1}}(s+1) &= E_{\omega^k} \circ E_{\omega^{k+1}}(s) \quad \text{for all } k \in \mathbb{N} \text{ and } s \in \mathbb{H}^{>\mathbb{R}} \qquad \text{(compare with (2.1))} \end{aligned}$$

The function L_{ω^k} can be thought of as a transfinite iterate of log of order k (see [20, Chapter 8]), and in particular each $L_{\omega^{k+1}}$ grows slowlier than any iterate $L_{\omega^k}^{\circ n}, n \in \mathbb{N}$ of the previous one.

In fact, it is easier to work with hyperlogarithms $L_{\omega^k}, k \in \mathbb{N}$ than with hyperexponentials, because of properties of the radii of formal convergence of their Taylor series. The various simplifications that are possible when doing so led van den Dries, van der Hoeven and Kaplan to construct [18] a field $\mathbb{L}$ of hyperseries of a purely logarithmic nature called the field of logarithmic hyperseries.

The field $\mathbb{L}$ is given as the Hahn series field $\mathbb{L} = \mathbb{R}[[\mathfrak{L}]]$ where $\mathfrak{L}$ is the group of formal products

$$\mathfrak{l} = \prod_{\gamma < \rho} \ell_\gamma^{\mathfrak{l}_\gamma}, \quad (\mathfrak{l}_\gamma)_{\gamma < \rho} \in \mathbb{R}^\rho$$

where ρ ranges in the class $\mathbf{On}$ of ordinals, ordered lexicographically: with

$$\mathfrak{l} \succ 1 \Longleftrightarrow ((\mathfrak{l} \neq 1) \quad \text{and} \quad \mathfrak{l}_\gamma > 0 \text{ where } \gamma := \min\{\gamma \in \mathbf{On} : \mathfrak{l}_\gamma \neq 0\}).$$

Setting $\ell_{\omega^\mu + \gamma} := \ell_\gamma \circ \ell_{\omega^\mu}$ for all $\gamma < \omega^{\mu+1}$ and

$$\begin{aligned} \ell_\rho{}' &:= \frac{1}{\prod_{\gamma < \rho} \ell_\gamma}, \\ \ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} &:= \ell_{\omega^{\mu+1}} - 1, \end{aligned}$$

it was proved that there is a unique extension of $'$ and $\circ$ into a derivation $\partial: \mathbb{L} \longrightarrow \mathbb{L}$ which commutes with transfinite sums and satisfies an infinite Leibniz rule, and a composition law $\circ: \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}$ which is associative and satisfies the chain rule with respect to ∂.

4.2 Hyperserial fields

In order to construct large fields of hyperseries, it is useful to work with an abstract notion of field of hyperseries, as Schmeling did [37] in the case of transseries. In view of the interesting properties of the field $\mathbb{L}$ of logarithmic hyperseries, our definition should include $\mathbb{L}$ itself and rely on its properties in order to simplify the treatment of these technically involved objects.

A *hyperserial field* is a Hahn series field $\mathbb{T}$ endowed with a function

$$\circ\colon \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$$

called the composition law, which satisfies (among other technical details: [8, Section 6]), for all $f \in \mathbb{L}$:

Compatibility. For all $s \in \mathbb{T}^{>\mathbb{R}}$, the function $\mathbb{L} \longrightarrow \mathbb{T}; h \mapsto h \circ s$: is a ring morphism which commutes with transfinite sums.

Associativity. $f \circ (g \circ s) = (f \circ g) \circ s$ for all $g \in \mathbb{L}^{>\mathbb{R}}$ and $s \in \mathbb{T}^{>\mathbb{R}}$.

Monotonicity. $\ell_{\omega^\mu} \circ s < \ell_{\omega^\mu} \circ t$ for all $\mu \in \mathbf{On}$ and $s, t \in \mathbb{T}^{>\mathbb{R}}$ with $s < t$.

Analyticity. $f \circ (s + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ for all $s \in \mathbb{T}^{>\mathbb{R}}$, and $\delta \in \mathbb{T}$ with $\delta \prec s$.

The structure $(\mathbb{L}, \circ)$ itself is a hyperserial field [18].

4.3 Hyperserial skeletons

Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a hyperserial field, let $f \in \mathbb{L}$ and $s \in \mathbb{T}^{>\mathbb{R}}$. We note that

1. By compatibility and associativity, the series $f \circ s$ is determined by the class of series $\ell_{\omega^\mu} \circ t$ for all $\mu \in \mathbf{On}$ and $t \in \mathbb{T}^{>\mathbb{R}}$.

2. By associativity and analyticity, the series $\ell_{\omega^\mu} \circ t$ is determined by the class of series $\ell_{\omega^\mu} \circ \mathfrak{a}$ where $\mathfrak{a}$ ranges in a given subclass $\mathfrak{M}_{\omega^\mu} \subset \mathfrak{M}$.

3. For $\mu \in \mathbf{On}$, we have the following self-contained definition of $(\mathfrak{M}_{\omega^\mu})_{\mu \in \mathbf{On}}$:

$$\begin{aligned} \mathfrak{M}_1 &= \mathfrak{M}^{\succ 1}, \\ \mathfrak{M}_{\omega^{\mu+1}} &= \{\mathfrak{m} \in \mathfrak{M}_{\omega^\mu} : \forall n \in \mathbb{N}, \ell_{\omega^\mu n} \circ \mathfrak{m} \in \mathfrak{M}_{\omega^\mu}\}, \quad \text{and} \\ \mathfrak{M}_{\omega^\mu} &= \bigcap_{\iota < \mu} \mathfrak{M}_{\omega^\iota} \quad \text{if } \mu > 0 \text{ is a limit.} \end{aligned}$$

It becomes advantageous when constructing hyperserial fields to work with the lighter structure $(\mathbb{T}, (\mathfrak{M}_{\omega^\mu})_{\mu \in \mathbf{On}}, (\mathfrak{a} \mapsto \ell_{\omega^\mu} \circ \mathfrak{a})_{\mu \in \mathbf{On}})$. Indeed, constructing composition laws entails proving many formal identities and showing that many families (involved for instance in the Analyticity axiom) are summable.

Goal: Defining $\mathfrak{M}_\alpha$ as in 3, find conditions on partial functions

$$\widetilde{L_{\omega^\mu}}\colon \mathfrak{M}_{\omega^\mu} \longrightarrow \mathbb{T}; \mathfrak{a} \mapsto \ell_{\omega^\mu} \circ \mathfrak{a}, \quad \mu \in \mathbf{On}$$

so that $(\widetilde{L_{\omega^\mu}})_{\mu \in \mathbf{On}}$ determine a lawful composition $\circ\colon \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$.

Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ and let $\widetilde{L_{\omega^\mu}}, \mu \in \mathbf{On}$ be partial functions $\widetilde{L_{\omega^\mu}}\colon \mathfrak{M}_{\omega^\mu} \longrightarrow \mathbb{T}$. Consider a law

$$\mathbb{R} \times \mathfrak{M} \longrightarrow \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^r$$

of ordered $\mathbb{R}$-vector space on $\mathfrak{M}$ ($\mathfrak{m}^r$ is interpreted as a real power of $\mathfrak{m}$).

For $\mu = 0$, i.e. $\omega^\mu = 1$, the function L_1 is a restriction of the logarithm, for which axiomatic conditions are well-known (see [30, 37]). Assume that $\mu > 0$. We then impose

A1. $\widetilde{L_{\omega^\mu}}(\mathfrak{a}) \prec \widetilde{L_{\omega^\eta}}(\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ and $\eta < \mu$.

A2. $\widetilde{L_{\omega^\mu}}(\mathfrak{a}) + \frac{1}{(\widetilde{L_{\omega^\eta}})^{\circ n}(\mathfrak{a})} < \widetilde{L_{\omega^\mu}}(\mathfrak{b})$ for all $n \in \mathbb{N}$ and $\nu < \mu$, whenever $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}_{\omega^\mu}$ and $\mathfrak{a} < \mathfrak{b}$.

A3. $\widetilde{L_{\omega^{\mu+1}}}(\widetilde{L_{\omega^\mu}}(\mathfrak{a})) = \widetilde{L_{\omega^{\mu+1}}}(\mathfrak{a}) - 1$ for all $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$.

We call a structure $(\mathbb{T}, (\widetilde{L_{\omega^\mu}})_{\mu \in \mathbf{On}})$ satisfying these conditions a hyperserial skeleton.

Theorem 4.1. [8, Theorem 1.1] *Let $(\mathbb{T}, (\widetilde{L_{\omega^\mu}})_{\mu \in \mathbf{On}})$ be a hyperserial skeleton and assume that for all $\mu \in \mathbf{On}$, each $s \in \mathbb{T}^{>\mathbb{R}}$ is « sufficiently close » to an $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$. Then there is a unique function $\circ: \mathbb{L} \times \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}$ with $\ell_{\omega^\mu} \circ \mathfrak{a} = \widetilde{L_{\omega^\mu}}(\mathfrak{a})$ for all $\mu \in \mathbf{On}$ and $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$, and such that $(\mathbb{T}, \circ)$ is a hyperserial field.*

4.4 Hyperexponential closure

A hyperserial field $(\mathbb{T}, \circ)$ is said *hyperexponentially closed* if for all $\mu \in \mathbf{On}$, the following function is bijective:

$$L_{\omega^\mu}: \mathbb{T}^{>\mathbb{R}} \longrightarrow \mathbb{T}^{>\mathbb{R}}; s \mapsto \ell_{\omega^\mu} \circ s.$$

In other words, a hyperserial field is hyperexponentially closed if each inverse function $E_{\omega^\mu} = L_{\omega^\mu}^{\mathrm{inv}}$ is totally defined on $\mathbb{T}^{>\mathbb{R}}$. We have a closure functor:

Theorem 4.2. [8, Theorem 1.3] *There is a hyperexponentially closed extension $\iota: \mathbb{T} \longrightarrow \tilde{\mathbb{T}}$ such that for each hyperexponentially closed extension $\varphi: \mathbb{T} \longrightarrow \mathbb{U}$, there is a unique embedding $\psi: \tilde{\mathbb{T}} \longrightarrow \mathbb{U}$ with $\varphi = \psi \circ \iota$.*

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \tilde{\mathbb{T}} \arrow[d, "\exists!\,\psi"] \\ & \mathbb{U} \end{tikzcd}$$

This universal condition entails that any element f of $\tilde{\mathbb{T}}$ can be written as a series involving elements in $\mathbb{T}$, transfinite sums, products, hyperexponentials and hyperlogarithms. It follows that operations defined on $\mathbb{T}$ extend uniquely to $\tilde{\mathbb{T}}$.

Theorem 4.3. [4, Theorem 6.7 and Corollary 7.24] *There are unique extensions $\tilde{\circ}: \tilde{\mathbb{L}} \times \tilde{\mathbb{T}}^{>\mathbb{R}} \rightarrow \tilde{\mathbb{T}}$ and $': \tilde{\mathbb{L}} \rightarrow \tilde{\mathbb{L}}$ of $\circ$ and $'$ with similar properties as $\circ$ and $'$.*

In particular, the structure $\tilde{\mathbb{L}}$ of so-called finitely nested hyperseries is equipped with an internal composition law $\tilde{\mathbb{L}} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \rightarrow \tilde{\mathbb{L}}$ and derivation $\tilde{\mathbb{L}} \rightarrow \tilde{\mathbb{L}}$ which extend the operations on $\mathbb{T}_{\mathrm{LE}}$ while extending the range of functional equations with formal solutions in $\tilde{\mathbb{L}}$:

Theorem 4.4. [4, Proposition 9.23 and Theorem 10.16] *The structure $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \circ, \ell_0, <)$ is a linearly bi-ordered group, and any two elements $f, g \in \tilde{\mathbb{L}}$ with $f, g > \ell_0$ are conjugate in $\tilde{\mathbb{L}}^{>\mathbb{R}}$.*

This solves an open problem regarding orderable groups [12, Problem 3.31].

5 Surreal numbers

Despite its strong closure properties, the structure $(\tilde{\mathbb{L}}, +, \cdot, \partial, \circ)$ of finitely nested hyperseries is not as "complete" as one may want. The extension of transseries with hyperlogarithms and then hyperexponentials is sufficient to fill all horizontal and vertical cuts, but not all nested cuts: indeed there is no finitely nested hyperseries f which fills the nested cut $\mathbf{N}$ of (3.2). This is related to the failure of the IVT for unary terms in the first-order language including composition, as the term

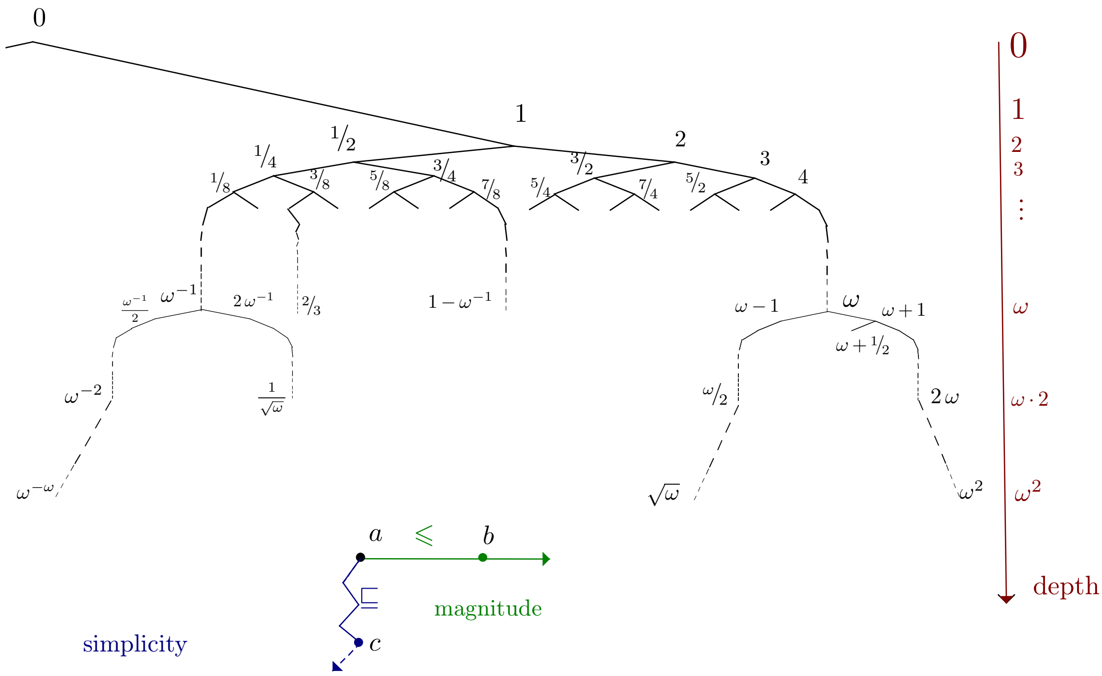
$$t(u) = \sqrt{\ell_0} + \mathrm{e}^{\ell_0} \circ u \circ \ell_1$$

does not have the intermediate value property in $\tilde{\mathbb{L}}$. In order to obtain a truly complete hyperserial fields, it is natural to turn to the very large and saturated field of surreal numbers as defined by Conway was a good candidate.

Indeed, as conjectured by van der Hoeven [26], surreal numbers should have natural representations as hyperseries, including infinitely nested ones. Let us give the intuition about this and describe our results.

5.1 The binary tree of surreal numbers

Conway's class **No** of surreal numbers can be represented as a binary tree. Its underlying order is lexicographically ordered complete binary tree $\{-1,1\}^{<\mathbf{On}}$ whose depths are arbitrary ordinals.



A number a is *simpler* than a number b, written $a \sqsubseteq b$, if there is a path from a to b in the tree.

$$\frac{1}{2} \sqsubseteq 1-\omega^{-1}, \qquad \frac{1}{2} \not\sqsubseteq 2, \qquad \text{and} \qquad 1 \sqsubseteq x \quad \text{for all} \quad x>0.$$

5.2 Inductive definitions on surreal numbers

The structure $(\mathbf{No}, \leqslant, \sqsubseteq)$ is determined, up to unique isomorphism, by the following properties:

i. The class $(\mathbf{No}, \sqsubseteq)$ is well-founded, and given an $a \in \mathbf{No}$, the classes

$$a_L := \{b \in \mathbf{No} : b \sqsubseteq a \wedge b < a\} \qquad \text{and} \qquad a_R := \{b \in \mathbf{No} : b \sqsubseteq a \wedge b > a\}$$

are linearly ordered (hence well-ordered) subsets of $(\mathbf{No}, \sqsubseteq)$.

ii. The class $(\mathbf{No}, \leqslant)$ is linearly ordered, and for all subsets $L, R \subseteq \mathbf{No}$ with $L < R$ (i.e. $l < r$ whenever $(l,r) \in L \times R$), the class $(L|R) := \{a \in \mathbf{No} : L < a < R\}$ has a minimum for $\sqsubseteq$.

The minimum of $(L|R)$ is usually denoted $\{L|R\}$. By definition, we have

$$a = \{a_L | a_R\}$$

for each number $a \in \mathbf{No}$. Since $(\mathbf{No}, \sqsubseteq)$ is well-founded, this allows for inductive definitions of functions on **No** and its powers. For instance, the "simplest" way to define a strictly increasing function $f: \mathbf{No} \longrightarrow \mathbf{No}$ by induction is to set

$$\forall a \in \mathbf{No}, f(a) := \{f(a_L) | f(a_R)\}. \tag{5.1}$$

An easy induction shows that f is well-defined, and is none other than the identity function on **No**. More interestingly, one may define the "simplest" binary operation $A: \mathbf{No} \times \mathbf{No} \longrightarrow \mathbf{No}$ which is strictly increasing at each variable as follows:

$$\forall a, b \in \mathbf{No}, S(a, b) := \{S(a_L, b), S(a, b_L) | S(a, b_R), S(a_R, b)\}. \tag{5.2}$$

Indeed, that A be strictly increasing in both variables forces $S(a, b) \in (S(a_L, b), S(a, b_L) | S(a, b_R), A(a_R, b))$, hence picking the simplest number of this class is natural. This is the definition that Conway found for the sum $a + b := S(a, b)$ of two surreal numbers a, b. That is, not only does the function satisfy the constraints from which its inductive definition emerges, but it also satisfies additional properties (associativity, symmetry, *etc*) that follow, in a way that is specific to this construction, from the inductive definition.

Using similar inductive definition, Conway also defined a product on **No**, such that $(\mathbf{No}, +, \cdot)$ is a real-closed extension of $\mathbb{R}$, which is identified here with the subset of surreal numbers a for which a_L, a_R are sets of dyadic numbers, a_L has a maximum or is empty if and only if a_R has a minimum or is empty.

Furthermore, the class of ordinals with their commutative arithmetic (i.e. Hessenberg arithmetic) is naturally contained in $(\mathbf{No}, +, \cdot)$. So we have surreal avatars ω, ω^ω, ε_0, ω_1 for all ordinals. See [14, Chapters 1 and 2] for more details.

5.3 Containment of transseries and derivation

Conway also established the identification between surreal numbers and Hahn series with real coefficient, and whose group of monomials is a subgroup **Mo** of $(\mathbf{No}^{>0}, \cdot, <)$ which is canonically isomorphic to $(\mathbf{No}, +, <)$.

Later, Gonshor defined [21, Chapter 10], again using a simple inductive definition, an exponential function $\exp: \mathbf{No} \longrightarrow \mathbf{No}^{>0}$ such that $(\mathbf{No}, +, \cdot, \exp)$ is an elementary extension [38] of the real exponential field. Taken together, these two ingredients of formal series representation and exponential give a natural embedding $f \mapsto f(\omega)$ of $\mathbb{T}_{\mathrm{LE}}$ into **No**. Indeed, it is unique to send $x \in \mathbb{T}_{\mathrm{LE}}$ to $\omega \in \mathbf{No}$ and to commute with transfinite sums and with the exponentials.

Finally, after giving a precise description of the behavior of the way exp interacts with the simplicity relation $\sqsubseteq$ on **No**, Berarducci and Mantova defined a derivation ∂_{BM}: $\mathbf{No} \longrightarrow \mathbf{No}$ which extends the derivation on $\mathbb{T}_{\mathrm{LE}}$. It was shown [3] that $(\mathbf{No}, +, \cdot, \partial, <, \prec)$ is in fact an elementary extension of $\mathbb{T}_{\mathrm{LE}}$.

5.4 Summing up

The properties of various ordered algebraic structures on **No** are summed up by the following table. Here "defining properties" refer to first-order properties which are explicitly implemented in the interpretation of the function symbols on **No**. The complete theory of the resulting structure $(\mathbf{No}, +, \ldots)$ is obtained by adding to the list of defining properties an axiom scheme of intermediate value theorem for unary terms in the language:

(**IVT** for $(\mathbf{No}, +, \ldots)$) If $t(u)$ is a unary term in the first-order language of $(\mathbf{No}, +, \ldots)$ and $a, b \in \mathbf{No}$ are such that $a < b$ and $t(a) < 0 < t(b)$, then there is a $c \in (a, b)$ with $t(c) = 0$.

Structure	Defining properties	Complete theory	Closure properties
$(\mathbf{No},+,<)$	ordered group	$\succcurlyeq(\mathbb{R},+)$	IVT for $(\mathbf{No},+)$
$(\mathbf{No},+,\times,<)$	ordered ring	$\succcurlyeq(\mathbb{R},+,\cdot)$	IVT for $(\mathbf{No},+,\times)$
$(\mathbf{No},+,\times,\exp,<)$	ordered exponential field	$\succcurlyeq\mathbb{R}_{\exp}$	IVT for $(\mathbf{No},+,\times,\exp)$
$(\mathbf{No},+,\times,\partial_{\mathrm{BM}},<)$	Liouville-closed H-field with small derivation	$\succcurlyeq\mathbb{T}_{\mathrm{LE}}$	IVT for $(\mathbf{No},+,\times,\partial_{\mathrm{BM}})$
$(\mathbf{No},+,\times,\partial,\circ,<)$	??	??	??

(In the case of the theory of ordered exponential field, it is unknown if this IVT scheme is sufficient to imply the complete theory.)

Our work with coauthors Joris van der Hoeven, Elliot Kaplan and Vincenzo Mantova consists in defining a composition law $\circ\colon \mathbf{No}\times\mathbf{No}^{>,\succ}\longrightarrow\mathbf{No}$ expanding the composition law on transseries, satisfying a chain rule $\partial(a\circ b)=\partial(a)\circ b\cdot\partial(b)$ with respect to the derivation ∂.

Since the derivation ∂_{BM} was defined without this goal in mind, it is in fact incompatible with the existence of such a composition law [11, Theorem 8.4]. Therefore, it is also necessary to define a distinct derivation ∂ on $\mathbf{No}$. Our approach is to define it in relation to a representation of each surreal number a as a formal hyperseries

$$a=f(\omega) \tag{5.3}$$

in ω, where f possibly involves infinite nesting. Then the way a acts with respect to ∂ and $\circ$ is entirely specified in the formal series f, which should allow us to define $\circ$ and ∂ on $\mathbf{No}$. In the sequel, we present the method used to establish the hyperserial representation (5.3) of surreal numbers.

6 Numbers as hyperseries

Let us see how to represent a number $a\in\mathbf{No}$ as a hyperseries in ω. We must first construe $\mathbf{No}$ as a hyperserial field so as to be able to give a meaning to the identification (5.3).

6.1 Surreal numbers as a hyperserial field

To construe $\mathbf{No}=\mathbb{R}[[\mathbf{Mo}]]$ as a hyperserial field is to define a composition law $\circ\colon\mathbb{L}\times\mathbf{No}^{>\mathbb{R}}\longrightarrow\mathbf{No}$ satisfying the properties in Section 4.2. Thanks to the hyperserial skeleton method (see Section 4.3), this reduces to defining partial functions $\widetilde{L}_{\omega^\mu}\colon\mathbf{Mo}_{\omega^\mu}\longrightarrow\mathbf{No}$, $\mu\in\mathbf{On}$ in an inductive manner. For $\mu=0$, this was already done by Gonshor [21]. Let us explain how $\widetilde{L}_\omega$ is defined. The class $\mathbf{Mo}_\omega$ is that of log-atomic numbers as per [37, 3]. This was identified [10] by Berarducci and Mantova as an isomorphic copy of $(\mathbf{No},\leqslant,\sqsubseteq)$ inside itself. Thus we may use an inductive definition of the same form as (5.1-5.2):

$$\forall\mathfrak{a}\in\mathbf{Mo}_\omega,\ \widetilde{L}_\omega(\mathfrak{a}):=\left\{\mathbb{R},\widetilde{L}_\omega(\mathfrak{a}')+\frac{1}{\log_n(\mathfrak{a}')}\,\middle|\,\widetilde{L}_\omega(\mathfrak{a}'')-\frac{1}{\log_n(\mathfrak{a}'')},\log_n(\mathfrak{a})\right\}, \tag{6.1}$$

where $\mathfrak{a}',\mathfrak{a}''$ range in $\mathbf{Mo}_\omega$ with the constraint that $\mathfrak{a}'<\mathfrak{a}<\mathfrak{a}''$, and n ranges in $\mathbb{N}$. $\mathfrak{a}'$, $\mathfrak{a}''\in\mathbf{Mo}_\omega$. The reader can see that (6.1) is a direct translation of the conditions **A1**, **A2** for $\widetilde{L}_\omega$. As is often the case for inductive definitions on $\mathbf{No}$, one also get the additional algebraic identity **A3**, as well as existential properties which insure that $\widetilde{L}_\omega$ extends canonically into a surjective strictly increasing function $\mathbf{No}^{>\mathbb{R}}\longrightarrow\mathbf{No}^{>\mathbb{R}}$. See [9] for more details.

Relying on earlier work [5], we showed that the inductive definition (6.1) generalizes to all ordinals $\mu > 0$ (see [6, Section 6]). We obtain the desired result:

Theorem 6.1. [6, Theorem 1.1] *There is a composition law* $\circ : \mathbb{L} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ *for which* $(\mathbf{No}, \circ)$ *is a hyperexponentially closed hyperserial field.*

6.2 Numbers as hyperseries

In order to represent numbers as hyperseries, we must first express each surreal monomial $\mathfrak{m} \in \mathbf{Mo}$ using hyperexponentials and hyperlogarithms of "less complex numbers", positing that ω is itself simple and cannot be further expanded. Indeed, we can see that every non-trivial monomial $\mathfrak{m} \in \mathbf{Mo} \setminus \{1\}$ admits a unique expansion as

$$\mathfrak{m} = \mathrm{e}^{\psi} \, (L_{\beta}(E_{\alpha}(u)))^{\iota}, \tag{6.2}$$

where $\mathrm{e}^{\psi} \in \mathbf{Mo}$, $\iota \in \{-1, 1\}$, $\beta \in \mathbf{On}, \alpha \in \omega^{\mathbf{On}}$ satisfy additional technical conditions (see [7, Section 5.1]). The precise conditions are irrelevant for the discussion here. We adopt the notations

$$\begin{aligned} L_{\beta} E_{\alpha}^{u} &:= L_{\beta}(E_{\alpha}(u)) \text{ and} \\ \operatorname{term} a &:= \{a_{\mathfrak{m}} \mathfrak{m} \colon \mathfrak{m} \in \operatorname{supp} a\}. \end{aligned}$$

Unfortunately, it is impossible to define a complexity measure for which the parameters u, ψ in the expansion of $\mathfrak{m}$ are always strictly less complex than $\mathfrak{m}$. This is related to the existence of infinite paths as defined next. An infinite path $P = (r_i \, \mathfrak{m}_i)_{i \in \mathbb{N}}$ in $a \in \mathbf{No}$ is thus defined as a sequence of non-zero terms $P = (r_i \, \mathfrak{m}_i)_{i \in \mathbb{N}} \in (\mathbb{R}^{\neq} \, \mathbf{Mo} \setminus \{1, \omega\})^{\mathbb{N}}$ with

$$\forall i \in \mathbb{N}, r_i \, \mathfrak{m}_i \in \operatorname{term} \psi_i \cup \operatorname{term} u_i,$$

where $(u_0, \psi_0) = (a, 0)$ and each $\mathfrak{m}_i$ expands as $\mathfrak{m}_i = \mathrm{e}^{\psi_{i+1}} \, (L_{\beta_i} \, E_{\alpha_i}^{u_{i+1}})^{\iota_i}$. We showed that each path $P = (r_i \, \mathfrak{m}_i)_{i \in \mathbb{N}}$ in a is such that for sufficiently large $j > 0$, we have

$$u_j = \varphi_j \pm \mathrm{e}^{\psi_{j+1}} \left(E_{\alpha_j}^{\varphi_{j+1} \pm \mathrm{e}^{\psi_{j+2}} \left(E_{\alpha_{j+1}}^{\cdot^{\cdot^{\cdot^{\varphi_{j+i} \pm \mathrm{e}^{\psi_{j+i+1}} \left(E_{\alpha_{j+i}}^{\cdot^{\cdot^{\cdot}}} \right)^{\iota_{j+i}}}}}} \right)^{\iota_{j+1}}} \right)^{\iota_j}. \tag{6.3}$$

In other words, for large enough $j \in \mathbb{N}$, the number u_j is an infinitely nested expansion. We proved [7, Theorem 1.1] that given the sequence of parameters $(\varphi_i, \psi_{i+1}, \pm 1, \iota_i, \alpha_i, \beta_i, u_{i+1})_{i \geqslant j}$, the class $\mathbf{Ne}$ of numbers that expands exactly as u_j is not a singleton, but a proper class. Moreover [7, Theorem 1.2], $(\mathbf{Ne}, <, \sqsubseteq)$ is uniquely isomorphic to $(\mathbf{No}, <, \sqsubseteq)$. So any element expanding as u_j may still be uniquely identified by the unique "label" $l(u_j) \in \mathbf{No}$ such that the isomorphism $\mathbf{No} \longrightarrow \mathbf{Ne}$ sends $l(u_j)$ to u_j. This allowed us to represent surreal numbers as trees labeled by real numbers, ordinal numbers and surreal numbers. We call such expressions hyperserial descriptions. The last main result is the following:

Theorem 6.2. [7, Theorem 1.3] *Every surreal number has a unique hyperserial description. Two numbers with the same hyperserial description are equal.*

In this way, every surreal number is seen as a hyperseries in ω. Furthermore, the description of paths in $\mathbf{No}$ is enough to give a natural way to define a derivation and composition law on $\mathbf{No}$. We plan to do this in further work and then study the first-order structure $(\mathbf{No}, +, \cdot, \partial, \circ, <)$, starting possibly with the simpler "almost reduct" $(\mathbf{No}^{>\mathbb{R}}, \circ, <)$.

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On Rayner Structures

Salma Kuhlmann

Abstract

In this note, we study substructures of generalized power series fields induced by families of well-ordered subsets of the group of exponents. We characterize the set-theoretic and algebraic properties of the induced substructures in terms of conditions on the families. We extend the work of Rayner (F. J. Rayner, *An algebraically closed field*; Glasgow Math. J. 9 (1968), 146151) by giving both necessary and sufficient conditions to obtain truncation closed subgroups, subrings and subfields.

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HYPERBOLIC POLYNOMIALS

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Abstract. Hyperbolic polynomials are real homogeneous polynomials in several variables with a certain reality condition on their roots. They orginate in PDE theory but have more recently been studied extensively in combinatorics, convex optimization and real algebraic geometry. A lot of their theory can be thought of as generalising symmetric determinants. In this talk, I will give a gentle introduction, survey several important results and point to some recent developments and open problems.

Results presented towards the end of this survey have been obtained in collaboration and have since been published:

[1] D. Plaumann, R. Sinn and S. Weis, *Kippenhahn's Theorem for joint numerical ranges and quantum states*, SIAM Journal on Applied Algebra and Geometry, **5,** 86-113 (2021).

[2] D. Plaumann, R. Sinn and J-L. Wesner, *Families of Faces and the Normal Cycle of a Convex Semi-algebraic Set*, in Beiträge zur Algebra und Geometrie (2022) (online).

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THE WONDERFUL GEOMETRY OF THE VANDERMONDE MAP

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Abstract. We study the geometry of the image of the nonnegative orthant under the power-sum and elementary symmetric polynomials maps. After analyzing the image in finite number of variables, we concentrate on the limit as the number of variables approaches infinity. We explain how the geometry of the limit plays a crucial role in undecidability results in nonnegativity of symmetric polynomials, deciding validity of trace inequalities in linear algebra, and extremal combinatorics (recently observed by by Blekherman, Raymond, and F. Wei [BRW22]). We verify the experimental observation that the image has the combinatorial geometry of a cyclic polytope made by Melánová, Sturmfels, and Winter [MSW22], and generalize results of Choi, Lam, and Reznick [CLR87] on nonnegative even symmetric polynomials. We also show that undecidability does not hold for the normalized power sum map.

1. Introduction

The main object of this article is the so called Vandermonde map which appears quite naturally in various contexts and thus providing connections between different mathematical domains. Our interest in this object is motivated by the following problem. Suppose that we are given a polynomial expression in traces of powers of symmetric matrices, such as

$$2\operatorname{tr}(A^2)\operatorname{tr}(B^6) - \operatorname{tr}(A^4)\operatorname{tr}(B^4),$$

is there an algorithm to decide whether this expression is nonnegative for all symmetric matrices A, B of all sizes? What happens if we replace trace by normalized trace $\widetilde{\operatorname{tr}}(A) = \frac{\operatorname{tr} A}{n}$, where n is the size of the matrix?

As one of the results of our work we show that the first (unnormalized) problem is *undecidable*, while the second one is *decidable*. The key to the hardness of the unnormalized problem is the fascinating geometry of *the image of the probability simplex under the Vandermonde map*. As we explain below, some geometric properties of this set were observed in different areas of mathematics making it an important and beautiful object to study.

For any $n \times n$ matrix A recall that $\operatorname{tr}(A^d) = \lambda_1^d + \cdots + \lambda_n^d$, where λ_i are the eigenvalues of A. We use p_d to denote the d-th power sum polynomial: $p_d(x) = x_1^d + \cdots + x_n^d$. We see that testing whether $2\operatorname{tr}(A^2)\operatorname{tr}(B^6) - \operatorname{tr}(A^4)\operatorname{tr}(B^4)$ is nonnegative on all symmetric matrices of all sizes is equivalent to understanding whether $2p_2(x)p_6(y) - p_4(x)p_4(y)$ is nonnegative on all real vectors x and y of any dimension. Define the *d-th Vandermonde map* $\nu_{n,d}$ by sending a point in $\mathbb{R}^n$ to its image under the first d power sums:

$$\nu_{n,d}(x) = (p_1(x), \ldots, p_d(x)).$$

Let Δ_{n-1} be the probability simplex in $\mathbb{R}^n$: Δ_{n-1} consists of all vectors with nonnegative coordinates with the sum of coordinates equal to 1. We call the image $\nu_{n,d}(\Delta_{n-1})$ of the probability simplex under the Vandermonde map *the* (n,d)-*Vandermonde cell* and denote it by $\Pi_{n,d}$. Observe that the first coordinate of $\Pi_{n,d}$ is identically 1, and so we may project it out, and see $\Pi_{n,d}$ as the subset of $\mathbb{R}^{d-1}$, which is the image of Δ_{n-1} under $(p_2, \ldots, p_d)$.

Key words and phrases. Vandermonde map, cyclic polytopes, trace polynomials, copositivity, undecidability.

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Since $2p_2(x)p_6(y) - p_4(x)p_4(y)$ is an even homogeneous polynomial, deciding whether it is nonnegative for all $x, y \in \mathbb{R}^n$ is equivalent to deciding whether the polynomial $2a_1b_3 - a_2b_2$ is nonnegative on the product $\Pi_{n,3} \times \Pi_{n,3}$, where $a_i = p_i(x)$ and $b_i = p_i(y)$.

We reach two important conclusions: first, we are interested in nonnegativity of polynomials on (products of) Vandermonde cells $\Pi_{n,d}$, and second, to consider matrices of all sizes we need to take the *limit of the Vandermonde cell* $\Pi_{n,d}$ *as* n *goes to infinity*.

The Vandermonde cell $\Pi_{n,d}$ is a compact subset of $\mathbb{R}^{d-1}$, and our first main result is that $\Pi_{n,d}$ has *the combinatorial structure of a cyclic polytope*, verifying an experimental observation of [MSW22].

For a fixed d the sets $\Pi_{n,d}$ form an increasing sequence of sets in $\mathbb{R}^{d-1}$. Let Π_d be the closure of the union of $\Pi_{n,d}$. We show that the set Π_d has the combinatorial structure of an *infinite cyclic polytope*, and that Π_d is not semialgebraic for all $d \geq 3$. The sets $\Pi_{n,3}$ and Π_3 are depicted in Figure 1. Reduction needed to show undecidability of the unnormalized trace problem is borrowed from the one used by Hatami and Norine in [HN11] in the context of homomorphism density inequalities in graph theory. The set used by Hatami and Norine is essentially a linear transformation of the set Π_3, and the reduction is based on the geometry of Π_3. In particular this shows that deciding validity of matrix power trace inequalities is already undecidable if we only use second, fourth and sixth matrix powers, and we need at most 11 matrix variables for the problem to become undecidable. We note that the geometry of Π_3 was also used directly by Blekherman, Raymond and Wei [BRW22] to show undecidability of homomorphism density inequalities with arbitrary edge weights.

We also consider the image of Δ_{n-1} under elementary symmetric polynomials. Our previous results on the boundary structure transfer over by using Newton's identities. We write $E_{n,d} := (e_1, \ldots, e_d)(\Delta_{n-1})$ and denote the limit image by E_d. We show that the convex hull of $E_{n,d}$ is an actual cyclic polytope. This helps us reprove and slightly generalize the result of Choi, Lam and Reznick [CLR87] on test sets for nonnegativity of even symmetric sextics. We note that the convex hull result can be traced to the work of Bollobás in extremal graph theory [Bol76].

Testing nonnegativity of univariate normalized trace polynomials was considered by Klep, Pascoe and Volčič [KPV21] where the authors proved a Positivstellensatz in the univariate case. Geometrically, such normalized trace polynomials correspond to power means. Nonnegativity of polynomials in power means was investigated by Blekherman and Riener in degree 4 [BR21] and more generally by Acevedo and Blekherman [AB23]. We briefly illustrate the connection with the Vandermonde map. Decidability of the normalized trace problem follows quickly from the work in [BR21]. As before we can consider the image of the normalized Vandermonde map, and fixing d take the (closure of the) limit as n goes to infinity. As explained in [AB23] the geometry of the limit is drastically different. For instance, the limit of the normalized Vandermonde map of the unit simplex Δ_{n-1} corresponds to the set of the first d moments of a probability measure supported on $\mathbb{R}_{\geq 0}$, and it is well-known that this set can be described by linear matrix inequalities [Sch17]. In particular, the limit is semialgebraic for all d.

1.1. **Previous Work and Main Results in Detail.**

The Vandermonde map has been studied from several different perspectives. Originating from the question of understanding univariate hyperbolic polynomials, Arnold, Givental and Kostov investigated the sets $(e_1, \ldots, e_d)(\mathbb{R}^n)$ [Arn86; Giv87; Kos89; Kos89; Kos99]. A detailed examination can also be found in [Meg92]. Kostov investigated the limit of the images of $\mathbb{R}^n$ for $d = 4$. The authors observed, that one can also allow positive *weights* in the definition of the Vandermonde map. Their description of the boundary of the image of the Vandermonde map and of fibers generalizes to the map

$$\mathbb{R}^n \longrightarrow \mathbb{R}^d, x \mapsto (a_1x_1 + \ldots + a_nx_n, \ldots, a_1x_1^d + \ldots + a_nx_n^d)$$

for any positive weights $a_1, \ldots, a_n > 0$. This is mainly due to the fact that Jacobians of the weighted and unweighted maps differ only by positive constant multiples.

The restriction of the Vandermonde map to the nonnegative orthant was investigated by Ursell [Urs59]. The paper contains several important results some of which we reprove. Ursell observed that the geometry of the Vandermonde map restricted to the nonnegative orthant generalizes further to arbitrary real positive exponents, i.e. to maps

$$\mathbb{R}^n \longrightarrow \mathbb{R}^d, x \mapsto (x_1^{\alpha_1} + \ldots + x_n^{\alpha_1}, \ldots, x_1^{\alpha_d} + \ldots + x_n^{\alpha_d})$$

for which $0 < \alpha_1 < \ldots < \alpha_d$. Ursell's original motivation came from studying valid inequalities in ℓ_p-norms.

Recently, there has been an interest in describing fibers and the image of the Vandermonde map using computational algebraic geometry [BCW21; MSW22]. Bik, Czapliński and Wageringe derived semialgebraic description of $\nu_{n,3}([0,1]^n)$ for all $n \geq 3$ which has applications in the study of L-functions and their zeros. Melánová, Sturmfels and Winter explored fibers and the image of the Vandermonde map over the complex numbers and real numbers.

Our first theorem is a result initially found by Ursell [Urs59]. We provide a different proof by adapting the techniques in Arnold's and Givental's work.

Theorem 2.4. *For integers $n \geq d$ the set $\operatorname{bd}\Pi_{n,d}$ is the set of evaluations of $\nu_{n,\alpha}$ at all points in Δ_{n-1} of the following two types:*

(1) $(\underbrace{0,\ldots,0}_{r_0}, \underbrace{x_1}_{r_1}, \underbrace{x_2,\ldots,x_2}_{r_2}, \ldots, \underbrace{x_{d-1},\ldots,x_{d-1}}_{r_{d-1}})$ *with $r_{2k-1} = 1$ and $r_0 \geq 0$, $r_{2k} \geq 1$ for all k,*

(2) $(\underbrace{x_1,\ldots,x_1}_{r_1}, \underbrace{x_2}_{r_2}, \ldots, \underbrace{x_{d-1},\ldots,x_{d-1}}_{r_{d-1}})$ *with $r_{2k} = 1$ and $r_{2k} \geq 1$ for all k.*

and $0 \leq x_1 \leq x_2 \leq \ldots \leq x_{d-1}$

We then investigate concretely the planar boundaries of $\Pi_{n,3}$ and the limit set Π_3 and derive consequences for all $d \geq 3$.

Corollary 2.20. *The sets Π_d and E_d are not semialgebraic for all $d \geq 3$.*

Let $\mu_d : \mathbb{R} \to \mathbb{R}^d, t \mapsto (t, t^2, \ldots, t^d)$ denote the d-dimensional moment curve and let $t_1 < \ldots < t_n$. For $n > d$ the *cyclic polytope* $C(n,d)$ is the convex polytope with vertices $\mu_d(t_i)$ for $1 \leq i \leq n$. The combinatorial structure of the cyclic polytope is independent of the chosen n points on the moment curve. Cyclic polytopes are the polytopes with maximal f-vector among all convex polytopes of given dimension and number of vertices [McM70; Sta75]. The facets of $C(n,d)$ are characterized by *Gale's evenness condition* [Gal63]. A subset $\{\mu_d(t_{i_1}), \ldots, \mu_d(t_{i_d})\}$ with $i_j < i_{j+1}$ for all $1 \leq j < d$ spans a facet if and only if any two elements in $\{t_1, \ldots, t_n\} \setminus \{t_{i_1}, \ldots, t_{i_d}\}$ are separated by an even number of elements $\{t_{i_1}, \ldots, t_{i_d}\}$. Answering a question in [MSW22] we prove that the set $\Pi_{n,d}$ has the combinatorial structure of a cyclic polytope, in the sense that the boundary is a gluing of patches where each patch is a curved simplex and the vertices of the patches are characterized by Gale's evenness condition. A set S is a curved simplex if it is the image of Δ_m under a continuous map f, such that f is a diffeomorphism when restricted to the relative interior of any face of Δ_m.

Theorem 3.1. *The set $\Pi_{n,d}$ has the combinatorial structure of the cyclic polytope $C(n, d-1)$, i.e. there is a homeomorphism $\operatorname{bd} C(n, d-1) \to \operatorname{bd}\Pi_{n,d}$ that is a diffeomorphism when restricted to the relative interior of any face of $\operatorname{bd} C(n, d-1)$.*

We provide an explicit map $\operatorname{bd} C(n, d-1) \to \operatorname{bd}\Pi_{n,d}$ in Section 3.
For $n \geq d$ it follows from Newton's identities

$$p_k = (-1)^{k-1} k e_k + \sum_{i=1}^{k-1} (-1)^{k-1+i} e_{k-i} p_i \qquad \text{for all } 1 \leq k \leq n \tag{1.1}$$

that the image of the even Vandermonde map is diffeomorphic to the image on the first d elementary symmetrics, i.e. $E_{n,d} \simeq \Pi_{n,d}$ under a polynomial diffeomorphism. We show that the convex set $\mathcal{E}_{n,d} := \operatorname{conv} E_{n,d}$ has nice properties which $\operatorname{conv}\Pi_{n,d}$ does not have.

Theorem 5.1. *For $d \geq 3$ the set $\mathcal{E}_{n,d}$ is a cyclic polytope, and it is the convex hull of the following finite set of points $\left(\binom{k}{2}\frac{1}{k^2}, \ldots, \binom{k}{d}\frac{1}{k^d}\right) : k \in [n]$.*

Recall that any symmetric polynomial can be written as a polynomial expression in elementary symmetrics or power sums. The vertices of $\mathcal{E}_{n,d}$ relate to copositivity of homogeneous symmetric polynomials in which only e_1 occurs nonlinearly, i.e. symmetric forms which can be written as

$$f = c_1 e_1^d + c_2 e_1^{d-2} e_2 + \ldots + c_m e_d$$

for some $c_1, \ldots, c_m \in \mathbb{R}$. We call such forms *hook-shaped symmetric polynomials.*

Theorem 5.7. *Let f be a hook-shaped symmetric polynomial in $n \geq d$ variables. Then f is copositive if and only if $f\left(1, \binom{k}{2}\frac{1}{k^2}, \ldots, \binom{k}{d}\frac{1}{k^d}\right) \geq 0$ for all $k \in [n]$.*

For $d = 3$ this test set was found by Choi, Lam and Reznick [CLR87] but formulated for even symmetric sextics.

Theorem 5.6. *[CLR87] Let $f(p_2, p_4, p_6)$ be an even symmetric form of degree 6 in $n \geq 3$ variables. Then, f is nonnegative if and only if $f\left(1, \frac{1}{k}, \frac{1}{k^2}\right)$ is nonnegative for all $k \in [n]$.*

Using Newton's identities and the fact that we can restrict to $p_2 = 1$ due to homogenicity and the nonnegative orthant, the test sets in elementary symmetrics and power sums are equivalent for $d \leq 3$ due to the linear relation of those families of polynomials on the probability simplex. Surprisingly, we show that the test set for copositivity in power sums does not generalize to any higher degree.

Proposition 5.5. *Let $n \geq d \geq 4$. Then the set $\operatorname{conv}\left\{\left(\frac{1}{k}, \frac{1}{k^2}, \cdots, \frac{1}{k^{d-1}}\right) : k \in [n]\right\}$ does not contain the set $\Pi_{n,d}$ and $\Pi_d \notin \operatorname{conv}\left\{(0, \ldots, 0), \left(\frac{1}{k}, \frac{1}{k^2}, \cdots, \frac{1}{k^{d-1}}\right) : k \in \mathbb{N}\right\}$.*

Finally, we prove undecidability of testing validity of inequalities of trace polynomials in all symmetric matrices of all sizes.

Theorem 6.2. *The following decision problem is undecidable.*

Instance: *A positive integer k and a trace polynomial $f(X_1, \ldots, X_k)$.*
Question: *Is $f(M_1, \ldots, M_k)$ nonnegative for all real symmetric matrices $M_1, \ldots, M_k$ of all sizes for all $1 \leq i \leq k$?*

When we replace the usual trace by the normalized trace, i.e. $\frac{\operatorname{tr}(A)}{n}$ for a symmetric matrix A of size $n \times n$, the problem becomes decidable.

Theorem 6.5. *The following decision problem is decidable.*

Instance: *A positive integer k and a normalized trace polynomial $f(X_1, \ldots, X_k)$.*
Question: *Is $f(M_1, \ldots, M_k)$ nonnegative for all symmetric matrices $M_1, \ldots, M_k$ of all sizes?*

2. The Vandermonde map

In this section we study the geometry of the boundary of the Vandermonde cell. We start with some definitions.

Definition 2.1. (1) For $n, k \in \mathbb{N}$, $n \geq k$ we write

$$e_k := \sum_{I \subset [n], |I| = k} \prod_{i \in I} x_i$$

for the *k-th elementary symmetric polynomial* in n variables.

(2) For $a \in \mathbb{R}_{>0}$ we consider

$$p_a := \sum_{i=1}^{n} x_i^a$$

the *power sum function*, which for $a \in \mathbb{N}$ is called a *power sum polynomial.*

(3) Given a sequence of strictly increasing positive real numbers $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_{>0}^d$ we consider the *α-Vandermonde map* in n variables to be the function

$$\begin{aligned} \nu_{n,\alpha} : \mathbb{R}_{\geq 0}^n &\longrightarrow \mathbb{R}^d \\ x &\longmapsto (p_{\alpha_1}(x), p_{\alpha_2}(x), \ldots, p_{\alpha_d}(x)) \end{aligned}$$

In the sequel we will restrict our study of Vandermonde maps to the probability simplex and power sum polynomials. This can be done without loss of generality which can be seen as follows.

Remark 2.2. *Given a sequence α as in Definition 2.1. We obtain the following normalized sequence*

$$\beta = (1, \frac{\alpha_2}{\alpha_1}, \dots, \frac{\alpha_d}{\alpha_1})$$

for which we find $\nu_{n,\alpha}(x) = \nu_{n,\beta}(x_1^{\alpha_1}, \dots, x_d^{\alpha_1})$ for all $x \in \mathbb{R}^n_{\geq 0}$. Therefore, given rational positive $\alpha \in \mathbb{Q}^d_{>0}$, i.e., the components of α are of the form $\alpha_1 = \frac{s_1}{t_1}, \dots, \alpha_d = \frac{s_d}{t_d}$ for some integers s_i, t_i one can set $q := t_1 \cdots t_d$ and $\beta := (q\alpha_1, \dots, q\alpha_d)$ to obtain $\nu_{n,\alpha}(x^q) = \nu_{n,\beta}(x)$. Taking into consideration that every real power sum function with irrational exponents can be be approximated by rational power sum functions, we can conclude that it is sufficient to study the image of $\nu_{n,\alpha}$ for integer exponents and $\alpha_1 = 1$ in order to explore the geometry of general Vandermonde mappings.

Definition 2.3. Let $\Delta_{n-1} := \left\{x \in \mathbb{R}^n_{\geq 0} : x_1 + \ldots + x_n = 1\right\}$ and $\alpha = (\alpha_1, \dots, \alpha_{d-1}) \in \mathbb{R}^{d-1}_{>1}$ be a strictly increasing sequence of real numbers larger than 1. The (n, α)-*Vandermonde cell* $\Pi_{n,\alpha}$ is the set $\nu_{n,\alpha}(\Delta_{n-1})$. Note for $\alpha = (2, \dots, d)$ we also write $\Pi_{n,\alpha} = \Pi_{n,d}$.

In Subsection 2.1 we investigate generally the boundary of the Vandermonde cell. In Subsection 2.2 we parameterize planar boundaries of the sets $\Pi_{n,3}$ and Π_3 and we show that the limit set Π_d is not semialgebraic for all $d \geq 3$.

2.1. **The boundary the Vandermonde cell.** In the sequel we will prove the following statement on points on the boundary of a Vandermonde cell.

Theorem 2.4. *For integers $n \geq d$ the set $\operatorname{bd} \Pi_{n,\alpha}$ is the set of evaluations of $\nu_{n,\alpha}$ at all points in Δ_{n-1} of the following two types:*

(1) $(\underbrace{0, \dots, 0}_{r_0}, \underbrace{x_1}_{r_1}, \underbrace{x_2, \dots, x_2}_{r_2}, \dots, \underbrace{x_{d-1}, \dots, x_{d-1}}_{r_{d-1}})$ *with $r_{2k-1} = 1$ and $r_0 \geq 0$, $r_{2k} \geq 1$ for all k,*

(2) $(\underbrace{x_1, \dots, x_1}_{r_1}, \underbrace{x_2}_{r_2}, \dots, \underbrace{x_{d-1}, \dots, x_{d-1}}_{r_{d-1}})$ *with $r_{2k} = 1$ and $r_{2k} \geq 1$ for all k.*

and $0 \leq x_1 \leq x_2 \leq \ldots \leq x_{d-1}$

Definition 2.5. The vector $r = (r_0, \dots, r_{d-1}) \in \mathbb{N}^d$ defined via Theorem 2.4 for every $x \in \Delta_{n-1}$ is called the *multiplicity vector* of x, where we set $r_0 = 0$ if the associated point is of type (2).

In order to understand the boundary of the Vandermonde cell we are considering the following notion of generalized positive Vandermonde variety. The study of such varieties goes back to work of Arnold, Givental and Kostov [Arn86; Giv87; Kos89] who had considered these in their study of hyperbolic polynomials. Our setup is a bit more evolved, as in contrast to the above named authors, we will consider general exponents and consider only the positive part. We fix $2 \leq d \leq n$ and a vector of integer exponents $\alpha = (\alpha_1, \dots, \alpha_{d-1})$ with $1 < \alpha_1 < \ldots < \alpha_{d-1}$ for the remaining part of this subsection.

Definition 2.6. For $2 \leq k \leq d$ and $c \in \{1\} \times \mathbb{R}^{k-1}_{\geq 0}$ we define the associated *generalized positive α-Vandermonde variety* to be the fiber over c of the corresponding Vandermonde map, i.e.,

$$V_k^\alpha(c) := \nu^{-1}_{n,(1,\alpha_1,\dots,\alpha_{k-1})}(c) \cap \mathbb{R}^n_{\geq 0}.$$

Some fundamental properties of these varieties had been shown already by the mentioned authors. We show here that their proofs almost directly can be generalized to the more general setup presented above. To this end we follow the proofs presented in [Meg92] (see also [Rai04] Section 3).

We begin with the following observation for the tangent space of a Vandermonde variety.

Lemma 2.7. *Let $c \in \{1\} \times \mathbb{R}^{k-1}_{\geq 0}$. Then a point $x \in V_k^\alpha(c) \cap \mathbb{R}^n_{\geq 0}$ with more than k distinct need non-zero coordinates is a smooth point.*

Proof. First, note that $\frac{\partial p_a}{\partial x_j} = a x_j^{a-1}$. Therefore, the Jacobian of the map $(p_1, p_{\alpha_1}, \dots, p_{\alpha_{k-1}})$ equals

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 x_1^{\alpha_1 - 1} & \alpha_1 x_2^{\alpha_1-1} & \dots & \alpha_1 x_n^{\alpha_1-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{k-1} x_1^{\alpha_{k-1}-1} & \alpha_{k-1} x_2^{\alpha_{k-1}-1} & \dots & \alpha_{k-1} x_n^{\alpha_{k-1}-1} \end{pmatrix}$$

Now suppose that the rows of the Jacobi matrix are linearly dependent, so there exist $l_0, \dots, l_{k-1}$ such then every column satisfies $l_0 \cdot 1 + l_2 \cdot \alpha_1 x_j^{\alpha_1 - 1} + \dots l_{k-1} \cdot \alpha_{k-1} x_j^{\alpha_k - 1} = 0$. Therefore, every coordinate x_j is a solution to the same univariate polynomial

$$f(t) = b_0 + b_1 t^{\alpha_1 - 1} + \dots + b_{k-1} t^{\alpha_{k-1} - 1}.$$

However, by Descarte's rule of signs f can have at most $k-1$ different roots in $\mathbb{R}^n_{>0}$. Therefore, in case x has more distinct non-zero absolute values of coordinates the Jacobi matrix has full rank. □

Lemma 2.8. *Let $c \in \{1\} \times \mathbb{R}^{d-2}_{\geq 0}$ be generic. The critical points of $p_{\alpha_{d-1}}$ on $V^{\alpha}_{d-1}(c) \cap \mathbb{R}^n_{\geq 0}$ are exactly the points with precisely $d-1$ distinct non-zero coordinates.*

Proof. For generic c the Vandermonde variety $V^{\alpha}_k(c)$ is smooth and by the Jacobian criterion ([Eis13, Thm 16.19]) $(n-d-1)$ equidimensional (or empty), therefore by the previous Lemma every point in $x \in V^{\alpha}_{d-1}(c)$ will have strictly more than $d-2$ distinct non-zero coordinates. Let $z = (z_1, \dots, z_n) \in \mathbb{R}^n_{\geq 0}$ be a critical point of $p_{\alpha_{d-1}}$ on $V^{\alpha}_{d-1}(c)$.

Then, there exists Lagrange multipliers $\lambda_1^*, \dots, \lambda_{d-1}^* \in \mathbb{R}$ such that all partial derivatives of the Lagrangian function

$$L(X) := p_{\alpha_{d-1}}(X) + \lambda_0 (p_1 - a_1) + \sum_{i=1}^{d-2} \lambda_i^* (p_{\alpha_i}(X) - a_{i+1}) \tag{2.1}$$

vanish at x. This yields

$$0 = \nabla p_{\alpha_{d-1}}(z) + \lambda_1^* \nabla p_1(z) + \dots + \lambda_{d-1}^* \nabla p_{\alpha_{d-2}}(z).$$

Again noting that $\frac{\partial p_a}{\partial x_j} = a x_j^{a-1}$ we observe that there exists a univariate polynomial

$$f(t) = \alpha_{d-1} t^{\alpha_{d-1} - 1} - \lambda_1 t - \dots - \lambda_{d-1} \alpha_{d-2} t^{\alpha_{d-2} - 1}$$

such that $f(z_i) = 0$ for all $1 \leq i \leq n$. However, by Descarte's rule of signs the polynomial f can have at most $d-1$ positive roots. Since $z \in \mathbb{R}^n$ is regular we have z must have exactly $d-1$ distinct absolute values of non-zero coordinates. Conversely, if $z \in V^{\alpha}_{d-1}(c) \cap \mathbb{R}^n_{\geq 0}$ has precisely $d-1$ distinct absolute values of non-zero coordinates the existence of the Lagrange multipliers follows from the observation that the rank of the $d-1 \times d$ matrix

$$A := \begin{pmatrix} z_1 & z_1^{\alpha_1 - 1} & \dots & z_1^{\alpha_{d-1}-1} \\ z_2 & z_2^{\alpha_1-1} & \dots & z_2^{\alpha_{d-1}-1} \\ \vdots & \vdots & & \vdots \\ z_{d-1} & z_{d-1}^{\alpha_1 - 1} & \dots & z_{d-1}^{\alpha_{d-1}-1} \end{pmatrix}$$

is $d-1$. Thus, the columns of this matrix being linearly dependent yields the existence of the Lagrange multipliers. □

Definition 2.9. To $x \in \mathbb{R}^n_{\geq 0}$ we associate the multiplicity vector $m = (m_0, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$ as follows: The number m_0 counts the number of zero coordinates, and m_i corresponds to the number of times the $i-th$ smallest coordinate appears. This means, assuming that x has k positive coordinates which are ordered increasingly, we have

$$x = (\underbrace{0, \dots, 0}_{m_0}, \underbrace{x_1, \dots, x_1}_{m_1}, \dots, \underbrace{x_k, \dots, x_k}_{m_k}).$$

Proposition 2.10. *Let $V^{\alpha}_{d-1}(c)$ be a smooth Vandermonde variety and $x \in \mathbb{R}^n_{\geq 0}$ be a critical point of $p_{\alpha_{d-1}}$ on $V^{\alpha}_{d-1}(c)$. Further set $r_i = m_i - 1$. Then, if d is odd (even), the Hessian of $p_{\alpha_{d-1}}$ on $V^{\alpha}_{d-1}(c)$ at the point x is the sum of a negative (positive) definite quadratic form on $\mathbb{R}^a$ and a positive (negative) definite form on $\mathbb{R}^b$, where $a = \sum_{i<d, i \notin 2\mathbb{N}} r_i$ and $b = m_0 + \sum_{i<d, i \in 2\mathbb{N}} r_i$. $p_{\alpha_{d-1}}$ is a Morse function on $V^{\alpha}_{d-1}(a)$ with Morse index b.*

Proof. Let x be a critical point which we assume without loss of generality to have only non-negative coordinates. By Lemma 2.8 we can assume

$$x = (\underbrace{0,\dots,0}_{m_0}, \underbrace{x_1,\dots,x_1}_{r_1}, \dots, \underbrace{x_{d-1},\dots,x_{d-1}}_{r_{d-1}}, x_1, \dots, x_{d-1})$$

for some positive pairwise distinct x_i's. Let

$$\tilde{x} = (\underbrace{0,\dots,0}_{m_0}, \underbrace{x_1,\dots,x_1}_{r_1}, \dots, \underbrace{x_{d-1},\dots,x_{d-1}}_{r_{d-1}}) \in \mathbb{R}^{n-d+1}$$

denote the point consisting of the first $n-d+1$ coordinates of x. Notice that since $x_1, \dots, x_{d-1}$ are pairwisely distinct, the last $d-1$ columns of the associated Jacobian will be of full rank. Thus, the first $n-d+1$ coordinates can be used as a system of local coordinates for $V^{\alpha}_{d-1}(c)$ in a neighborhood of x. Since x is a critical point we know from the proof of Lemma 2.8 that every coordinate of x satisfies the same univariate polynomial equation $f(x_i) = 0$. By the intermediate value theorem the roots of the derivative $f'(t)$ interlace the roots of f. Noting that the leading coefficient of g is positive, we find that the function values of the derivative f' at the roots of f satisfy

$$f'(x_{d-1}) > 0, f'(x_{d-2}) < 0, f'(x_{d-3}) > 0, \dots, (-1)^q f'(x_1) < 0, (-1)^q f'(0) > 0,$$

where $q = d - 1 \mod 2$. Now, since we have that the Hessian of Lagrange function in (2.1)

$$\frac{\partial^2 L}{\partial X_i \partial X_j} = 0 \text{ for } i \neq j, \text{ and } \frac{\partial^2 L}{\partial X_i \partial X_i} = f'(x_i),$$

we can conclude that the Hessian of $p_{\alpha_{d-1}}$ on $V^{\alpha}_{d-1}(c)$ at x has indeed the claimed form. □

Thus, we immediately obtain:

Corollary 2.11. *Let x be a critical point of $p_{\alpha_{d-1}}$ on $V^{\alpha}_{d-1}(c)$. Then x is a strict local minimum/maximum if x is of type (1)/(2) if d is odd and of type (2)/(1) if d is even.*

Lemma 2.12. *Let $n \geq d \geq 2$. The image of the function $p_{\alpha_{d-1}} : \mathbb{R}^n \to \mathbb{R}$ on the set $V^{\alpha}_{d-1}(c)$ is either empty or an interval for all $c \in \mathbb{R}^{d-1}$.*

The proof of this Lemma essentially follows from the following statement, which originally had been shown bu Givental [Giv87, p. 275] for the case of Vandermonde varieties $V^{\alpha}_{d-1}(c)$, with $\alpha = (2, \dots, d)$.

Proposition 2.13. *For generic c the set*

$$\{0 \leq x_1 \leq \dots \leq x_n\} \bigcap V^{\alpha}_{d-1}(c)$$

is either contractible or empty.

Although the statement in Givental's article [Giv87] is not stated in this generality the proof follows verbatim using Proposition 2.10 and the following general insight the on the local topology of functions with non-degenerate Hessians.

Proposition 2.14. *[Giv87, Lemma 2] The reconstruction of the topology of a level set of a function f on $\mathbb{R}^a_+ \times \mathbb{R}^b_-$ in the neighborhood of the critical point $(0,0)$ with non-degenerate Hessian $F = Q_+ + Q_-$ is trivial if $a, b > 0$ and consists of the birth (death) of a simplex otherwise.*

A reader interested in more details about the above statements might also consult [Rai04; Meg92] for more detailed proofs. With these preparations the proof of Lemma 2.12 follows almost directly.

Proof of Lemma 2.12. We remark first that it is sufficient to show the claimed statement for generic $c \in \mathbb{R}^{d-1}$ since it follows from [Kos89, Lemma 2.6] that from the generic case the statement follows for all c. Thus, we can assume that c is generic. Then, following Proposition 2.13 the image of $p_{\alpha_{d-1}}$ on the restriction is connected and compact and thus, if it is non-empty, an interval. □

We are now in the position to give the proof of Theorem 2.4.

Proof of Theorem 2.4. We conduct the proof in three steps starting with integer exponents and then deal with positive rational and general positive exponents.

(1) First, we suppose $\alpha \in \mathbb{Z}_{\geq 2}^{d-1}$ is an integer exponent vector. Any point of type (1) or (2) with $d-1$ distinct non-zero coordinates is indeed mapped to the boundary by Corollary 2.11. However, any point of type (1) or (2) with less than $d-1$ distinct non-zero coordinates is then also mapped to the boundary by continuity. Now, we assume that $c := (p_{\alpha_1}(x), \dots, p_{\alpha_{d-1}}(x))$ is contained in the boundary of the set $\Pi_{n,\alpha}$. If c is non-singular the set $p_{\alpha_{d-1}}(V_{d-1}^{\alpha}(c))$ is an interval by Lemma 2.12. We observe that $p_{\alpha_{d-1}}$ is either minimized or maximized at x. We can apply Corollary 2.11 and obtain that x must be of type (1) or (2). If c is a singular point then x can be obtained as the limit of a sequence of such points.
(2) Second, we suppose $\alpha \in \mathbb{Q}_{>1}^{d-1}$ is a rational exponent vector. There exists $q \in \mathbb{N}$ such that $\beta := q\alpha \in \mathbb{Z}_{\geq 2}^{d-1}$ is an integer exponent vector and $\nu_{n,\alpha}(x) = \nu_{n,\beta}(x^{1/q})$ for all $x \in \mathbb{R}_{\geq 0}^n$, where $x^{1/q} = (x_1^{1/q}, \dots, x_n^{1/q})$. We already know that the claim for the Vandermonde cell $\Pi_{n,\beta}$. However, since $\nu_{n,\beta}$ is weighted homogeneous, we have an analogous description of $\operatorname{bd} \nu_{n,\beta}(\mathbb{R}_{\geq 0}^n) = \nu_{n,\alpha}(\mathbb{R}_{\geq 0}^n)$. This shows the claim for $\Pi_{n,\alpha}$ since also $\nu_{n,\alpha}$ is weighted homogeneous with rational weights.
(3) Third, we suppose $\alpha \in \mathbb{R}_{>1}^{d-1}$ is a real exponent vector. Then there exists a sequence of rational exponents $\alpha^m \in \mathbb{Q}_{>1}^{d-1}$ which converges to α. We have $c \in \operatorname{bd} \Pi_{n,\alpha}$ if and only if there exists a sequence $c_m \in \operatorname{bd} \Pi_{n,\alpha^n}$ with $c_m \to c$ for $m \to \infty$. Then the claim follows by continuity.

□

Remark 2.15. *The description of the boundary of the (n,α)-Vandermonde cell in Theorem 2.4 transfers to α-Vandermonde maps with positive weight vector $w \in \mathbb{R}_{>0}^n$, i.e. to maps*

$$x \mapsto (w_1 x_1^{\alpha_1} + \ldots + w_n x_n^{\alpha_1}, \ldots, w_1 x_1^{\alpha_{d-1}} + \ldots + w_n x_n^{\alpha_{d-1}})$$

This is since the Jacobi matrix of a weighted α-Vandermonde map differs only by positive scalars of the Jacobi matrix of an α-Vandermonde map and thus of a generalized Vandermonde matrix.

2.2. **Boundary of the Vandermonde Cell $\Pi_{n,3}$.** In this subsection we investigate planar parametrizations of $\operatorname{bd}\Pi_{n,3}$. Newton's identities imply $\Pi_{n,d} \simeq E_{n,d}$ up to a polynomial diffeomorphism for $n \geq d$. Thus, the parametrization transfers to the image in elementary symmetric polynomials.

Theorem 2.16. *For $n \geq 3$, a parametrization of $\operatorname{bd}\Pi_{n,3}$ is given by the following n arcs. The upper part of the boundary is parametrized by the arc*

$$\left(\frac{(1-t)^2}{n-1} + t^2, \frac{(1-t)^3}{(n-1)^2} + t^3\right) : \frac{1}{n} \leq t \leq 1 \tag{2.2}$$

while the lower part is parameterized by the $n-1$ arcs

$$\left(\left(\frac{(1-t)^2}{n-k-1} + t^2, \frac{(1-t)^3}{(n-k-1)^2} + t^3\right) : 0 \leq t \leq \frac{1}{n-k}\right)_{0 \leq k \leq n-2}. \tag{2.3}$$

Proof. We apply Theorem 2.4 to determine the boundary. The boundary consists of the closure of the set of all point evaluations in (p_2, p_3) at all points $(0, \dots, 0, x_1, \dots, x_1, x_2, \dots, x_2) \in \Delta_{n-1}$ of the form $0 < x_1 < x_2$ of type (1) or (2).
Note that any point of type (2) must be of the form $(a, \dots, a, b)$ with $0 < a < b$ and $(n-1)a + b = 1$.

Thus, $a = \frac{1-b}{n-1}$, $\frac{1}{n} < b < 1$ and we observe that the upper part of the boundary is indeed parameterized by the curve in (2.2).

We note that there are essentially $n-1$ points of type (1). Namely, points of the form

$$(\underbrace{0,\dots,0}_{\#=k}, a, \underbrace{b,\dots,b}_{\#=n-k-1})$$

for $0 \le k \le n-2$ satisfying $b = \frac{1-a}{n-k-1}$ and $a \le \frac{1}{n-k}$. We obtain precisely the parametrizations (2.3) of the lower part of the boundary. □

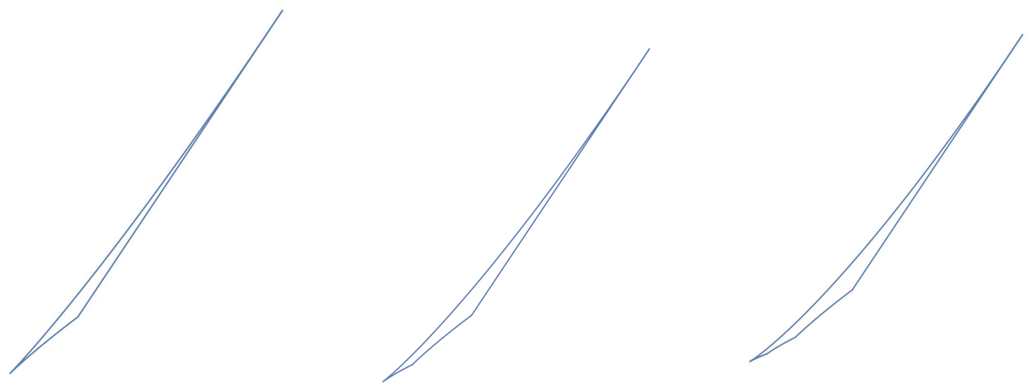

FIGURE 1. The sets $\operatorname{bd}\Pi_{n,3}$ for $3 \le n \le 5$

We immediately obtain parametrizations of the boundary of the sets $E_{n,3}$ by Newton's identities. See Figure 2 for a visualisation of these boundaries.

FIGURE 2. The sets $\operatorname{bd} E_{n,3}$ for $n \in \{3,5\}$

Theorem 2.16 generalizes to a parametrization of the boundary of the set

$$\{(p_k(x), p_m(x)) : x \in \Delta_{n-1}\}$$

for $2 \le k < m$. However, we note that the upper part of the boundary cannot be described by just one smooth parametrizitation. This is since there are essentially more points of type *(1)* resp. *(2)* than for $k = 2$ and $m = 3$, where there is only 1. However, a careful analysis can lead to a description of the boundary.

Example 2.17. *For $2 \le k \le 3$, the lower part of the boundary of the set*

$$\{(p_k(x), p_4(x)) : x \in \Delta_3\}$$

is the union of the images of the following two parametrizations

$$\left(2s^k + t^k + (1-2s-t)^k, 2s^4 + t^4 + (1-2s-t)^4\right) : 0 \le s \le t \le \frac{1}{2} - s \ , \tag{2.4}$$

$$\left(s^k + t^k + \frac{1}{2^{k-1}}(1-s-t)^k, s^4 + t^4 + \frac{1}{8}(1-s-t)^4\right) : 0 \le s \le t < \frac{1}{3} - \frac{1}{3}s \ . \tag{2.5}$$

Parametrization (2.4) *comes from the points with multiplicity vector* (x_1, x_1, x_2, x_3) *and* (2.5) *from the points* (x_1, x_2, x_3, x_3).
The upper part of the boundary is the union of the images of the following two parametrizations

$$\left(s^k + 2t^k + (1-s-2t)^k, s^4 + 2t^4 + (1-s-2t)^4\right) : 0 \le s \le t \le \frac{1}{3} - \frac{1}{3}s \ , \tag{2.6}$$

$$\left(s^k + t^k + (1-s-t)^k, s^4 + t^4 + (1-s-t)^4)\right) : 0 \le s \le t \le \frac{1}{2} - \frac{1}{2}t \ . \tag{2.7}$$

Parametrization (2.6) *comes from the points with multiplicity vector* (x_1, x_2, x_2, x_3) *and* (2.7) *from those with* $(0, x_1, x_2, x_3)$.

In the transition from $\Pi_{n,3}$ to $\Pi_{n+1,3}$ in Theorem 2.16 the arc describing the upper part of the boundary grows and converges. Its limit has the parametrization $(t, t^{3/2})$, $0 \le t \le 1$. Moreover, any point on the lower part of the boundary for n remains on the boundary for $n+1$, but a single new smooth curve is added. Namely, the smooth curve with parametrization

$$\left(\frac{(1-t)^2}{n} + t^2, \frac{(1-t)^3}{n^2} + t^3\right) \ , \ 0 \le t \le \frac{1}{n+1} \ .$$

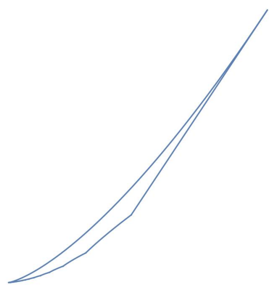

FIGURE 3. The boundary of the set $\Pi_{20,3}$

Corollary 2.18. *The boundary of the set* Π_3 *equals*

$$\left\{(t, t^{3/2}) : 0 \le t \le 1\right\} \cup \bigcup_{k \in \mathbb{N}_{>1}} \left\{\left(\frac{(1-t)^2}{k} + t^2, \frac{(1-t)^3}{k^2} + t^3\right) : 0 \le t \le \frac{1}{k+1}\right\} \ .$$

We note that two different parametrizations of the lower part of the boundary

$$\left(\frac{(1-t)^2}{k} + t^2, \frac{(1-t)^3}{k^2} + t^3\right) \ : \ 0 \le t \le \frac{1}{k+1} \text{ and}$$
$$\left(\frac{(1-s)^2}{l} + s^2, \frac{(1-s)^3}{l^2} + s^3\right) \ : \ 0 \le s \le \frac{1}{l+1}$$

intersect if and only if $k = l-1$ or $k = l+1$. Without loss of generality be $k = l-1$. The intersection is a point and the curves meet at $\left(\frac{1}{l}, \frac{1}{l^2}\right)$ for $t = \frac{1}{l}$ and $s = 0$. Moreover, the gradients $(0,0)$ and $\left(\frac{-2}{l}, \frac{-3}{l}\right)$ differ at this point which shows that $\left(\frac{1}{l}, \frac{1}{l^2}\right)$ is a singular point of $\operatorname{bd} \Pi_3$.

Corollary 2.19. *The limit Vandermonde cell* Π_3 *has countably infinitely many isolated singular points which are the points of the form*

$$\left(\frac{1}{k}, \frac{1}{k^2}\right), \ k \in \mathbb{N}_{>0} \text{ and } (0,0) \ .$$

Proof. It follows from the discussion above that only neighboring parametrizations of the lower part of the boundary intersect and their intersection point is a singular point of the boundary. The intersection points are all of the form $\left(\frac{1}{k+1}, \frac{1}{(k+1)^2}\right)$ for all $k \in \mathbb{N}$. However, $(1,1)$ is an intersection of the parametriztation $(t, t^{3/2}), 0 \le t \le 1$ of the upper part of the boundary and

$((1-s)^2+s^2,(1-s)^3+s^3):0\le s\le 1/2$ of the lower part. For $t=1$ and $s=0$, but again the gradients are different which shows that $(1,1)$ is a singular point.
Moreover, any singular point must be an intersection of two parametrizations. But the intersection points are precisely the points of the claimed form and the limit point $(0,0)$.
Since all the singular points lie in the rational moment curve (t,t^2) the points are indeed isolated (see e.g. ([Bar02, Chapter II.9.])). □

Corollary 2.20. *The sets Π_d and E_d are not semialgebraic for all $d\ge 3$.*

Proof. We show that Π_3 is not semialgebraic. Then for $d\ge 4$ the set Π_d is not semialgebraic, since for $d\ge 3$ we have $\Pi_3=\pi(\Pi_d)$, where $\pi:\mathbb{R}^d\to\mathbb{R}^2$ denotes the projection onto the first 2 coordinates. Moreover, E_d is a polynomial image of the set Π_d which must also be non semialgebraic.

Suppose hat the set Π_3 is semialgebraic. However, by Corollary 2.19 the semialgebraic set Π_3 has countably infinitely many isolated singular points. Let T denote the union of these singular points. The union of all singular points of a semialgebraic set is again semialgebraic since this condition can be formalized as the vanishing and non-vanishing of certain polynomial equalities. Thus, T is semialgebraic. By ([BCR13, Theorem 2.4.4]) every semialgebraic set is the disjoint union of a finite number of semialgebraically connected semialgebraic sets. However, there are countable infinitely isolated points in T which contradicts T being semialgebraic. In particular, Π_3 cannot be semialgebraic. □

3. Combinatorial properties of the boundary of $\Pi_{n,d}$

Our main result in this section is the following theorem, which provides a combinatorial description of the boundary of the Vandermonde cell.

Theorem 3.1. *The set $\Pi_{n,d}$ has the combinatorial structure of the cyclic polytope $C(n,d-1)$.*

Cyclic polytopes are well studied objects in polyhedral combinatorics.

Definition 3.2. For $n>d\ge 2$ the *cyclic polytope* $C(n,d)$ is the convex polytope with n vertices which are points on the real d-dimensional moment curve $(t,t^2,\dots,t^d)$.

The combinatorial structure, e.g. the f-vector, of $C(n,d)$ is independent of the chosen points and its boundary is a $d-1$-dimensional *simplicial polytope*. Thus, we can speak about *the* cyclic polytope $C(n,d)$. Cyclic polytopes are interesting objects. For instance, the upper bound theorem says that $C(n,d)$ has the component-wise maximal f-vector among all d-dimensional convex polytopes with n vertices [McM70; Sta75]. We refer to ([Zie12, Section 0]) for more background on cyclic polytopes. For all $n\ge d\ge 3$ we have $\operatorname{conv}\left\{\left(\frac{1}{k},\frac{1}{k^2},\dots,\frac{1}{k^{d-1}}\right):k\in[n]\right\}$ is the cyclic polytope $C(n,d-1)$ and this is the choice of vertices of $C(n,d-1)$ we will usually use.

Definition 3.3. A set $S\subset\mathbb{R}^d$ has the *combinatorial structure* of the cyclic polytope $C(n,d)$ if there exists a homeomorphism $\Phi:\operatorname{bd}C(n,d)\to\operatorname{bd}S$ which is a diffeomorphism when restricted to the relative interior of any face of $\operatorname{bd}C(n,d)$. The *vertices* of S are the images of the vertices of $C(n,d)$.
We call a set $S\subset\mathbb{R}^n$ a *curved m-simplex* if S is the image of Δ_m under a continuous map f, such that f is a diffeomorphism when restricted to the relative interior of any face of Δ_m. The *vertices* of a curved m-simplex are the images of the vertices of the simplex Δ_m.

Note that the boundary of a set which has the combinatorial structure of a cyclic polytope is the gluing of $\#\{\text{facets of } C(n,d)\}$ many patches and each patch is a curved d-simplex such that the vertices of the patches are labelled by Gale's evenness condition. Moreover, patches of the boundary intersect if and only if the intersection of their sets of vertices is non-empty.

The facets of a cyclic polytope $C(n,d)$ are characterized by *Gale's evenness condition.* For an integer k we write $\hat{k}:=\left(\frac{1}{k},\frac{1}{k^2},\dots,\frac{1}{k^d}\right)$.

Theorem 3.4 ([Gal63]). *The facets of $C(n,d)$ are precisely given by all $\{\hat{k} : k \in S\}$, where $S \subset [n]$ is any set of size d satisfying*

i) If d is even, then S is either a disjoint union of consecutive pairs $\{i, i+1\}$, or a disjoint union of consecutive pairs $\{i, i+1\}$ and $\{1, n\}$.

ii) If d is odd, then S is a disjoint union of consecutive pairs $\{i, i+1\}$ and either the singleton $\{1\}$ or $\{n\}$.

The standard formulation of Gale's evenness condition is the following. Let $n > d$, $k_1 < \ldots < k_n \in \mathbb{R}$ and $T = \{\hat{k}_1, \ldots, \hat{k}_n\}$ be the vertices of $\operatorname{conv}\{\hat{k}_i : 1 \leq i \leq n\}$. Then a set $T_d \subset T$ of size d spans a facet of $C(n,d)$ if and only if any two elements in $T \setminus T_d$ are separated by an even number of elements from T_d in the sequence $(k_1, k_2, \ldots, k_n)$.

We briefly present an outline of our proof of Theorem 3.1. By Theorem 2.4 we have $\operatorname{bd}\Pi_{n,d} = \{(p_2, \ldots, p_d)(x) : x \in \Delta_{n-1}, x_i \leq x_{i+1}, \forall i, x \text{ is of type (1) or (2)}\}$. We associate the multiplicity vectors r of type (1) and (2) with $d-2$-dimensional simplices Δ_*^r and the simplices correspond to the facets of $C(n, d-1)$ (see Proposition 3.8). We show that there are natural homeomorphisms $\Psi_r : \Delta_*^r \to \operatorname{bd}\Pi_{n,d}$ which are diffeomorphisms when restricted to the relative interior of any face of Δ_*^r in Lemma 3.5. Then we show that there is a one-to-one map between the union of all the associated simplices Δ_*^r and $\operatorname{bd}\Pi_{n,d}$ in Theorem 3.10.

For a sequence $r = (r_0, r_1, \ldots, r_{d-1}) \in \mathbb{N}^d$ we define the $d-2$-dimensional simplex

$$\Delta_*^r := \{(\underbrace{0, \ldots, 0}_{r_0}, \underbrace{x_1, \ldots, x_1}_{r_1}, \ldots, \underbrace{x_{d-1}, \ldots, x_{d-1}}_{r_{d-1}}) \in \mathbb{R}^n_{\geq 0} : x_i \leq x_{i+1} \forall i, \sum_{i=1}^{d-1} r_i x_i = 1\}$$

and the map

$$\begin{array}{cccc} \psi_r : & \Delta_*^r & \longrightarrow & \mathbb{R}^{d-1}_{\geq 0} \\ & z & \longmapsto & (p_2(z), \ldots, p_d(z)) \end{array}.$$

Lemma 3.5. *Let $r \in \mathbb{N}^d$ with $\sum_{i=0}^{d-1} r_i = n$. Then the map $\psi_r : \Delta_*^r \to \operatorname{Im}(\psi_r)$ is a homeomorphism and a diffeomorphism when restricted to the relative interior of any face of Δ_*^r.*

Proof of Lemma 3.5. First, we want to show that ψ_r is a homeomorphism. It follows from ([Mas79, Theorem 1]) that we only need to verify that the Jacobian of ψ_r is positive on the interior of Δ_*^r and positive on the restriction of ψ_r to any face. Note, this is true since

$$\det(\operatorname{Jac}\psi_r) = r_1 r_2 \cdots r_{d-1} \cdot (d-1)! \prod_{1 \leq i < j \leq d-1} (x_i - x_j)$$

is a positive scalar of the Vandermonde determinant and the Vandermonde matrix is totally positive [Ste02], i.e. all the minors are positive.
Second, we show that ψ_r restricted to the relative interior of any face is a diffeomorphism. Since the Jacobian of ψ_r restricted to the relative interior of any face of Δ_*^r is always non-singular a local inverse of the Jacobi matrix exists, by the inverse function theorem. However, the Jacobi matrix at any point in the interior of Δ_*^r is a one-to-one map and differentiable in any local neighborhood. Thus, the local inverse must be a global inverse and ψ_r is a diffeomorphism when restricted to the relative interior of any face. ☐

Corollary 3.6. *Each simplex Δ_*^r is mapped to a curved $d-2$-simplex in $\mathbb{R}^d$ via ψ_r and the vertices of the curved simplex $\psi_r(\Delta_*^r)$ are the points $(\frac{1}{k}, \frac{1}{k^2}, \ldots, \frac{1}{k^{d-1}})$ for all vertices $\overline{k}$ of Δ_*^r.*

It follows from Theorem 2.4 that the set $\operatorname{bd}\Pi_{n,d}$ is the union of the curved simplices $\psi_r(\Delta_*^r)$ for all multiplicity vectors of type (1) and (2). We still have to show that the vertices of each simplex satisfy Gale's evenness condition and the curved simplices are indeed correctly arranged patches of $\operatorname{bd}\Pi n, d$.

Recall, a multiplicity vector r of type (1) has the form $r_0 \geq 0$, $r_{2k-1} = 1$ and $r_{2k} \geq 1$ for all $1 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$ and a multiplicity vector of type (2) is of the form $r_0 = 0, r_{2k-1} \geq 1$ and $r_{2k} = 1$ for all $1 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$. For $1 \leq k \leq n$ we write $\overline{k} := (\underbrace{0, \dots, 0}_{n-k}, \underbrace{\tfrac{1}{k}, \dots, \tfrac{1}{k}}_{k})$.

Lemma 3.7. *Let $r \in \mathbb{N}^d$ with $\sum_{i=0}^{d-1} = n$ be a multiplicity vector of type (1) or (2). The vertices of Δ_*^r are*

$$\begin{cases} \overline{n - r_0}, \overline{n - r_0 - 1}, \overline{n - r_0 - r_2 - 1}, \overline{n - r_0 - r_2 - 2}, \dots & \text{if } r \text{ is of type } (1) \\ \overline{n}, \overline{n - r_1}, \overline{n - r_1 - 1}, \overline{n - r_1 - r_3 - 1}, \overline{n - r_1 - r_3 - 2}, \dots & \text{if } r \text{ is of type } (2) \end{cases}$$

Proof. The vertices are the points where all but one of the defining inequalities of the simplex Δ_*^r are tight. Thus, the vertices are the $d-1$ points for which $x_i = 0, 1 \leq i \leq k-1$ and $x_k = \dots = x_{d-1}$ for all $0 \leq k \leq d-2$. □

In the following Proposition we observe that multiplicity vectors of type (1) and (2) encode Gale's evenness condition with the identification $k \leftrightarrow \overline{k}$.

Proposition 3.8. *Let $r = (r_0, \dots, r_{d-1}) \in \mathbb{N}^d$ with $\sum_{i=0}^{d-1} r = n$ be a multiplicity vector of type (1) or (2). Then, the set*

$$\{k \in [n] : \overline{k} \text{ vertex of } \Delta_*^r\}$$

satisfies Gale's evenness condition. Moreover, any set $S \subset [n]$ of size $d-1$ satisfying Gale's evenness condition gives rise to a simplex Δ_*^s such that $|s| = n$ and the multiplicity vector s is of type (1) or (2).

Proof. We distinguish between $d-1$ odd and even. We first consider the case $d-1$ is odd. In this situation we have by Lemma 3.7 that the vertices of Δ_*^r are

$$\begin{cases} \overline{n} \text{ and } \overline{r_{d-1}}, \overline{r_{d-1} + 1}, \dots, \overline{n - r_1 - 1}, \overline{n - r_1} & \text{if } r \text{ is of type } (1) \\ \overline{1} \text{ and } \overline{1 + r_{d-1}}, \overline{2 + r_{d-1}}, \dots, \overline{n - r_0 - 1}, \overline{n - r_0} & \text{if } r \text{ is of type } (2) \end{cases}$$

The corresponding set of integers satisfies Gale's evenness condition in both cases. Conversely, we suppose $J \subset [n]$ with $|J| = d-1$ and J satisfies Gale's evenness condition. Given a set $S \subset [n]$ we construct the associated multiplicity vector s.

(1) First, we suppose $J = \biguplus_{j=1}^{\frac{d}{2}-1} \{i_j, i_j + 1\} \uplus \{1\}$ and $1 < i_1 < \dots < i_{\frac{d}{2}-1} < n$. We note $d-2$ is even and define $s_{d-2} := i_1 - 1 \geq 1$ and $s_{d-2j} := i_j - i_{j-1} - 1 \geq 1$ for all $1 < j \leq \frac{d}{2} - 1$. Then,

$$s_2 + s_4 + \dots + s_{d-2} = i_{\frac{d}{2}-1} - \frac{d}{2} + 1 \leq n - \frac{d}{2}.$$

We set $s_1 = s_3 = \dots = s_{d-1} := 1$ and $0 \leq s_0 := n - s_1 + s_2 + \dots + s_{d-1}$. We note $s_1 + \dots + s_{d-1} \leq n - \frac{d}{2} + \lceil \frac{d-1}{2} \rceil \leq n$. Thus, the vector s is indeed of type (2) and the simplex Δ_*^s has vertex set $\{\overline{k} : k \in J\}$

(2) Second, we suppose $J = \biguplus_{j=1}^{\frac{d}{2}-1} \{i_j, i_j + 1\} \uplus \{n\}$ and $1 \leq i_1 < \dots < i_{\frac{d}{2}-1} < n$. We define $s_{d-1} := i_1$, $s_{d-2j+1} := i_j - i_{j-1} - 1 \geq 1$ for all $2 \leq j \leq \frac{d}{2} - 1$. Then,

$$s_{d-1} + s_{d-3} + \dots + s_3 = i_{\frac{d}{2}-1} - \frac{d}{2} + 2 \leq n - \frac{d}{2}.$$

We define $s_2 = s_4 = \dots = s_{d-2} := 1$ and $s_1 := n - (s_{d-1} + s_{d-1} + \dots + s_2) \geq 1$. We have s is of type (2) and the simplex Δ_*^s has vertex set $\{\overline{k} : k \in J\}$.

We now turn to the case with $d-1$ even. Again by Lemma 3.7 the vertices of Δ_*^r are

$$\begin{cases} \overline{r_{d-1}}, \overline{r_{d-1} + 1}, \dots, \overline{n - r_0 - 1}, \overline{n - r_0} & \text{if } r \text{ is of type } (1) \\ \overline{1}, \overline{n} \text{ and } \overline{r_{d-2} + 1}, \overline{r_{d-2} + 2}, \dots, \overline{n - r_1 - 1}, \overline{n - r_1} & \text{if } r \text{ is of type } (2) \end{cases}$$

Also in this case the corresponding set of integers satisfies Gale's evenness condition. Conversely, let $J \subset [n]$ be of size $d-1$ satisfying Gale's evenness condition. Given a set $S \subset [n]$ we construct the associated multiplicity vector s.

(1) First, we suppose

$$J = \{1,n\} \uplus \biguplus_{j=1}^{\frac{d-3}{2}} \{i_j, i_j+1\} \text{ with } < i_1 < \ldots < i_{\frac{d-3}{2}} < n-1.$$

We define $s_2 = s_4 = \ldots = s_{d-1} := 1$, $s_{d-2} := i_1 - 1 \geq 1$, $s_{d-2k} := i_k - i_{k-1} - 1 \geq 1$ for all $2 \leq k \leq \frac{d-3}{2}$, and $s_1 := n - (s_2 + \ldots + s_{d-1}) \geq 1$. The vector s is of type (2) and the simplex Δ_*^s has the vertex set $\{\overline{k} : k \in J\}$.

(2) Second, we suppose

$$J = \biguplus_{j=1}^{\frac{d-1}{2}} \{i_j, i_j+1\} \text{ and } 1 \leq i_1 < \ldots < i_{\frac{d-1}{2}} < n.$$

We set $s_1 = \ldots = s_{d-2} := 1$, $s_{d-1} := i_1 \geq 1$, $s_{d-2k+1} := i_k - i_{k-1} - 1 \geq 1$ for all $2 \leq k \leq \frac{d-1}{2}$ and $s_0 := n - (s_1 + \ldots + s_{d-1}) \geq 0$. Then, the vector s is of type (1) and the vertex set of the simplex Δ_*^s is $\{\overline{k} : k \in J\}$.

$\square$

Corollary 3.9. *The map*

$$\begin{array}{rcl} \kappa_{n,d} : \operatorname{bd}\left(\operatorname{conv}\{\left(\frac{1}{i}, \ldots, \frac{1}{i^{d-1}}\right) : 1 \leq i \leq n\}\right) & \longrightarrow & \bigcup_{r \text{ has type } (1),(2)} \Delta_*^r \\ \sum_{j=1}^d \lambda_{i_j}\left(\frac{1}{i_j}, \ldots, \frac{1}{i_j^{d-1}}\right) & \longmapsto & \sum_{j=1}^d \lambda_{i_j}\left(0, \ldots, 0, \frac{1}{i_j}, \ldots, \frac{1}{i_j}\right) \end{array}$$

is a homeomorphism and a diffeomorphism when restricted to the relative interior of any face of $\operatorname{bd}\left(\operatorname{conv}\{\left(\frac{1}{i}, \ldots, \frac{1}{i^{d-1}}\right) : 1 \leq i \leq n\}\right)$.

The map $\operatorname{bd} C(n, d-1) \to \operatorname{bd} \Pi_{n,d}$ in Theorem 3.1 will be the composition $\nu_{n,d} \circ \kappa_{n,d}$.

Proof. Since any facet of the cyclic polytope $\operatorname{conv}\{\left(\frac{1}{i}, \ldots, \frac{1}{i^{d-1}}\right) : 1 \leq i \leq n\}$ is the convex hull of $d-1$ points on the moment curve, these points are convexly independent. Moreover, the facet defining sets of vertices correspond to the multiplicity vectors r of type (1) and (2) by Proposition 3.8. Thus the map $\kappa_{n,d}$ is well-defined. However, the map is clearly a homeomorphism and a diffeomorphism when restricted to the relative interior of any face of $\operatorname{bd}\left(\operatorname{conv}\{\left(\frac{1}{i}, \ldots, \frac{1}{i^{d-1}}\right) : 1 \leq i \leq n\}\right)$ since it is a linear map on any facet of $C(n, d-1)$. $\square$

The following Theorem is used to show that the curved simplices $\psi_r(\Delta_*^r)$ can be arranged according to Gale's evenness condition as patches of $\operatorname{bd} \Pi_{n,d}$.

Theorem 3.10. *The Vandermonde map maps* $\nu_{n,d}^{-1}(\operatorname{bd} \Pi_{n,d}) \cap \{x \in \mathbb{R}^n_{\geq 0} : 0 \leq x_1 \leq \ldots \leq x_n\}$ *one-to-one to* $\operatorname{bd} \Pi_{n,d}$.

Thus, any point in $\operatorname{bd} \Pi_{n,d}$ has a unique preimage in $\{x \in \Delta_{n-1} : 0 \leq x_1 \leq \ldots \leq x_n\}$. Theorem 3.10 is actually an adaption of ([Kos89], Theorem 1.12.) and follows from restricting the domain of the Vandermonde map from $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq \ldots \leq x_n\}$ to its intersection with the nonnegative orthant.

Proof of Theorem 3.10. The proof follows exactly the same steps as Kostov's proof of ([Kos89], Theorem 1.12.). Since the Vandermonde map is weighted homogeneous we actually can prove the claim for the domain $\{x \in \mathbb{R}^n : 0 \leq x_1 \leq x_2 \ldots \leq x_n\}$. Instead of considering $(p_1, \ldots, p_d)(\{x \in \mathbb{R}^n_{\geq 0} : x_i \leq x_{i+1}\})$ we consider $(p_2, \ldots, p_{2d})(\{x \in \mathbb{R}^n : 0 \leq x_1 \leq x_2 \ldots \leq x_n\})$.

All statements in [Kos89] needed to prove an adapted version of ([Kos89], Theorem 1.12) are already proved in Subsection 2.1 with the exception of ([Kos89], Theorem 1.8), i.e. the image of the closure of any $k \leq d$ dimensional stratum of $\{x \in \mathbb{R}^n_{\geq 0} : 0 \leq x_1 \leq \ldots x_n\}$ under $(p_2, \ldots, p_{2d})$ is a stratified manifold and the graph of a $d-k$ dimensional vector function. However, working with even power sums restricted to $\{x \in \mathbb{R}^n : 0 \leq x_1 \leq \ldots \leq x_n\}$ instead of the Vandermonde map on $\{x \in \mathbb{R}^n : x_1 \leq \ldots \leq x_n\}$, the determinant occurring in Kostov's proof of ([Kos89], Theorem 1.8) must be replaced by $\prod_{i=1}^k x_i \prod_{1 \leq q < r \leq k}(x_q^2 - x_r^2)$ which is up to a positive scalar equal to the

determinant of $(\frac{\partial p_{2i}}{\partial x_j})_{1\le i,j\le k}$. This determinant vanishes at $a \in \{x \in \mathbb{R}^n : 0 \le x_1 \le x_2 \le \ldots \le x_n\}$ if and only if a is contained in the boundary of a stratum of $\{x \in \mathbb{R}^n_{\ge 0} : 0 \le x_1 \le \ldots \le x_n\}$.
This way we obtain an adapted version of Theorem ([Kos89], Theorem 1.8). $\square$

We are ready to give a proof of Theorem 3.1.

Proof of Theorem 3.1. We suppose $C(n,d-1) = \operatorname{conv}\{(\frac{1}{i},\ldots,\frac{1}{i^{d-1}}) : 1 \le i \le n\}$. By Corollary 3.9 the map $\kappa_{n,d} : \operatorname{bd} C(n,d-1) \to \bigcup_{r \text{ has type } (1),(2)} \Delta_*^r$ is a homeomorphism and a diffeomorphism when restricted to the relative interior of any face of $\operatorname{bd} C(n,d-1)$.
We consider the map $\nu_{n,d} \circ \kappa_{n,d} : \operatorname{bd} C(n,d-1) \to \operatorname{bd} \Pi_{n,d}$. The map $\nu_{n,d}$ is surjective by Theorem 2.4. On each simplex Δ_*^r the map ψ_r is a homeomorphism and a diffeomorphism when restricted to the relative interior of any face by Lemma 3.5. Thus, $\nu_{n,d} \circ \kappa_{n,d}$ is a diffeomorphism on the restriction of any face of $\operatorname{bd} C(n,d-1)$. The claim follows, since $\nu_{n,d}^{-1}(\operatorname{bd} \Pi_{n,d}) \to \operatorname{bd} \Pi_{n,d}$ is one-to-one by Theorem 3.10. $\square$

We conclude the section with two observations.

Remark 3.11. *Although the set $\Pi_{n,d}$ has the combinatorial structure of a cyclic polytope the natural extension κ of $\kappa_{n,d}$ to the interior of $C(n,d-1)$*

$$\begin{array}{rcl}\kappa : \operatorname{conv}\{(\frac{1}{i},\ldots,\frac{1}{i^{d-1}}) : 1 \le i \le n\} & \longrightarrow & \Pi_{n,d} \\ \sum_{j=1}^d \lambda_{i_j}\left(\frac{1}{i_j},\ldots,\frac{1}{i_j^{d-1}}\right) & \longmapsto & \nu_{n,d}\left(\sum_{j=1}^d \lambda_{i_j}\left(0,\ldots,0,\frac{1}{i_j},\ldots,\frac{1}{i_j}\right)\right)\end{array}$$

is not well defined. For instance, $\frac{1}{13}(\frac{1}{2},\frac{1}{4}) + \frac{12}{13}(\frac{1}{4},\frac{1}{16}) = \frac{27}{52}(\frac{1}{3},\frac{1}{9}) + \frac{25}{52}(\frac{1}{5},\frac{1}{25})$ *but*

$$\begin{array}{rcl}\nu_{5,3}\,(1/13\,(0,0,0,1/2,1/2) + 12/13\,(0,1/4,1/4,1/4,1/4)) & = & (1,85/338) \\ \nu_{5,3}\,(27/52\,(0,0,1/3,1/3,1/3) + 25/52\,(1/5,1/5,1/5,1/5,1/5)) & = & (1,319/1352)\,.\end{array}$$

Similarly, we obtain from ([Kos89, Theorem 1.14]) that $\operatorname{bd}(\nu_{n,d}(\mathbb{R}^n))$ is a gluing of patches

$$\nu_{n,d}(\{(\underbrace{x_1,\ldots,x_1}_{t_1},\ldots,\underbrace{x_{d-1},\ldots,x_{d-1}}_{t_{d-1}}) \in \mathbb{R}^n : x_1 \le x_2 \le \ldots \le x_{d-1}\})$$

where $t_{2i} \ge 1, t_{2i-1} = 1$ or $t_{2i} = 1, t_{2i-1} \ge 1$ for all $1 \le i \le d-1$. Moreover,

$$\bigcup_{t \text{ eligible}} \{(\underbrace{x_1,\ldots,x_1}_{t_1},\ldots,\underbrace{x_{d-1},\ldots,x_{d-1}}_{t_{d-1}}) \in \mathbb{R}^n : x_1 \le x_2 \le \ldots \le x_{d-1}\} \to \operatorname{bd} \nu_{n,d}(\mathbb{R}^n)$$

is one-to-one. Recall that the patches on the boundary of the Vandermonde cell are of type (1) or (2), while the patches on $\operatorname{bd}(\nu_{n,d}(\mathbb{R}^n))$ are all of type (2). In general there are less patches than facets of the cyclic polytope $C(n,d-1)$. For instance, the facets of $C(4,3)$ correspond to $\{1,2,3\},\{1,3,4\},\{1,2,4\},\{2,3,4\}$. But the multiplicity patterns of points on the boundary of $\mu_{4,4}(\mathbb{R}^4)$ are (x_1,x_1,x_2,x_3), (x_1,x_2,x_3,x_3) and (x_1,x_2,x_2,x_3)

4. The boundary at infinity

We show that the set $\operatorname{bd} \Pi_d$ is a gluing of countably infinitely many patches and each patch is a curved $(d-2)$-simplex. The vertices of any patch satisfy Gale's evenness condition. We begin with investigating properties of Π_d.
We write $\mathfrak{p}_k := \sum_{i\in\mathbb{N}} x_i^k$ for the power sum function in countably many variables. If $x \in \mathbb{R}^{\mathbb{N}}$ contains only finitely many non-zero coordinates we could also write $p_k(x)$ instead of $\mathfrak{p}_k(x)$ for a power sum polynomial in sufficiently many variables.

Lemma 4.1. *For $x = (x_1,\ldots,x_{d-1}) \in \Pi_d$ we have $(t^2x_1,\ldots,t^dx_{d-1}) \in \Pi_d$ for all $0 \le t \le 1$.*

Proof. Let $x = (p_2\ldots,p_d)(z)$ for a point $z \in \Delta_{n-1}$ and consider the point z':

$$z' = \left(tz, \frac{1-t}{n},\ldots,\frac{1-t}{n}\right) \in \Delta_{2n-1}.$$

We see that

$$\nu_{n,d}(z') = (t^2 x_1, \dots, t^d x_{d-1}) + \left(\frac{(1-t)^2}{n}, \dots, \frac{(1-t)^d}{n^{d-2}} \right) \in \Pi_d,$$

which implies $(t^2 x_1, \dots, t^d x_{d-1}) \in \Pi_d$ for $n \to \infty$. $\square$

Let Δ'_{n-1} be the convex hull of Δ_{n-1} and the origin, i.e. Δ'_{n-1} consists of all points with nonnegative coordinates, with the sum of coordinates at most 1.

Lemma 4.2. *The set Π_d is the closure of the limit of the sets $\nu_{n,d}(\Delta'_{n-1})$ as n goes to infinity, i.e. $\Pi_d = \mathrm{cl}\left(\bigcup_{n \geq d} \nu_{n,d}(\Delta'_{n-1})\right)$.*

Proof. Clearly, we have $\Pi_d \subset \mathrm{cl}(\bigcup_{n \geq d} \nu_{n,d}(\Delta'_{n-1}))$. If $x = (x_1, \dots, x_n) \in \Delta'_{n-1}$, then $\theta := \sum_{j=1}^n x_j \leq 1$ and thus $x^{[m]} := (x, (1-\theta)/m, \dots, (1-\theta)/m) \in \Delta_{n+m-1}$. We have

$$\Pi_d \ni \lim_{m \to \infty} \nu_{n,d}(x^{[m]}) = \nu_{n,d}(x)$$

and the remaining inclusion follows since Π_d is closed. $\square$

Lemma 4.3. *Let $d \geq 3$. Then $p \in \mathrm{bd}\,\Pi_d$ if and only if there exists a sequence (p_n) such that $p_n \in \mathrm{bd}\,\Pi_{n,d}$ and $p_n \to p$ as $n \to \infty$.*

For a set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we denote the *distance* from x to A by $\mathrm{d}(x, A)$, i.e. $\mathrm{d}(x, A) = \inf \{\|x - a\| : a \in A\}$.

Proof. First, we suppose that $p \in \mathrm{bd}\,\Pi_d$. Then, since the sets $\Pi_{n,d}$ are nested increasingly, we have $\mathrm{d}(p, \Pi_{n,d})$ is a decreasing sequence in n. However, $p \in \mathrm{bd}\,\Pi_d$ implies that $p \notin \mathrm{int}\,\Pi_{n,d}$ for all n, and thus $\mathrm{d}(p, \Pi_{n,d}) = \mathrm{d}(p, \mathrm{bd}\,\Pi_{n,d})$. Hence $\mathrm{d}(p, \mathrm{bd}\,\Pi_{n,d}) \to 0$ which implies that there exists a sequence $(p_n)_n$ with $p_n \in \mathrm{bd}\,\Pi_{n,d}$ and $p_n \to p$.

Now suppose that $p \notin \mathrm{bd}\,\Pi_d$. If $p \notin \Pi_d$, then clearly there does not exist a sequence of points in $\Pi_{n,d}$ approaching p. The only remaining case is $p \in \mathrm{int}\,\Pi_d$. Suppose that for some $\varepsilon > 0$ and some n' we have $B_\varepsilon(p) \subset \Pi_{n',d}$. Then, $\varepsilon \leq \mathrm{d}(p, \mathrm{bd}\,\Pi_n)$ for all $n \geq n'$, which shows that there cannot exist a sequence $p_n \in \mathrm{bd}\,\Pi_{n,d}$ with $p_n \to p$. Since we have $\mathrm{d}(p, \Pi_{n,d}) \to 0$, it follows that for any $\varepsilon > 0$, the ball $B_\varepsilon(p)$ contains a boundary point of $\Pi_{n,d}$ for all n sufficiently large. $\square$

Recall that $\mathrm{bd}\,\Pi_{n,d}$ is a gluing of patches, where each patch is a curved $(d-2)$-simplex whose vertices satisfy Gale's evenness condition for the set $[n]$ by Theorem 3.1. We aim to show in Theorem 4.4 that the boundary of Π_d consists of two types of patches. The first type comes from a finite subset J of $\mathbb{N}$ which satisfies Gale's evenness condition. We call such patches *stable patches*. Let m be the maximal element of J. Note that the patch corresponding to J is a boundary patch of $\Pi_{n,d}$ for all $n \geq m$, and therefore it lies on the boundary of Π_d. The second type comes from limits of patches of $\Pi_{n,d}$, and we call such patches *limit patches*.

For $n \geq d \geq 3$, we say that a set $J \subset [n]$ of size $d-1$ satisfying Gale's evenness condition contains n as an *end point*, if $J = \biguplus_{i \in \tilde{J}} \{i, i+1\} \uplus \{n\}$ or $J = \biguplus_{i \in \tilde{J}} \{i, i+1\} \uplus \{1, n\}$. Note that d must be even in the first case and odd in the second case. Let $I \subset [m]$ be a set of size $d-1$ satisfying Gale's evenness condition which contains m as an end point. Then, for $n \geq m$ we define $I_n := I \uplus \{n\} \setminus \{m\}$. The set $I_n \subset [n]$ also satisfies Gale's evenness condition and contains n as an end point. We can take the limit of the patches I_n, and we say that the resulting limit patch corresponds to the set $I_\infty \subset \mathbb{N} \cup \{\infty\}$, where $I_\infty := I \uplus \{\infty\} \setminus \{m\}$. We denote this limit patch by P_{I_∞}. We note that a point q belongs to P_{I_∞} if and only if there exists a set $I \subset [m]$ of size $d-1$ satisfying Gale's evenness condition which contains m as an end point, and a sequence $(q_n)_{n \geq m}$ such that $q_n \in P_{I_n}$ for all $n \geq m$ and $q_n \to q$ for $n \to \infty$.

More formally, we can naturally extend Gale's evenness condition to subsets of $\mathbb{N} \cup \{\infty\}$, and we see that I_∞ does indeed satisfy Gale's evenness condition. Recall that the vertices of the patch of $\Pi_{n,d}$ corresponding to $J \subset [n]$ have the form $(\frac{1}{j}, \frac{1}{j^2}, \dots, \frac{1}{j^{d-1}})$ for $j \in J$. Therefore the endpoints of I_n converge to $0 \in \mathbb{R}^d$ as n goes to ∞. For a finite finite set $J \subset \mathbb{N} \cup \{\infty\}$ we define

$$P_J := (\mathfrak{p}_2, \dots, \mathfrak{p}_d)(\operatorname{conv}\{(0, \dots, 0, \underbrace{1/j, \dots, 1/j}_{j \text{ times}}) : j \in J\})$$

where we say that $1/j = 0$ if $j = \infty$.

Theorem 4.4. *The set* $\operatorname{bd} \Pi_d$ *is the union of all stable and limit patches. It consists of curved* $d-2$*-simplices* P_I*, where the index set* $I \subset \mathbb{N} \cup \{\infty\}$ *ranges over all sets of size* $d-1$ *which satisfy Gale's evenness condition i.e. all sets* $I \subset \mathbb{N} \cup \{\infty\}$ *of size* $d-1$ *of the form*

$$I = \biguplus_i \{i, i+1\} \text{ or } I = \{1, \infty\} \uplus \biguplus_i \{i, i+1\} \text{ or } I = \{\infty\} \uplus \biguplus_i \{i, i+1\} \text{ or } I = \{1\} \uplus \biguplus_i \{i, i+1\}.$$

Proof. By Lemma 4.3 the boundary of Π_d consists of the points q for which there exists a sequence $(q_n)_n$ with $q_n \in \operatorname{bd} \Pi_{n,d}$ and $q_n \to q$. In particular, if $I \subset \mathbb{N} \cup \{\infty\}$ satisfies Gale's evenness condition, we have $P_I \subset \operatorname{bd} \Pi_d$. To prove the theorem we need to show that any limit of a converging sequence $(q_n)_n$ with $q_n \in \operatorname{bd} \Pi_{n,d}$ is contained in a patch P_I for a set $I \subset \mathbb{N} \cup \{\infty\}$ of size $d-1$ which satisfies Gale's evenness condition.

Suppose that (q_n) is a sequence with limit q, and $q_n \in P_{J_n}$ where $J_n \subset [n]$ satisfies Gale's evenness condition. We proceed by a case distinction.

(1) There exists an integer N and a subsequence of index sets (J_{n_k}) such that $J_{n_k} \subset [N]$ for all k. Then, by the pigeonhole principle (J_n) contains a constant subsequence (J). Since P_J is closed, we have $q \in P_J$ and $J \subset \mathbb{N} \cup \{\infty\}$ satisfies Gale's evenness condition.

(2) There does not exist a subsequence of bounded index sets. Then there are two options:
(a) The sequence (α_n), where α_n is the smallest element of J_n, has a subsequence which monotonously diverges to ∞.
(b) There exists an integer $K \in \mathbb{N}$ and a subsequence (J_{n_k}) of (J_n) with $|[K] \cap J_{n_k}| = m$ is equal for all k and the sequence (α_{n_k}), where α_{n_k} is the smallest element of $J_{n_k} \cap [K]^c$, monotonously diverges to ∞.
We investigate the cases 2) (a) and (b) below.
   (a) We must have $q_{n_k} \to 0$ for $k \to \infty$. Recall that $0 \in P_I$ for any set $I \subset \mathbb{N} \cup \{\infty\}$ of size $d-1$ which contains ∞. In particular, there exists a set $I \subset \mathbb{N} \cup \{\infty\}$ of size $d-1$ of the form

$$I = \{\infty\} \uplus \biguplus_i \{i, i+1\} \text{ or } I = \{1, \infty\} \uplus \biguplus_i \{i, i+1\}$$

   and $0 \in P_I \subset \operatorname{bd} \Pi_d$.
   (b) In the second case we can restrict to a subsequence (J_{n_ℓ}) of (J_n) which intersection with $[K]$ is the same set for all $n_\ell \in \mathbb{N}$. This follows from the pigeonhole principle. We claim that we can extend the set $J_{n_\ell} \cap [K]$ to a set $I \subset \mathbb{N} \cup \{\infty\}$ of size $d-1$ satisfying Gale's evenness condition and $q \in P_I$. It must be

$$J_{n_\ell} \cap [K] = \biguplus \{i, i+1\} \text{ or } J_{n_\ell} \cap [K] = \{1\} \uplus \biguplus \{i, i+1\}$$

   since J_{n_ℓ} satisfies Gale's evenness condition and $K+1 \notin J_{n_\ell}$. We have to distinguish between d even and odd.
      (i) If $d-1$ is odd and $J_{n_\ell} \cap [K] = \biguplus \{i, i+1\}$ we consider

$$I := \biguplus \{i, i+1\} \uplus \{\infty\} \uplus \biguplus \{j, j+1\}$$

      for some (possibly none) large integers j which is possible since $|J_{n_\ell} \cap [K]|$ is even and cannot satisfy Gale's evenness condition.
      (ii) If $d-1$ is even and $J_{n_\ell} \cap [K] = \biguplus \{i, i+1\}$ we know that $d-1 > |J_{n_\ell} \cap [K]|$. Let $k \leq K+1$ be minimal with $k \notin J_{n_\ell} \cap [K]$.
         (A) If $k = 1$ we set

$$I := \{1, \infty\} \uplus \biguplus \{i, i+1\} \uplus \biguplus \{j, j+1\}$$

         for some (possibly none) large integers j.

(B) Otherwise

$$I := \{1,\infty\} \uplus \{2,3\} \uplus \ldots \uplus \{k-1,k\} \uplus \biguplus_{k<i\leq K-1,\, i,i+1\in J_n} \{i,i+1\} \uplus \biguplus \{j,j+1\}$$

for some (possibly none) large integers j.

In (A) and (B) we added an even number of integers to $J_{n_\ell} \cap [K]$. This is possible since $|J_{n_\ell}|$ is even and contains at least two integers larger than K.

(iii) If $d-1$ is even and $J_{n_\ell} \cap [K] = \{1\} \biguplus \{i,i+1\}$ one proceeds analogously to (ii).

(iv) If $d-1$ is odd and $J_{n_\ell} \cap [K] = \{1\} \biguplus \{i,i+1\}$ one proceeds analogously to (i).

Finally, we point out that q is indeed contained in P_I since the limit of the sequence q_n equals $(\mathfrak{p}_2,\ldots,\mathfrak{p}_d)(y)$ for an $y \in \operatorname{conv}\{(0,\ldots,0),(0,\ldots,0,1/j,\ldots,1/j) : j \in J_{n_\ell} \cap [K]\}$.

$\square$

Remark 4.5. *Sequences of patches of* $\operatorname{bd}\Pi_{n,d}$ *indexed by sets* $I_n \subset [n]$ *satisfying Gale's evenness condition which do not contain* n *as a boundary point but contain* n*, i.e* $\{n-1,n\} \subset I$ *is a disjoint part of* I_n*, converge to lower dimensional cells in* $\operatorname{bd}\Pi_d$*. Any such lower dimensional cell is contained in a patch* P_I *of* $\operatorname{bd}\Pi_d$*.*

5. Convex hull for elementary symmetrics and test sets for copositivity

In this section we analyze the convex hulls of the sets $E_{n,d}, \Pi_{n,d}, E_d$ and Π_d. Although $E_{n,d} \simeq \Pi_{n,d}$ and $E_d \simeq \Pi_d$ are diffeomorphic, we show that $\operatorname{conv} E_{n,d}$ has nice properties which are not shared by $\operatorname{conv}\Pi_{n,d}$. We relate the study of the convex hulls to copositivty of certain symmetric forms. The vertex representation of $\operatorname{conv} E_{n,d}$ can be reformulated in terms of test sets which geometrically explains and slightly generalizes the case $d=3$ investigated by Choi, Lam and Reznick [CLR87].

We embed $\Delta_{n-1} \subset \Delta_n$ via $a \mapsto (a,0)$, and denote by $\Delta := \operatorname{cl}(\bigcup_{n\in\mathbb{N}} \Delta_n)$ the *infinite probability simplex* which can be viewed as the limit of the Δ_n's. For $n \geq d$ we write $\mathcal{E}_{n,d} := \operatorname{conv} E_{n,d}$ and $\mathcal{E}_d := \operatorname{cl}(\bigcup_{n\geq d} \mathcal{E}_{n,d})$.

The following observation about extreme points of $\mathcal{E}_{n,d}$ appeared for the first time in the context of extremal combinatorics. In the planar setting it was proven by Bollobás to give a description of the convex hull of the range of edge versus triangle densities of graphs [Bol76]. The result was extended to larger dimensions shortly afterwards and new proofs appeared for instance also in [For87; KKR12; Rie12; RS22]. The cyclic polytope observation appears to be new.

Theorem 5.1. *The set* $\mathcal{E}_{n,d}$ *is a cyclic polytope and it is the convex hull of the following finite set of points* $\left(\binom{k}{2}\frac{1}{k^2},\ldots,\binom{k}{d}\frac{1}{k^d}\right) : k \in [n]$.

We present a short proof using the following two Lemmas. The following short proof is a formalization of Bollobás's original argument, which we borrow from [Zha22] and provide for completeness.

Lemma 5.2 ([Zha22], Lemma 5.4.3)**.** *For* $n \geq d$ *a non-constant symmetric map of the form* $c_1e_1 + c_2e_2 + \ldots + c_de_d : \mathbb{R}^n \to \mathbb{R}$ *attains its extremal values on* Δ_{n-1} *at points of the form* $(0,\ldots,0,\underbrace{1/k,\ldots,1/k}_{k\text{ times}})$ *for* $1 \leq k \leq n$.

Proof. Let $n \geq d$ and $\phi(e_2,\ldots,e_d) = c_1 + c_2e_2 + \ldots + c_ne_d : \mathbb{R}^n \to \mathbb{R}$ be an affine non-constant linear map on $\mathcal{E}_{n,d}$ and let x^* be a mininizer of $\phi^* = \phi(e_2,\ldots,e_d)$ on Δ_{n-1}. We show that $x^* = (1/k,\ldots,1/k,0,\ldots,0)$ up to permutation for a $1 \leq k \leq n$. If x^* is not the vector containing only 0's and one 1 we suppose without loss generality that $x_1, x_2 > 0$ and write $\phi^*(x) = x_1A + x_2B + x_1x_2C + D$, where A,B,C,D are functions in $x_3,\ldots,x_n$. Then, since ϕ^* is symmetric we have $A = B$ and by fixing $x_1 + x_2 = x_1^* + x_2^*$ we obtain $\phi^*(x) = x_1x_2C + D'$. If $C(x^*) \geq 0$ we set either $x_1 = 0$ or $x_2 = 0$ with holding $x_1 + x_2 = x_1^* + x_2^*$ fixed and obtain that x^* was not a minimum. If $C(x^*) < 0$ we obtain $\phi^*(x^*)$ is minimized at $x_1^* = x_2^*$. Iteratively, we must have $x^* = (1/k, \dots, 1/k, 0, \dots, 0)$. □

Lemma 5.3. *For $n \geq d$, the map*

$$\begin{array}{rccc} \Phi_d : & \left\{\left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right) : k \in [n]\right\} & \longrightarrow & \left\{\left(\frac{1}{k}, \frac{1}{k^2}, \dots, \frac{1}{k^{d-1}}\right) : k \in [n]\right\} \\ & \left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right) & \longmapsto & \left(\frac{1}{k}, \frac{1}{k^2}, \dots, \frac{1}{k^{d-1}}\right) \end{array}$$

induces an affine isomorphism $\mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$.

Proof. Let $m \geq 2$ and $k \in [n]$ be integers and $z_m := (1, 2, \dots, m-1, 0, \dots, 0) \in \mathbb{R}^n$. Then by Vieta's formula we have

$$\begin{aligned} \binom{k}{m}\frac{1}{k^m} &= \frac{\prod_{i=1}^{m-1}(k-i)}{m! \cdot k^{m-1}} \\ &= \frac{1}{m! \cdot k^{m-1}}\left(k^{m-1} - e_1(z_m)k^{m-2} \pm \dots + (-1)^{m-1}(e_{m-1}(z_m))\right) \\ &= \frac{1}{m!} - \frac{1}{2(m-2)!}\frac{1}{k} + \dots + \frac{(-1)^{m-1}}{m}\frac{1}{k^{m-1}} \end{aligned}$$

which shows that for all $k \in [n]$ the same affine linear relation of the m-th coordinates of points in the sets $\left\{\left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right) : k \in [n]\right\}$ and $\left\{\left(\frac{1}{k}, \frac{1}{k^2}, \dots, \frac{1}{k^{d-1}}\right) : k \in [n]\right\}$ is satisfied. □

Proof of Theorem 5.1. By Minkowski's theorem a compact, convex set is the convex hull of its extreme points. Extreme points of $\mathcal{E}_{n,d}$ are precisely the minima of affine linear maps on $\mathcal{E}_{n,d}$. Through evaluating we obtain $(e_2, \dots, e_d)(0, \dots, 0, 1/k, \dots, 1/k) = \left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right)$ for all $1 \leq k \leq n$. It follows from Lemma 5.3 that all the claimed points are indeed vertices of $\mathcal{E}_{n,d}$, since points on the moment curve are in convex position and that $\mathcal{E}_{n,d}$ is a cyclic polytope. □

Since

$$\lim_{k\to\infty}\left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right) = \left(\frac{1}{2!}, \dots, \frac{1}{d!}\right)$$

we immediately obtain a description of the limit set $\mathcal{E}_d$. Figure 5 visualizes how the additional vertices eventually accumulate around the point $\left(\frac{1}{2!}, \frac{1}{3!}\right)$.

Proposition 5.4. $\mathcal{E}_d = \operatorname{conv}\left\{\left\{\left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right) : k \in \mathbb{N}\right\} \uplus \left\{\left(\frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{d!}\right)\right\}\right\}$.

Proof. We observe that the set $\bigcup_{n\geq d}\mathcal{E}_{n,d}$ is convex: if $v, w \in \bigcup_{n\geq d}\mathcal{E}_{n,d}$ then for some integer N we have v, w are contained in the convex set $\mathcal{E}_{n,d}$, because the sets $\mathcal{E}_{n,d}$ are nested. Thus, $\mathcal{E}_d$ is convex as the closure of the convex set $\bigcup_{n\geq d}\mathcal{E}_{n,d}$. We note

$$\left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right), \left(\frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{d!}\right) \in \mathcal{E}_d$$

per definition and since $\mathcal{E}_d$ is closed. Thus, the set on the right hand side is contained in $\mathcal{E}_d$. Moreover, we have $\mathcal{E}_{n,d} \subset \operatorname{conv}\left\{\left\{\left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right) : k \in \mathbb{N}\right\} \uplus \left\{\left(\frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{d!}\right)\right\}\right\}$ for all $n \geq d$ and thus $\operatorname{cl}\left(\bigcup_{n\geq d}\mathcal{E}_{n,d}\right) \subset \operatorname{conv}\left\{\left\{\left(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}\right) : k \in \mathbb{N}\right\} \uplus \left\{\left(\frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{d!}\right)\right\}\right\}$, since the set on the right-hand side is closed. □

We note $\left(p_i\left(0, \dots, 0, \frac{1}{k}, \dots, \frac{1}{k}\right)\right)_{2\leq i\leq d} = \left(\frac{1}{k}, \frac{1}{k^2}, \dots, \frac{1}{k^{d-1}}\right)$ and recall that also $\operatorname{bd}\Pi_{n,d}$ has isolated singularities at $(p_2, \dots, p_d)(0, \dots, 0, 1/k, \dots, 1/k)$ for all $1 \leq k \leq n$.

For $n \geq d \geq 4$ Newton's identities (1.1) provide polynomial (but not linear) transition maps between $E_{n,d}$ and $\Pi_{n,d}$. Already the power sum p_4 is quadratic in e_2. However, Lemma 5.3 shows that for any degree $d \geq 2$ there still exists an isomorphism between the isolated singularities of $E_{n,d}$ and $\Pi_{n,d}$, i.e

$$(e_2, \dots, e_d)(0, \dots, 0, 1/k, \dots, 1/k) \mapsto (p_2, \dots, p_d)(0, \dots, 0, 1/k, \dots, 1/k) \quad \text{for } 1 \leq k \leq n$$

is an isomorphism.

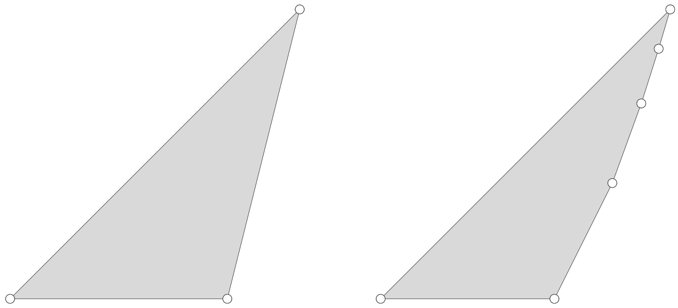

FIGURE 4. The sets $\mathcal{E}_{3,3}$ (left) and $\mathcal{E}_{6,3}$ (right)

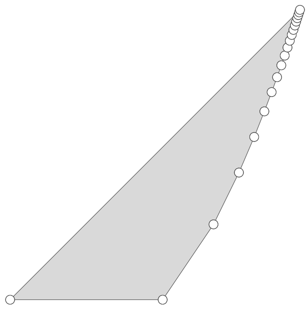

FIGURE 5. The set $\mathcal{E}_{20,3}$

We observed in Corollary 2.19 that $\Pi_{n,3} \subset \operatorname{conv}\{(\frac{1}{k}, \frac{1}{k^2}) : k \in [n]\}$. So it seems natural to ask whether an analogous result to Theorem 5.1 in terms of the power sums and the rational points on the moment curve generalizes to $d \geq 4$. We provide a negative answer.

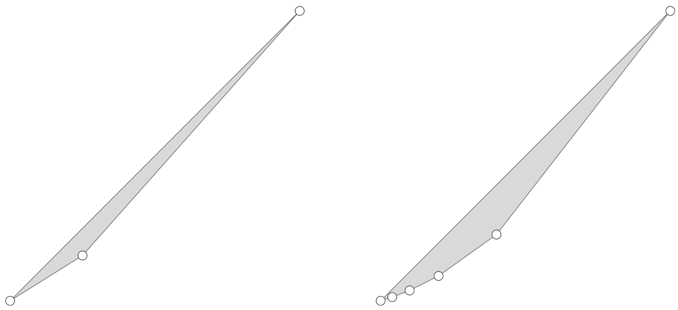

FIGURE 6. The convex polytopes $\operatorname{conv}\Pi_{n,3}$ for $n = 3$ (left) and $n = 6$ (right)

Proposition 5.5. *Let $n \geq d \geq 4$. Then the set $\operatorname{conv}\left\{\left(\frac{1}{k}, \frac{1}{k^2}, \cdots, \frac{1}{k^{d-1}}\right) : k \in [n]\right\}$ does not contain the set $\Pi_{n,d}$. Moreover, $\Pi_d \notin \operatorname{conv}\left\{(0, \dots, 0), \left(\frac{1}{k}, \frac{1}{k^2}, \cdots, \frac{1}{k^{d-1}}\right) : k \in \mathbb{N}\right\}$*

Proof. We consider $f(p_1, \dots, p_4) = 2p_4 - 3p_{(3,1)} + p_{(2,1^2)}$. For $n = m + 1$ we have

$$g_m(a) := f(p_1, \dots, p_4)(a, \underbrace{1, \dots, 1}_{\#1's=m}) = -ma^3 + a^2m^2 + a^2m + 2am^2 - 3am + m^3 - 3m^2 + 2m.$$

Thus, for fixed m we observe that the univariate polynomial $g_m(a)$ has a negative leading coefficient which shows that for sufficiently large $a > 0$ we must have $f(p_1, \dots, p_4)(a, 1, \dots, 1) < 0$. Therefore, f cannot be nonnegative on $\mathbb{R}^n_{\geq 0}$ and since f is homogeneous f cannot be nonnegative on Δ_{n-1}.

However, the form $f(1,p_2,p_3,p_4)$ is nonnegative on the rational points on the moment curve of the form $(1/k,1/k^2,1/k^3)$, since

$$f(1,1/k,1/k^2,1/k^3)=\frac{(k-3/2)^2-1/4}{k^3}\geq 0$$

for all $k\in\mathbb{N}$.

To conclude the proof we suppose

$$\Pi_{n,d}\subset\operatorname{conv}\left\{\left(\frac{1}{k},\frac{1}{k^2},\cdots,\frac{1}{k^{d-1}}\right):k\in[n]\right\}.$$

But since $f(1,p_2,p_3,p_4)$ is linear in the p_i's and $f(1,1/k,1/k^2,1/k^3)\geq 0$ we have f is nonnegative on $\operatorname{conv}\left\{\left(1,\frac{1}{k},\frac{1}{k^2},\cdots,\frac{1}{k^{d-1}}\right):k\in[n]\right\}$ and thus nonnegative on $\Pi_{n,d}$ which is a contradiction. $\square$

5.1. **Test sets for copositivity.** Choi, Lam and Reznick investigated in their paper [CLR87] nonnegative even symmetric sextics in any number of variables ≥ 3. They found finite test sets for nonnegativity. Note that any even symmetric sextic is of the form $f(p_2,p_4,p_6)=c_1p_2^3+c_2p_2p_4+c_3p_6$ for some $c_1,c_2,c_3\in\mathbb{R}$.

Theorem 5.6 ([CLR87], Theorem 3.7)**.** *Let $f(p_2,p_4,p_6)$ be an even symmetric sextic in $n\geq 3$ variables. Then f is nonnegative if and only if $f\left(1,\frac{1}{k},\frac{1}{k^2}\right)$ is nonnegative for all $k\in[n]$.*

Choi, Lam and Reznick derived their result by induction and using trigonometric functions. We want to geometrically explain and slightly expand Theorem 5.6 to a certain set of symmetric polynomials. We call a symmetric polynomial *hook-shaped* if it can be written as a linear combination of elementary symmetrics of the form $e_{(d-i,1^i)}$, i.e. $f=\sum_{i=1}^{d}c_ie_{(d-i,1^i)}$ for some scalars $c_i\in\mathbb{R}$. A hook-shaped symmetric polynomial is homogeneous and thus copositive if and only if it is nonnegative on the probability simplex Δ_{n-1}. By setting $e_1=1$ we obtain linear polynomials in $e_2,\dots,e_d$. Hence nonnegativity of $c_1+\sum_{i=2}^{d}c_ie_{d-i}$ on $E_{n,d}$ is equivalent to nonnegativity on the vertices of $\mathcal{E}_{n,d}$. Alternatively, we can consider even symmetric polynomials of the form $\sum_{i=1}^{d}c_ie_i(x^2)e_1(x^2)^{d-i}$ and present test sets for global nonnegativity.

Theorem 5.7. *Let f be a hook-shaped symmetric form in $n\geq d$ variables. Then f is copositive if and only if $f\left(1,\binom{k}{2}\frac{1}{k^2},\dots,\binom{k}{d}\frac{1}{k^d}\right)$ is nonnegative for all $k\in[n]$.*

Proof. Since $f(e_1,e_2,\dots,e_d)$ is homogeneous we can restrict to the domain Δ_{n-1} where e_1 is the constant 1 function. As $f(1,e_2,\dots,e_d)$ is linear in the remaining elementary symmetrics e_i for $2\leq i\leq d$, we observe that $f(e_1,\dots,e_d)$ is copositive if and only if $f(1,x)$ is nonnegative $\mathcal{E}_{n,d}$. In particular, $f(1,x)$ is nonnegative on $\mathcal{E}_{n,d}$ if and only if $f(1,x)$ is nonnegative on the vertices of $\mathcal{E}_{n,d}$, which are precisely the claimed points by Theorem 5.1. $\square$

Corollary 5.8. *Let $\mathfrak{f}(e_1,e_2,\dots,e_d)=\sum_{i=1}^{d}c_ie_ie_1^{d-i}$ be a symmetric form. Then $\mathfrak{f}(e_1,\dots,e_d)$ is nonnegative in any number of variables $\geq d$ if and only if $\mathfrak{f}$ is nonnegative on the discrete set $\left\{\left(1,\binom{k}{2}\frac{1}{k^2},\dots,\binom{k}{d}\frac{1}{k^d}\right):k\in\mathbb{N}\right\}$.*

Remark 5.9. *A generalization of the discrete test sets in power sums to degrees ≥ 4 cannot be given. This follows from Proposition 5.5.*

Recall that for $d\geq 3$ the sets Π_d and E_d are not semialgebraic by Corollary 2.20. Thus $\mathcal{E}_d$ cannot be semialgebraic as the convex hull of the set E_d.

5.2. **The convex body $\mathcal{E}_d$.** We show that the convex body $\mathcal{E}_d$ behaves like an infinite cyclic polytope also from the dual point of view: it has countably infinitely many facets which can be described by Gale's evenness condition.

The following results on $\operatorname{bd}\mathcal{E}_d$ can be derived completely analogously to our examination of $\operatorname{bd}\Pi_d$ in Section 4. We omit the proofs as they are even simpler, since we work with convex polytopes instead of curved simplices.

Lemma 5.10. $\operatorname{bd}\mathcal{E}_d=\{q\in\mathbb{R}^{d-1}:\forall n\geq d\ \exists q_n\in\operatorname{bd}\mathcal{E}_{n,d}\text{ with }q_n\to\infty\}$.

Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N} \cup \{\infty\})$ denote the set of all sets $I \subset \mathbb{N} \cup \{\infty\}$ of size $d-1$ satisfying Gale's evenness condition with respect to the end points 1 and ∞. Furthermore, for $I \in \mathcal{I}$ let F_I denote the convex hull of $\{(\binom{k}{2}\frac{1}{k^2}, \dots, \binom{k}{d}\frac{1}{k^d}) : k \in I\}$, where we use the limit point $(\frac{1}{2!}, \dots, \frac{1}{d!})$ if $k = \infty$.

Theorem 5.11. $\operatorname{bd} \mathcal{E}_d = \bigcup_{I \in \mathcal{I}} F_I$.

Corollary 5.12. *The convex set $\mathcal{E}_d$ contains countably infinitely many facets indexed by Gale's evenness condition.*

Proof. The sets F_I can only intersect on their boundary and not in their interior since they are cuts of hyperplanes. □

To conclude the section we briefly present a H-representation of $\mathcal{E}_{n,d}$ and show that the convex body $\mathcal{E}_d$ can be defined as the intersection of countably many halfspaces. We follow ([Zie12, Page 14]) where the H-representation of a cyclic polytope is given. For $S = \{k_1, \dots, k_{d-1}\} \subset [n]$ of size $d-1$ we define the linear map

$$\tilde{\ell}_S : \mathbb{R}^{d-1} \to \mathbb{R}, X \mapsto \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ X_1 & k_1 & \dots & k_{d-1} \\ \vdots & \vdots & & \vdots \\ X_{d-1} & k_1^{d-1} & \dots & k_{d-1}^{d-1} \end{pmatrix}.$$

By properties of the Vandermonde determinant we have $\tilde{\ell}_S(k, k^2, \dots, k^{d-1}) = 0$ if and only if $k \in S$. Thus, the kernel of $\tilde{\ell}_S$ equals $\langle (k, k^2, \dots, k^{d-1}) : k \in S \rangle_{\mathbb{R}}$ and the H-representation of $C(n, d-1)$ is given by inequalities of the form $\pm \tilde{\ell}_S(X) \le r_S$ for all facet defining sets $S \subset [n]$ and some $r_S \in \mathbb{R}$. We write ℓ_S for $\tilde{\ell}_S$ multiplied by -1 to the correct power such that the inequality reads $\ell_S(X) \le r_S$.

Proposition 5.13. *Let $n \ge d \ge 3$ be nonnegative integers and let $\mathcal{C}_{d-1}$ denote the collection of facet defining sets of $C(n, d-1)$. Then the H-representation of $\mathcal{E}_{n,d}$ is $\{\ell_S \circ \Phi_d(X) \le r_S \ : \ S \in \mathcal{C}_{d-1}\}$.*

Proof. The claim follows from the discussion above and since

$$\mathcal{E}_{n,d} = \Phi_d^{-1}(\operatorname{conv}\{(1/k, \dots, 1/k^{d-1}) : k \in [n]\}) = \Phi_d^{-1}(\{x \in \mathbb{R}^{d-1} : \ell_S(x) \le r_S, S \in \mathcal{C}_{d-1}\})$$

by Proposition 5.3. We have $\mathcal{E}_{n,d} = \{x \in \mathbb{R}^{d-1} : \ell_S \circ \Phi_d(x) \le r_S, S \in \mathcal{C}_{d-1}\}$. □

Example 5.14. *Using Sage ([Ste07]) we calculate the H-representations of $\mathcal{E}_{n,3}$ for $3 \le n \le 5$.*

$$\mathcal{E}_{3,3} = \{x \in \mathbb{R}^2 : x_2 \ge 0,\ x_1 - 9x_2 \ge 0,\ -4x_1 + 9x_2 \ge -1\},$$
$$\mathcal{E}_{4,3} = \{x \in \mathbb{R}^2 : x_2 \ge 0,\ x_1 - 6x_2 \ge 0,\ -11x_1 + 18x_3 \ge -3,\ -4x_1 + 9x_2 \ge -1\},$$
$$\mathcal{E}_{5,3} = \{x \in \mathbb{R}^2 : x_2 \ge 0,\ -4x_1 + 9x_2 \ge -1,\ -11x_1 + 18x_2 \ge -3,\ -7x_1 + 10x_2 \ge -2,\ x_1 - 5x_2 \ge 0\}$$

We recall that for integers $n > m$ and a set $I \subset [m]$ we write $I_n = I \setminus \{m\} \uplus \{n\}$.

Lemma 5.15. *Let $m \ge d \ge 3$ and $I \subset [m]$ be a facet defining set of indices of $\mathcal{E}_{m,d}$ containing m as an end point. Then, for all $n \ge m$ the inequalities $\ell_{I_n} \circ \Phi_d \le r_{I_n}$ corresponding to a facet of $\mathcal{E}_{n,d}$ converge to an inequality $\ell_{I_\infty} \circ \Phi_d \le r_{I_\infty}$ defining a facet of $\mathcal{E}_d$.*

This is to be understood in the sense that the sequence $(a_{n,1}, \dots, a_{n,d-1}, r_{I_n})$ containing the coefficients of ℓ_{I_n} and r_{I_n} converges to a limit inequality $(a_1, \dots, a_{d-1}, r_{I_\infty})$.

Proof. Since all but one of the vertices of the facets corresponding to I_n are equal, the remaining sequence of changing vertices converges to the limit vertex

$$\left(\binom{n}{2}\frac{1}{n^2}, \dots, \binom{n}{d}\frac{1}{n^d}\right) \to \left(\frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{d!}\right) \in \mathcal{E}_d,\ n \to \infty.$$

Thus, the facets corresponding to I_n in $\mathcal{E}_{n,d}$ converge to the facet indexed by I_∞ in $\mathcal{E}_d$. By continuity the defining linear inequalities must also converge which was to show. □

It follows that $\mathcal{E}_d$ can be defined as an intersection of countably infinitely many halfspaces.

Proposition 5.16. *Let $d \ge 3$, then $\mathcal{E}_d = \left\{x \in \mathbb{R}^{d-1} : \ell_I \circ \Phi_d(x) \le r_I : I \in \mathcal{I}_d\right\}$.*

6. Undecidability of nonnegativity of trace polynomials

In this Section we show that the problem of deciding nonnegativity of trace polynomials in symmetric matrices of all sizes is undecidable (see Theorem 6.2). This result stays in sharp contrast to the case of finitely many variables. Surprisingly, we then prove that the analogous problem defined with normalized traces is decidable (see Theorem 6.5). The key for the undecidability lies in the geometry of Π_3. To prove Theorem 6.2 we show that deciding copositivity of homogeneous product symmetric polynomials in any number of variables is an undecidable problem (see Theorem 6.6) which proof follows from [HN11; BRW22] on undecidability in graph homomorphism densities.

Definition 6.1. For a variable X we denote by $\operatorname{tr}(X)$ the *formal trace symbol* on X. A *trace polynomial* in the variables $X_1, \dots, X_k$ is a polynomial expression in formal trace symbols of powers of the variables $X_1, \dots, X_k$. A trace polynomial is *univariate* if $k = 1$.

For instance, $2\operatorname{tr}(X_1^4)\operatorname{tr}(X_2) - \operatorname{tr}(X_2^5)^3$ is a trace polynomial in the variables X_1, X_2, while $4\operatorname{tr}(X_1X_2^2) - 4\operatorname{tr}(X_1^3)$ is not a trace polynomial. A trace polynomial can naturally be evaluated on square matrices of all sizes. We call a trace polynomial $f(X_1, \dots, X_k)$ *nonnegative* if $f(A_1, \dots, A_k) \geq 0$ for all symmetric matrices $A_1, \dots, A_k$ of all sizes. We show that establishing nonnegativity of a trace polynomial is an undecidable problem.

Theorem 6.2. *The following decision problem is undecidable.*

Instance: *A positive integer k and a trace polynomial $f(X_1, \dots, X_k)$.*
Question: *Is $f(M_1, \dots, M_k)$ nonnegative for all real symmetric matrices $M_1, \dots, M_k$ of all sizes with $\operatorname{tr}(M_i^2) = 1$ for all $1 \leq i \leq k$?*

We now give an intuitive explanation of undecidability, and relate trace nonnegativity to the geometry of the limit Vandermonde cell. Recall that for any matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ we have $\operatorname{tr}(A^m) = \sum_{i=1}^n \lambda_i^m$. The problem of establishing trace nonnegativity is already undecidable when we consider only trace polynomials with traces in the first three even powers of symmetric matrices, i.e. we only consider even power sums $\sum \lambda_i^2, \sum \lambda_i^4$ and $\sum \lambda_i^6$ of eigenvalues. If we restirct to matrices A such that $\operatorname{tr} A^2 = 1$, then the image of all symmetric matrices of all sizes is simply the limit Vandermonde cell Π_3. The key to the hardness of the problem is the geometry of Π_3 which we investigated in Section 2. Recall that the set $\operatorname{bd}\Pi_3$ contains countably infinitely many isolated singularities on the rational moment curve (t, t^2) (see Corollary 2.19). A k-variate trace polynomial can be viewed as a polynomial on k-fold direct product $(\Pi_3)^k$. We can reduce testing nonnegativity of certain integer polynomials on $(\Pi_3)^k$ to just testing nonnegativity on products of the isolated points on the moment curve. Then deciding nonnegativity of such trace polynomials reduces to deciding nonnegativity of k-variate polynomials on $\mathbb{N}^k$ which is known to be undecidable [HN11].

Remark 6.3. *We deduce from Theorem 6.2 that there cannot exist a unified algorithm or an effective certificate to determine the validity of polynomial inequalities in traces of powers of symmetric matrices of all sizes. Note that for a finite number of variables it follows by Artin's solution to Hilbert's 17th problem [Art27] that validity of polynomial inequalities on semialgebraic sets is decidable.*

Nonnegativity of trace polynomials is investigated in the context of non-commutative real algebraic geometry. There one usually considers normalized trace polynomials. In [KPV21] the authors prove a Positivstellensatz for univariate normalized trace polynomials.

Definition 6.4. For a variable X we denote by $\widetilde{\operatorname{tr}}(X)$ the *normalized formal trace symbol* on X. A *normalized trace polynomial* in the variables $X_1, \dots, X_k$ is a polynomial expression in normalized formal trace symbols of powers of the variables $X_1, \dots, X_k$. A normalized trace polynomial is univariate if $k = 1$.

As the name normalized trace operator indicates, for a matrix $A \in \mathbb{R}^{n\times n}$ we define the evaluation $\widetilde{\mathrm{tr}}(A) := \frac{1}{n}\operatorname{tr}(A)$. A normalized trace polynomial is *nonnegative* if its evaluation on all symmetric matrices of all sizes is nonnegative.

Theorem 6.5. *The following decision problem is decidable.*

Instance: *A positive integer k and a normalized trace polynomial $f(X_1,\dots,X_k)$.*
Question: *Is $f(M_1,\dots,M_k)$ nonnegative for all symmetric matrices $M_1,\dots,M_k$ of all sizes?*

For matrices of fixed size deciding nonnegativity of normalized trace polynomials and trace polynomials is equivalent. The sharp contrast appears when we ask about nonnegativity for matrices of all sizes. Geometrically, the limit of the normalized Vandermonde map of the unit simplex Δ_{n-1} corresponds to the set of the first d moments of a probability measure supported on $\mathbb{R}_{\geq 0}$, and it is well-know that this set can be described by linear matrix inequalities [Sch17]. In particular, the limit is semilagebraic for all d. The phenomenon of decidability for normalized trace can also be explained with the half-degree principle ([Tim03], Corollary 2.1), and we follow this direction in our proof.

6.1. **Proof of Theorem 6.2.** We show that the subproblem of deciding copositivity of polynomial expressions in $p_1(X_i)$, $p_2(X_i)$ and $p_3(X_i)$ for all $1 \leq i \leq k$ on Δ_{n-1} for all n is undecidable. Recall, we can also work with the first 3 elementary symmetric polynomials on the probability simplex. Nonnegativity of a symmetric polynomial in any number of variables can also be formulated as nonnegativity of an associated symmetric function. A *symmetric function* f is a formal power series in countably infinitely many variables which is invariant under the action of the group $S_\infty = \bigcup_{n\in\mathbb{N}} S_n$ and for which the set of degrees of the monomials in f is finite (see e.g. [Mac98, §I.2] for details). The ring of symmetric functions $\mathbb{R}[x]^{\mathcal{S}_\infty} := \mathbb{R}[x_1,x_2,\dots]^{\mathcal{S}_\infty}$ can be constructed as the inverse limit of the rings of symmetric polynomials with respect to the transition maps

$$\mathbb{R}[x]^{\mathcal{S}_{n+1}} \to \mathbb{R}[x]^{\mathcal{S}_n},\ f(x_1,\dots,x_{n+1}) \mapsto f(x_1,\dots,x_n,0). \tag{6.1}$$

For $n \geq d$ the transition map implies

$$f(x_1,\dots,x_{n+1}) = g(p_1,\dots,p_d) \mapsto f(x_1,\dots,x_n,0) = g(p_1,\dots,p_d),$$

where the power sums are polynomials in a different number of variables. The analogous to elementary symmetric and power sum polynomials in $\mathbb{R}[x]^{\mathcal{S}_\infty}$ are the *elementary symmetric function* $\mathfrak{e}_k := \sum_{I\subset\mathbb{N},|I|=k}\prod_{i\in I} X_i$ and the *power sum function* $\mathfrak{p}_k := \sum_{i\in\mathbb{N}} X_i^k$.

A homogeneous symmetric polynomial $f = \sum_\alpha c_\alpha e_1^{\alpha_1}\cdots e_d^{\alpha_d}$ of degree d is nonnegative in any number of variables $n \geq d$ if and only if the symmetric function $\mathfrak{f} = \sum_\alpha c_\alpha \mathfrak{e}_1^{\alpha_1}\cdots\mathfrak{e}_d^{\alpha_d}$ is nonnegative on the infinite probability simplex Δ.

To prove Theorem 6.6 which follows from [HN11], we require access to polynomials with domain E_d^k. Therefore, we need *product symmetric functions*, i.e. symmetric functions in several groups of countably infinitely many variables which are invariant under the diagonal action of $\mathcal{S}_\infty^k$. We denote the by Δ^k the k-copies of the infinite probability simplex.

Theorem 6.6. *The following problem is undecidable.*

Instance: *A positive integer k and a product symmetric function $\mathfrak{f}$ in k groups of variables.*
Question: *Does the inequality $\mathfrak{f}(a) \geq 0$ hold for all $a \in \Delta^k$?*

We follow ([HN11, § 5]) and use their notation. Hatami and Norin's work concerns undecidability of determining the validity of linear inequalities in graph homomorphism densities for graphons and answers negative a question of Lovász ([Lov08, Problem 17]). By adapting only very few parts of Hatami and Norin's proof we show that an undecidable problem can be embedded into the problem of deciding copositivity of product symmetric homogeneous functions in $\mathfrak{e}_1,\mathfrak{e}_2,\mathfrak{e}_3$. We write $\mathfrak{e}_{j,(i)}$ for the j-th elementary symmetry functions in the i-th group of variables.

Proof of Theorem 6.6. By ([HN11, Lemma 5.1]) it follows from Matiyasevich's solution to Hilbert's tenth problem that the following validity problem is undecidable:

Instance: A positive integer k and a polynomial $p \in \mathbb{Z}[Y_1, \dots, Y_k]$.
Question: Do there exist $x_1, \dots, x_k \in \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ with $p(x_1, \dots, x_k) < 0$?

We define $C := \operatorname{conv}(2\mathfrak{e}_2, 6\mathfrak{e}_3)(\Delta)$, $g(x) := 2x^2 - x$ and the piecewise linear function

$$L(x) := \frac{3t^2 - t - 2}{t(t+1)} x - \frac{2(t-1)}{t+1}$$

on the interval $[0,1]$, where $t \in [0,1)$ is chosen such that $x \in \left[1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right]$ for some $t \in \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$, and $L(1) := 1$. By Corollary 5.4 we have $C = \operatorname{conv}\left\{(1,1), \left(1 - \frac{1}{n}, \frac{(n-1)(n-2)}{n^2}\right) : n \in \mathbb{N}\right\}$. The piecewise linear function L takes the same value as g on all the endpoints of the intervals $\left[1 - \frac{1}{t}, 1 - \frac{1}{t+1}\right]$ and we have $L(x) \geq g(x)$ for all $x \in [0,1]$. Further, we define $R := \{(x,y) \in [0,1]^2 : y \geq L(x)\}$. The images of each piecewise linear part of L on $[0,1]$ are precisely the facets of the lower part of the boundary of C.
Let $p \in \mathbb{R}[Y_1, \dots, Y_k]$ be a polynomial and let M be the sum of the absolute values of its coefficients multiplied by $100 \deg(p)$. We consider the real auxiliary polynomial

$$q(Y_1, \dots, Y_k, Z_1, \dots, Z_k) := p \prod_{i=1}^k (1 - Y_i)^6 + M \left(\sum_{i=1}^k Z_i - g(Y_i) \right).$$

Then, by ([HN11, Lemma 5.4]) and the observation $(1,1) \in R$ the following are equivalent:

(i) $q(x_1, \dots, x_k, y_1, \dots, y_k) < 0$ for some $x_1, \dots, x_k, y_1, \dots, y_k$ with $(x_i, y_i) \in R$ for all $1 \leq i \leq k$;
(ii) $p(x_1, \dots, x_k) < 0$ for some $x_1, \dots, x_k \in \{1, 1 - \frac{1}{n} : n \in \mathbb{N}\}$.

Now, we consider the map

$$\begin{aligned} \tau : \mathbb{R}[Y_1, \dots, Y_k, Z_1, \dots, Z_k] &\longrightarrow \mathbb{R}[X^k]^{S^k} \\ f(Y_1, \dots, Y_k, Z_1, \dots, Z_k) &\longmapsto \prod_{i=1}^k \mathfrak{e}_{1,(i)}^{3 \deg f} \cdot f\left(\frac{\mathfrak{e}_{2,(1)}}{\mathfrak{e}_{1,(1)}^2}, \dots, \frac{\mathfrak{e}_{2,(k)}}{\mathfrak{e}_{1,(k)}^2}, \frac{\mathfrak{e}_{3,(1)}}{\mathfrak{e}_{1,(1)}^3}, \dots, \frac{\mathfrak{e}_{3,(k)}}{\mathfrak{e}_{1,(k)}^3} \right). \end{aligned}$$

For $f \in \mathbb{R}[Y_1, \dots, Y_k, Z_1, \dots, Z_k]$ the rational function $\tau(f)$ is actually a homogeneous product symmetric function. This is, since $\mathfrak{e}_{2,(i)}$ and $\mathfrak{e}_{1,(i)}^2$ (resp. $\mathfrak{e}_{3,(i)}$ and $\mathfrak{e}_{1,(i)}^3$) have degree 2 (resp. 3) and thus every monomial in the rational product symmetric function

$$f\left(\frac{\mathfrak{e}_{2,(1)}}{\mathfrak{e}_{1,(1)}^2}, \dots, \frac{\mathfrak{e}_{2,(k)}}{\mathfrak{e}_{1,(k)}^2}, \frac{\mathfrak{e}_{3,(1)}}{\mathfrak{e}_{1,(1)}^3}, \dots, \frac{\mathfrak{e}_{3,(k)}}{\mathfrak{e}_{1,(k)}^3} \right)$$

has degree 0. Multiplying by $\mathfrak{e}_{1,(i)}^{3 \deg f}$ ensures that $\tau(f)$ has always nonnegative exponent in $\mathfrak{e}_{1,(i)}$ for all $1 \leq i \leq k$.

As in ([HN11], Claims 5.7 & 5.8) we claim that the following assertions are equivalent

(a) $q(x_1, \dots, x_k, y_1, \dots, y_k) < 0$ for some $x_1, \dots, x_k, y_1, \dots, y_k$ with $(x_i, y_i) \in R$ for all $1 \leq i \leq k$;
(b) $\tau(q)$ attains a negative value on Δ^k.

First, we suppose (a). Hatami and Norine show in the proof of ([HN11, Lemma 5.4]) that if $q(x_1, \dots, x_k, y_1, \dots, y_k) < 0$ for some $x_1, \dots, x_k, y_1, \dots, y_k$ with $(x_i, y_i) \in R$ for all $1 \leq i \leq k$ then the x_i's can be chosen as $x_1, \dots, x_k \in \{1, 1 - \frac{1}{n} : n \in \mathbb{N}\}$, and $y_i = L(x_i)$. Thus, $\tau(q)$ is negative on Δ^k by Corollary 5.4. More precisely, $\mathfrak{e}_{1,(i)} = 1, 2\mathfrak{e}_{2,(i)} = x_i$ and $6\mathfrak{e}_{3,(i)} = y_i$ for all $1 \leq i \leq k$ is feasible and thus $\tau(q)$ is not nonnegative.
Second, we suppose $q(x_1, \dots, x_k, y_1, \dots, y_k) \geq 0$ for all x_i, y_i with $(x_i, y_i) \in R$ for all $1 \leq i \leq k$, then $\tau(q)$ is nonnegative on Δ^k, since $C^k \subset R^k$.

So the assertions (ii) and (b) are equivalent. In particular, the question to determine if a given Diophantine set is non-empty was reformulated as asking whether a product symmetric polynomial in $\mathfrak{e}_{1,(i)}$, $\mathfrak{e}_{2,(i)}$ and $\mathfrak{e}_{3,(i)}$ for $1 \leq i \leq k$ is nonnegative on Δ^k. This proves the Theorem. $\square$

We are ready to prove the main theorem on undecidability of nonnegativity of trace polynomials.

Proof of Theorem 6.2. For a symmetric matrix A with trace 1 of size $n \times n$ with eigenvalues $\lambda_1, \dots, \lambda_n$ we have $\operatorname{tr}(A^k) = \sum_{i=1}^n \lambda_i^k$. We identify a subproblem which is already known to be undecidable. Thus the general problem must also be undecidable.
Consider the subproblem of determining validity of nonnegativity of homogeneous trace polynomials $f(X_1, \dots, X_k)$ in which any formal trace symbol is in an even square of a variable up to degree 6, i.e. f is a polynomial expression in $\operatorname{tr}(X_i^2), \operatorname{tr}(X_i^4), \operatorname{tr}(X_i^6)$ for $1 \le i \le k$. Then deciding nonnegativity of f for all symmetric matrices $M_1, \dots, M_k$ of all sizes is equivalent to deciding nonnegativity of the product symmetric function $f(\mathfrak{p}_{2,(1)}, \mathfrak{p}_{4,(1)}, \mathfrak{p}_{6,(1)}, \dots, \mathfrak{p}_{2,(k)}, \mathfrak{p}_{4,(k)}, \mathfrak{p}_{6,(k)})$. Its nonnegativity is equivalent to copositivity of $f(\mathfrak{p}_{1,(1)}, \mathfrak{p}_{2,(1)}, \mathfrak{p}_{3,(1)}, \dots, \mathfrak{p}_{1,(k)}, \mathfrak{p}_{2,(k)}, \mathfrak{p}_{3,(k)})$. However, this problem is undecidable by Theorem 6.6 since Newton's identities provide a linear relation between the power sums and elementary symmetrics up to degree 3 when $\mathfrak{p}_1 = 1$. □

6.2. **Proof of Theorem 6.5.** The small adjustment of using normalized traces makes the problem decidable. An important role is played by Timofte's half degree principle. The decidability was implicitly observed by Blekherman and Riener in [BR21].

Theorem 6.7 ([Tim03])**.** *A symmetric polynomial $f \in \mathbb{R}[x]^{S_n}$ is nonnegative if and only if $f(a) \ge 0$ for any $a \in \mathbb{R}^n$ with $\#\{a_1, \dots, a_n\} \le \max\{\lfloor \frac{\deg f}{2} \rfloor, 2\}$.*

We briefly illustrate why the normalized problem is decidable. Suppose we are given a power sum $p_d = x_1^d + \dots + x_n^d$ in n variables of degree $d \ge 4$. To test nonnegativity we can equivalently test nonnegativity of the $\lfloor \frac{\deg f}{2} \rfloor$-variate polynomials $(p_d)_\alpha = \alpha_1 x_1^d + \dots + \alpha_{\lfloor \frac{\deg f}{2} \rfloor} x^d_{\lfloor \frac{\deg f}{2} \rfloor}$ for all sequences $\alpha \in \mathbb{N}^{\lfloor \frac{\deg f}{2} \rfloor}$ with $\sum_{i=1}^{\lfloor \frac{\deg f}{2} \rfloor} \alpha_i = n$ by Theorem 6.7. Thus, testing nonnegativity of normalized $\frac{p_d}{n}$ power sums is equivalent to testing nonnegativity of $(\frac{p_d}{n})_\alpha = \frac{\alpha_1}{n} x_1^d + \dots + \frac{\alpha_{\lfloor \frac{\deg f}{2} \rfloor}}{n} x^d_{\lfloor \frac{\deg f}{2} \rfloor}$ for all α's. We observe that nonnegativity of $\frac{p_d}{n}$ for all n is equivalent to nonnegativity of $\beta_1 x_1^d + \dots + \beta_{\lfloor \frac{\deg f}{2} \rfloor} x^d_{\lfloor \frac{\deg f}{2} \rfloor}$ for all $(\beta_1, \dots, \beta_{\frac{d}{2}}) \in \Delta_{\lfloor \frac{\deg f}{2} \rfloor - 1} \times \mathbb{R}^d$ due to the density of $\mathbb{Q}$ in $\mathbb{R}$.

Definition 6.8. Let $\mathfrak{f} = (\sum_\lambda c_\lambda \frac{p_{\lambda_1} \cdots p_{\lambda_l}}{n^l})_{n \in \mathbb{N}}$ be a sequence of symmetric polynomials of degree $2d$ where $\mathfrak{f}_n$, the n-th element in the sequence, is a polynomial in n variables. We define the associated $2d$-variate function $\Phi_\mathfrak{f}$ as

$$\Phi_\mathfrak{f}(s, t) = \sum_\lambda c_\lambda \prod_{i=1}^{l} (s_1 t_1^{\lambda_i} + \dots + s_d t_d^{\lambda_i})$$

The following Lemma generalizes the application of Timofte's half degree principle from the discussion above to arbitrary normalized symmetric polynomials.

Lemma 6.9 ([BR21] Theorem 3.4)**.** $\mathfrak{f} = (\sum_\lambda c_\lambda \frac{p_{\lambda_1} \cdots p_{\lambda_l}}{n^l})_{n \in \mathbb{N}}$ *be a sequence of symmetric polynomials of degree $2d$ where $\mathfrak{f}_n$ is a polynomial in n variables. Then $\mathfrak{f}_n$ is nonnegative for all $n \in \mathbb{N}$ if and only if $\Phi_\mathfrak{f}$ is nonnegative on $\Delta_{d-1} \times \mathbb{R}^d$.*

We are ready to prove Theorem 6.5.

Proof of Theorem 6.5. We note, for a symmetric matrix $M \in \mathbb{R}^{n \times n}$ and eigenvalues $\lambda_1, \dots, \lambda_n$ we have

$$\widetilde{\operatorname{tr}}\left(M^k\right) = \frac{1}{n} \operatorname{tr}(M^k) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i^k = \frac{1}{n} p_k(\lambda).$$

Thus verifying nonnegativity of a univariate normalized trace polynomial is equivalent to verifying nonnegativity of the associated sequence of normalized symmetric polynomials in any number of variables. By Lemma 6.9 this is equivalent to nonnegativity of a polynomial on the semialgebraic set $\Delta_{d-1} \times \mathbb{R}^d$ and thus decidable.

For a multivariate normalized trace polynomial we proceed analogously and have that nonnegativity of a normalized trace polynomial in k variables is equivalent to nonnegativity of an associated polynomial on the semialgebraic set $(\Delta_{d-1} \times \mathbb{R}^d)^k$. $\square$

7. Conclusion and open questions

In this article we have studied the wonderful geometry of the Vandermonde map in the finite and infinite setup. In particular, we have shown how a connection to trace polynomials allows to show that the problem to determine if a given multivariate trace polynomial is nonnegative is undecidable. Our proof inspired by Hatami-Norine's proof [HN11] relied on Matiyasevich work on Hilbert's tenth problem [Mat70] which showed that it is not possible to computationally decide if a Diophantine equation in several variables has an integer solution. In this context it is worth noticing that asserting that a given univariate polynomial has a root in the integers is a decidable task. Our construction used to prove Theorem 6.2 does not apply if we restrict to univariate trace polynomials and therefor it remains a natural question whether verification of nonnegativity of univariate trace polynomials is decidable.

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LINEAR SLICES OF HYPERBOLIC POLYNOMIALS AND POSITIVITY OF SYMMETRIC POLYNOMIAL FUNCTIONS

CORDIAN RIENER AND ROBIN SCHABERT

ABSTRACT. A real univariate polynomial of degree n is called hyperbolic if all of its n roots are on the real line. Such polynomials appear quite naturally in different applications, for example, in combinatorics and optimization. The focus of this article are families of hyperbolic polynomials which are determined through k linear conditions on the coefficients. The coefficients corresponding to such a family of hyperbolic polynomials form a semi-algebraic set which we call a *hyperbolic slice*. We initiate here the study of the geometry of these objects in more detail. The set of hyperbolic polynomials is naturally stratified with respect to the multiplicities of the real zeros and this stratification induces also a stratification on the hyperbolic slices. Our main focus here is on the *local extreme points* of hyperbolic slices, i.e., the local extreme points of linear functionals, and we show that these correspond precisely to those hyperbolic polynomials in the hyperbolic slice which have at most k distinct roots and we can show that generically the convex hull of such a family is a polyhedron. Building on these results, we give consequences of our results to the study of symmetric real varieties and symmetric semi-algebraic sets. Here, we show that sets defined by symmetric polynomials which can be expressed sparsely in terms of elementary symmetric polynomials can be sampled on points with few distinct coordinates. This in turn allows for algorithmic simplifications, for example,to verify that such polynomials are non-negative or that a semi-algebraic set defined by such polynomials is empty.

1. INTRODUCTION

A monic real univariate polynomial f which has only real roots is classically called a hyperbolic polynomial. Such polynomials and their multivariate relatives appear naturally in various mathematical contexts from differential equations to combinatorics, real algebraic geometry and optimization (see for example [16, 15, 23, 6]). By identifying monic polynomials of degree n with the list of coefficients, one can describe hyperbolic polynomials of degree n as a semi-algebraic subset of $\mathbb{R}^n$. We consider linear slices, i.e., intersections with linear subspaces, of this semi-algebraic set, which is in fact the closure of one connected component of the complement of the discriminant variety. The study of these *hyperbolic slices* is inspired by the works of Arnold who considered families of hyperbolic polynomials where the first k coefficients were fixed. Arnold [2] and Givental [14] showed that these sets are topologically contractible (see also [25, 24]) and have a rich geometric structure as was shown by Kostov [18] (see also [20, 19] for more related results). In a similar spirit to the works of Arnold and Meguerditchian we study the local extreme points of these sets (see Definition 2.5). In analogy to their result, we show in Theorem 2.8 that these points correspond to hyperbolic polynomials with few distinct roots. Furthermore, we show in Theorem 2.14 that a generic hyperbolic slice only has finitely many local extreme points. This signifies in particular that the convex hull of each of its connected components is in fact a polyhedron. In contrast to the case considered by Arnold, our slices are in general not contractible and not compact. However, we are able to give some sufficient condition to decide if a hyperbolic slice is compact or has at least a local extreme point.

One of our main interests for the study of these hyperbolic slices stems from an application to symmetric real polynomial functions, i.e., polynomial functions that are left invariant by any permutation of the variables. Real symmetric functions are related to hyperbolic polynomials via the so called *Vieta map*: Recall that for $1 \leq i \leq n$ the i-th elementary

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symmetric polynomial in n variables is defined by

$$e_i := \sum_{1 \le j_1 < j_2 < \dots < j_i \le n} X_{j_1} \cdots X_{j_i}.$$

By Vieta's formula the coefficients of a univariate monic polynomial of degree n are given by evaluating these elementary symmetric polynomials at the corresponding roots. Conversely, it is also classical that the roots depend continuously on the coefficients and the natural action of S_n permuting the roots does not effect the coefficients. Therefore, the polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by the above connection effectuates a homeomorphism from $\mathbb{R}^n/S_n$ to its image called the *Vieta map*. Since it is classically known that every symmetric polynomial can be uniquely written as a polynomial in the elementary symmetric polynomials one can view real symmetric polynomial functions as functions on the image of the Vieta map. This connection between univariate monic polynomials and symmetric polynomials in n variables gives rise to an application of our results on hyperbolic slices in the context of symmetric polynomial functions: We are interested in the question to what extend the global behavior of symmetric functions is determined by its behavior of symmetrical points or points with a large stabilizer. For example, several authors (e.g. [17, 36]) have studied families of symmetric polynomials which attain their minimal values on symmetric points, i.e., points where all coordinates are equal. More generally, it has been shown that symmetric polynomial functions of a given degree $2d$ assume only non-negative values if and only if they have this property on point with at most d distinct coordinates [34, 30]. To further this line of ideas, we introduce the notion of k-complete symmetric polynomial functions. Those are polynomial functions whose set of values is already obtained by evaluation only on points which have at most k distinct coordinates (see Definition 3.1). Using the geometry of hyperbolic slices we are able to identify a new class of k-complete functions in Theorem 3.8 which is given by functions that are constant or linear along a hyperbolic slice (see Definition 3.5 for the technical definition). The results we give here also include the mentioned findings of [34, 30] which can be interpreted by saying that every symmetric polynomial of degree $d \ge 4$ is $\lfloor \frac{d}{2} \rfloor$-complete. The class of k-complete symmetric functions allows for significant algorithmic simplifications in several algorithmic tasks related to polynomial functions. For example, it is known (see [28]) that checking if a real multivariate polynomial f is non-negative is in general NP-hard, already in the case of polynomials of degree 4. However, as we discuss in this article, the complexity of verifying non-negativity for a k-complete symmetric polynomial can be drastically reduced if $k < n$, since the set of points that need to be considered is of dimension k. We highlight this and several related results in the second part of the article.

Outline: In Section 2 we introduce the notion of hyperbolic slices as families of hyperbolic polynomials defined by linear conditions on the coefficients. Our main result in this section is that the local extreme points of such slices correspond to hyperbolic polynomials with few distinct roots (Theorem 2.8) and that generically there are only finitely many such local extreme points (Theorem 2.14). Finally, we give sufficient criteria for the existence of such local extreme points in the cases when a slice is not compact. In Section 3 we study symmetric polynomials which attain their minima on points with few distinct coordinates, i.e., on points with a non trivial and potentially large stabilizer. Our main results there (Theorem 3.8 and Corollary 3.10) provide a large class of such functions based on the results from Section 2. We furthermore highlight how to efficiently verify that a given symmetric polynomial satisfies the conditions needed to apply these results. The following Section 4 highlights the applicability of our results. We show that our findings allow for simple proofs for different symmetric inequalities and also recover the mentioned known results. Furthermore, we in particular highlight in Theorem 4.6 a family of symmetric polynomials which attain their minimum on symmetric points. Finally, we close with some concluding remarks and outlooks in Section 5.

Notation: Throughout the article, we fix $n \in \mathbb{N}$ and denote by $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$ the polynomial ring in n variables over $\mathbb{R}$.

2. Hyperbolic slices

In this section we define and analyze the notion of a hyperbolic slice. To begin we formalize the notion of hyperbolic polynomials as used in the article.

Definition 2.1. *We will denote by*

$$\mathcal{H} := \left\{ z \in \mathbb{R}^n \;\middle|\; T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n \text{ only has real roots} \right\}$$

*the set of hyperbolic polynomials of degree at most n, and for $1 \le m \le n$ the m-*boundary *of $\mathcal{H}$*

$$\mathcal{H}^m := \left\{ z \in \mathcal{H} \;\middle|\; T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n \text{ has at most } m \text{ distinct roots} \right\}.$$

As described above we are interested in families of univariate monic hyperbolic polynomials whose coefficients are restricted by linear conditions. In order to define this more concretely, we fix throughout this section an integer $1 \le k \le n$, a real point $a \in \mathbb{R}^k$, and a surjective linear map $L: \mathbb{R}^n \longrightarrow \mathbb{R}^k$. This choice of a linear map and a point characterizes the linear conditions we aim to impose on hyperbolic polynomials and the *hyperbolic slices* corresponding to these choices can be defined as follows.

Definition 2.2. *With the notation introduced above, the* hyperbolic slice *associated to L and a is the affine linear slice*

$$\mathcal{H}_L(a) := \mathcal{H} \cap L^{-1}(a).$$

Furthermore, for $1 \le m \le n$ we define by

$$\mathcal{H}_L^m(a) := \mathcal{H}^m \cap L^{-1}(a),$$

its restriction to the m-boundary.

We briefly discuss one possible connection of the above definition to polynomial interpolation for which our results might be interesting in their own rights: For $k \in \mathbb{N}$ consider $a_1, b_1, \dots, a_k, b_k \in \mathbb{R}$. Then the space of polynomials f of degree n which satisfy $f(a_i) = b_i$ for $1 \le i \le k$ is called a polynomial interpolation space. Now, since evaluations at given points define linear maps, an interpolation problem for which one is interested in hyperbolic polynomials only constitutes one example of a hyperbolic slice defined above.

Clearly, the assumption that L is surjective is only for convenience in the notation. As mentioned above the set of hyperbolic polynomials is tightly connected to the Vieta map.

Remark 2.3. *The set $\mathcal{H}$ of hyperbolic polynomials is the image of the so-called Vieta map*

$$\Gamma: \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathcal{H} \\ x = (x_1, \dots, x_n) & \longmapsto & (e_1(x), \dots, e_n(x)) \end{array},$$

and the restriction of Γ to the polyhedral cone

$$\mathcal{W} := \{ x \in \mathbb{R}^n \,|\, x_1 \le x_2 \le \ldots \le x_n \}$$

is a homeomorphism. In particular, the roots of a univariate polynomial depend continuously on its coefficients. $\mathcal{H}$ is in fact a basic closed semi-algebraic subset of $\mathbb{R}^n$. Clearly, $\mathcal{H} = \mathcal{H}^n \supset \mathcal{H}^{n-1} \supseteq \cdots \supseteq \mathcal{H}^1$ and $\mathcal{H}^{n-1}$ is the topological boundary of $\mathcal{H}$. Furthermore, for $1 \le m \le n$ the m-boundary $\mathcal{H}^m$ is the image of the union of the m-faces of $\mathcal{W}$ under Γ and therefore of dimension m. For more details, we refer to [37, Appendix V.4].

The next example shows one of the simplest situations of a hyperbolic slice obtained by fixing the first two coefficients of a monic polynomial of degree 4.

Example 2.4. *For $k \ge 2$ we can fix the first k coefficients of a monic polynomial. The set of hyperbolic polynomials in such a family defines a hyperbolic slice and this setup corresponds to the situation studied by Arnold* [2] *and Kostov* [18]*. For example, we can consider $\mathcal{H}_L(0, -6)$, where*

$$L: \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ (z_1, z_2, z_3, z_4) & \longmapsto & (z_1, z_2) \end{array}.$$

This choice yields the hyperbolic slice in the plane shown in Figure 1.

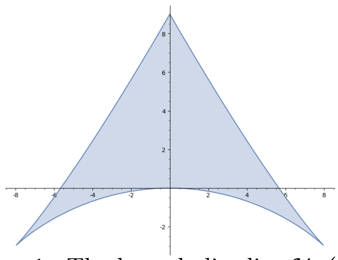


FIGURE 1. The hyperbolic slice $\mathcal{H}_L(0,-6)$

As can be seen from the example above, a hyperbolic slice is not convex but bears some resemblance to a polytope. By the connection via the Vieta map, we have that $\mathcal{H}$ is homeomorphic to the polyhedral cone $\mathcal{W}$. Furthermore, one finds three extreme points/ vertices in the above picture. For convex sets in $\mathbb{R}^n$ the extreme points contain important information about the set. To generalize this notion to the sets defined above, we will be interested in the following local notion of extreme points.

Definition 2.5. *Let $A \subseteq \mathbb{R}^n$. We call $z \in A$ a* local extreme point *of A, if there is a neighborhood $U \subseteq \mathbb{R}^n$ of z such that z is an extreme point of* $\operatorname{conv}(A \cap U)$. *We denote the set of all local extreme points of A by* $\operatorname{locextr}(A)$.

Classically, in convex optimization, the interest in extreme points stems from the fact that linear functions attain their minimum or maximum on these points. Similarly, the following holds for local extreme points.

Remark 2.6. *Let $A \subseteq \mathbb{R}^n$, and $\varphi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ and $z_\varphi \in A$ a (strict) local minimal point of φ in A. Then z_φ is also a local extreme point of A. Conversely, let $z \in A$ be a local extreme point of A, then there is $\varphi_z \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ such that z is a local minimal point of φ in A.*

Example 2.7. *We more generally examine the local extreme points of the hyperbolic slices discussed above which are similar to the one in Figure 1. We consider again the linear map*

$$L: \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ (z_1, z_2, z_3, z_4) & \longmapsto & (z_1, z_2) \end{array},$$

and we examine local extreme points of the family of slices $\mathcal{H}_L(0,a)$, with $a \in \mathbb{R}$. Then we find that the local extreme points in this case are

$$\operatorname{locextr}(\mathcal{H}_L(0,a)) = \mathcal{H}_L^2(0,a) = \left\{ \left(0, a, \pm\left(\sqrt{-\frac{2a}{3}}\right)^3, -\frac{a^2}{12}\right), \left(0, a, 0, \frac{a}{2}\right) \right\}.$$

By examining the resultants of the corresponding quartic polynomials and their second derivative, one finds that each of these local extreme points corresponds to hyperbolic polynomials with at most two distinct roots.

As a first result, we are now going to establish that the above example generalizes in the following sense. For a general hyperbolic slice, defined through k linear conditions, the local extreme points can be characterized as hyperbolic polynomials of the k-boundary. This generalizes Theorem [30, Theorem 4.2] to general hyperbolic slices.

Theorem 2.8. *The local extreme points of a hyperbolic slice are contained in the k-boundary, i.e.,*

$$\operatorname{locextr}(\mathcal{H}_L(a)) \subseteq \mathcal{H}_L^k(a).$$

Proof. Let $z \in \mathcal{H}_L(a)$ be a local extreme point, i.e., there is a neighborhood U of z such that z is an extreme point of $\operatorname{conv}(\mathcal{H}_L(a) \cap U)$. We assume that $z \notin \mathcal{H}_L^k(a)$ and want to find a contradiction. To this end, we want to find $c \in \ker L$ non-zero such that $z \pm \varepsilon c \in \mathcal{H}_L(a)$ for all $\varepsilon > 0$ small enough. Consider $f := T^n - z_1 T^{n-1} + \dots + (-1)^n z_n$ with distinct roots

$x_1, \ldots, x_m$ where $m > k$ and factor as follows:

$$f = \underbrace{\prod_{i=1}^{m}(T - x_i)}_{=:p} \cdot q,$$

where the set of zeros of q contains only elements from $\{x_1, \ldots, x_m\}$ and q is of degree $n - m$. Write $q = T^{n-m} + q_1 T^{n-m-1} + \cdots + q_{n-m}$ and define $q_0 := 1$ and consider the linear map

$$\begin{array}{rccc} \chi : & \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ & y & \longmapsto & \left(\sum_{i+j=1} q_i y_j, \ldots, \sum_{i+j=n} q_i y_j\right) \end{array}.$$

Since $m > k$, there is $b \in \ker(L \circ \chi) \setminus \{0\}$. We define $h := b_1 T^{m-1} + \cdots + b_m$ and $g := h \cdot q = c_1 T^{n-1} + \ldots + c_n \neq 0$, where $c = \chi(b)$ by construction and therefore $c \in \ker L$. Now, because p has no multiple roots, $p \pm \varepsilon h$ is hyperbolic for $\varepsilon > 0$ small enough: the roots depend continuously on the coefficients and complex roots come as conjugated pairs (see Remark 2.3). Hence

$$(p \pm \varepsilon h) \cdot q = f \pm \varepsilon h \cdot q = f \pm \varepsilon g$$

is hyperbolic for all $\varepsilon > 0$ small enough, i.e., $z \pm \varepsilon c \in \mathcal{H}_L(a)$. If we choose $\varepsilon > 0$ small enough we can ensure also that $z \pm \varepsilon c \in U$. But then

$$z = \frac{z + \varepsilon c + z - \varepsilon c}{2},$$

a contradiction to z being an extreme point of $\operatorname{conv}(\mathcal{H}_L(a) \cap U)$. $\square$

Remark 2.9. *If the map L is not surjective, one can obtain similar results by replacing k with* $\operatorname{rank} L$.

In view of Remark 2.6 we get the following.

Corollary 2.10. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a linear or concave function and consider the optimization problem*

$$\min_{z \in \mathcal{H}_L(a)} g(z).$$

Let M denote the set of minimizers of this problem. If $\mathcal{H}_L(a)$ is non-empty and compact, then we have $M \cap H_L^k(a) \neq \varnothing$. In particular $H_L(a)$ contains a point $z \in \mathcal{H}_L^k(a)$.

Proof. Since $\mathcal{H}_L(a)$ is compact, there is a minimizer $z \in M$ such that z is an extreme point of the convex hull of $\mathcal{H}_L(a)$. In particular, z is a local extreme point of $\mathcal{H}_L(a)$ and therefore on the k-boundary of $\mathcal{H}_L(a)$ by Theorem 2.8, i.e., $z \in M \cap \mathcal{H}_L^k(a)$. $\square$

As can be observed in the example shown in Figure 1 connected components of hyperbolic slices appear to have a similarity to polytopes. They are not convex, but appear to be "deflated" polytopes. To make this a bit more concrete we show that a generic hyperbolic slice has only finitely many local extreme points. This in particular implies that their convex hull, or in fact the convex hull of each of its connected components, is a polytope. The proof uses elementary properties of subdiscriminants. The relevance of subdiscriminants for counting roots of real univariate polynomials is explained in [3, Chapter 4].

Definition 2.11. *Let $f \in \mathbb{R}[T]$ be a monic polynomial of degree n with roots $x_1, \ldots, x_n$ in $\mathbb{C}$. Then the* $(n-m)$-subdiscriminant, *$1 \le m \le n$, of f is defined as*

$$\operatorname{sDisc}_{n-m}(f) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = m}} \prod_{\substack{i,j \in I \\ j > i}} (x_i - x_j)^2.$$

Remark 2.12. *Each $(n-m)$-subdiscriminant of f is defined above as a polynomial of degree $m(m-1)$ in terms of the roots of f. Noticing that each of the expressions is in turn symmetric in the roots, one immediately obtains that each subdiscriminant of f can be expressed in the elementary symmetric polynomials evaluated at the roots, i.e., in the coefficients of f. Indeed, the subdiscriminants of f can be obtained directly by minors of the Sylvester matrix - also called subresultants - of f and f'. So the degree of each $(n-m)$-subdiscriminant expressed in the coefficients is $2m-2$* [3, Proposition 4.27].

Proposition 2.13. [3, Remark 4.6 and Proposition 4.50] *A monic polynomial $f \in \mathbb{R}[T]$ of degree n has exactly k distinct roots if and only if*

$$\mathrm{sDisc}_0(f) = \cdots = \mathrm{sDisc}_{n-k-1}(f) = 0,\ \ \mathrm{sDisc}_{n-k}(f) \neq 0.$$

Moreover, if and only if additionally

$$\mathrm{sDisc}_{n-k}(f) > 0, \ldots, \mathrm{sDisc}_{n-1}(f) > 0,$$

then f has only real roots.

Theorem 2.14. *The k-boundary $\mathcal{H}_L^k(a)$ of a generic hyperbolic slice is finite. In particular, a generic hyperbolic slice has only finitely many local extreme points. The number of those points is bounded by*

$$\min\left\{2^{n-k}\frac{(n-1)!}{(k-1)!}, \binom{n}{k}\frac{(n-1)!}{(k-1)!}\right\}.$$

Proof. First, we establish that for a generic hyperbolic slice the k-boundary $\mathcal{H}_L^k(a)$ is finite. For this recall that the set of hyperbolic polynomials with at most k distinct roots, $\mathcal{H}^k$, is of dimension k by Remark 2.3. Therefore, a generic $(n-k)$-dimensional affine linear subspace will intersect $\mathcal{H}^k$ in only finitely many points. Furthermore, in view of Proposition 2.13 we see further that $\mathcal{H}^k$ is contained in the algebraic set defined by the vanishing of $n-k$ polynomials. On the one hand, each of the subdiscriminants describing this algebraic set is a homogeneous polynomial of degree $(2n-2), (2n-4), \ldots, (2k)$ expressed in the elementary symmetric polynomials by Remark 2.12 and we can apply Bézout's Theorem to obtain the bound

$$2^{n-k}\frac{(n-1)!}{(k-1)!}.$$

On the other hand, we can apply the weighted Bézout's Theorem (see [27, chapter VIII]): We assign to the i-th elementary symmetric polynomial e_i the weight i. Then each subdiscriminant is weighted homogeneous of degree $n(n-1), (n-1)(n-2), \ldots, (k+1)k$. Indeed, this is exactly the degree of the subdiscriminants expressed in the roots. Furthermore, we can bound the weighted degree of each of the k affine hyperplanes describing our slice by $n, n-1, \ldots, n-k+1$. So we obtain the bound

$$\frac{1}{n!}\frac{n!}{(n-k)!} \cdot \frac{n!(n-1)!}{k!(k-1)!} = \binom{n}{k}\frac{(n-1)!}{(k-1)!}.$$

□

Remark 2.15. *The second bound obtained in 2.14 by the weighted Bézout's Theorem can even be refined, when one considers the coefficients appearing in $L(z)$ for $z \in \mathcal{H}$. For example, if just the first coefficients are fixed, i.e., $L(z) = (z_1, \ldots, z_k)$, then $\binom{n}{k}$ can be replaced by* 1.

Since the extreme points of the convex hull of a set are local extreme points, we can deduce the following.

Corollary 2.16. *The convex hull of a generic hyperbolic slice is a polyhedron. The same applies to any of its connected components.*

Note that the proof of Theorem 2.14 together with Proposition 2.13 gives an explicit description of the k-boundary of a hyperbolic slice as a semi-algebraic set. The following example shows that the k-boundary of a hyperbolic slice can be infinite. But even in this case, there might only be finitely many local extreme points.

Example 2.17. *Consider $L\colon\ \mathbb{R}^4 \to \mathbb{R}^3,\ (z_1, z_2, z_3, z_4) \mapsto (z_1, z_3, z_4)$ and $a \in \mathbb{R}$. Then*

$$\mathcal{H}_L(a,0,0) = \left\{(a, z_2, 0, 0) \;\middle|\; z_2 \in \mathbb{R},\ z_2 \leq \frac{a^2}{4}\right\} = \mathcal{H}_L^3(a,0,0)$$

is not finite. But $\mathcal{H}_L(a,0,0)$ is obviously convex with only local extreme point

$$\left(a, \frac{a^2}{4}, 0, 0\right) \in \mathcal{H}_L^2(a,0,0).$$

Next, we will give sufficient conditions on L for the compactness of a hyperbolic slice and for the existence of local extreme points. For that, we will need the following definition.

Definition 2.18. *Let $f, g \in \mathbb{R}[T]$ be hyperbolic polynomials with real roots $\alpha_n \le \cdots \le \alpha_1$ and $\beta_m \le \ldots \le \beta_1$ respectively. We say that g* interlaces *f if $\alpha_n \le \beta_m \le \alpha_{n-1} \le \ldots \le \alpha_1$ or $\beta_m \le \alpha_n \le \beta_{m-1} \le \ldots \le \alpha_1$. Furthermore, we say f and g are* interlacing, *if f interlaces g or g interlaces f.*

Remark 2.19. *If g interlaces f, then clearly f and g either have the same degree, i.e., $n = m$ or the degree of g is smaller by one, i.e., $m = n - 1$.*

The following classical result (see [10, Theorem 4.1.]) connects interlacing polynomials to linear pencils of hyperbolic polynomials.

Theorem 2.20 (Dedieu)**.** *Let $f, g \in \mathbb{R}[T]$ be hyperbolic, non-zero polynomials of degree at most n. Then the following statements are equivalent:*
(1) f and g are interlacing.
(2) $f + \xi \cdot g$ is hyperbolic for any $\xi \in \mathbb{R}$.

From now on we express L in terms of k linearly independent linear forms $l_1, \ldots, l_k \in \mathbb{R}[Z_1, \ldots, Z_n]_1$ as $L : \mathbb{R}^n \to \mathbb{R}^k$, $z \mapsto (l_1(z), \ldots, l_k(z))$. We can use the results above to give a sufficient condition on $l_1, \ldots, l_k$ for the existence of local extreme points of a hyperbolic slice.

Lemma 2.21. *If $Z_1 \in \operatorname{span}(l_1, \ldots, l_k)$ and $\mathcal{H}_L(a) \neq \varnothing$, then $\mathcal{H}_L(a)$ has a local extreme point.*

Proof. Let $z \in \mathcal{H}_L(a)$ and write $Z_1 = \sum_{i=1}^k \lambda_i l_i$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Furthermore, denote by $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the roots of

$$f_z := T^n - z_1 T^{n-1} + \cdots + (-1)^n z_n.$$

Then $e_1(x) = z_1 = \sum_{i=1}^k \lambda_i l_i(z) = \sum_{i=1}^k \lambda_i a_i$ and hence

$$z_2 = e_2(x) = \frac{1}{2}\left(e_1(x)^2 - \sum_{i=1}^n x_i^2\right) \le \frac{1}{2} e_1(x)^2 = \frac{1}{2}\left(\sum_{i=1}^k \lambda_i a_i\right)^2.$$

So the optimization problem

$$\max_{z \in \mathcal{H}_L(a)} z_2$$

has a non-empty set of maximizers M. Suppose $\mathcal{H}_L(a)$ has no local extreme point. Then M contains a line, i.e., there is a maximizer $m = (m_1, \ldots, m_n) \in M$ and a $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ non-zero such that $y_1 = y_2 = 0$ and $m + \xi y \in \mathcal{H}$ for all $\xi \in \mathbb{R}$. This means $f := T^n - m_1 T^{n-1} + \cdots + (-1)^n m_n$ and $g := -y_3 T^{n-3} + \cdots + (-1)^n y_n$ are interlacing by 2.20, which is not possible because of degree reasons. ☐

We can use the existence of an extreme point, for example, to obtain the following result which connects to polynomial interpolation.

Corollary 2.22. *Consider the set of polynomials of degree n, which are monic, have the second coefficient fixed, and solve a k-points interpolation problem. Then there exists a hyperbolic polynomial in this set if and only if there exists one with at most k distinct roots.*

Proof. Under the conditions, the corresponding hyperbolic slice has at least one extreme point by Lemma 2.21. ☐

By prescribing not only the first but also the second-highest coefficient of a monic polynomial, one directly obtains a sufficient condition for the compactness of a hyperbolic slice.

Lemma 2.23. *If $Z_1, Z_2 \in \operatorname{span}(l_1, \ldots, l_k)$, then $\mathcal{H}_L(a)$ is compact.*

Proof. As the empty set is compact we can assume that there is $z \in \mathcal{H}_L(a)$. Furthermore we write $Z_1 = \sum_{i=1}^k \lambda_i l_i$ and $Z_2 = \sum_{i=1}^k \chi_i l_i$ for some $\lambda_1, \dots, \lambda_k, \chi_1, \dots, \chi_k \in \mathbb{R}$ and denote by $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the roots of

$$f_z := T^n - z_1 T^{n-1} + \dots + (-1)^n z_n.$$

Then $e_1(x) = z_1 = \sum_{i=1}^k \lambda_i l_i(x) = \sum_{i=1}^k \lambda_i a_i$ and $e_2(x) = \sum_{i=1}^k \chi_i a_i$ and hence

$$\sum_{i=1}^n x_i^2 = e_1(x)^2 - 2e_2(x) = \left(\sum_{i=1}^k \lambda_i a_i\right)^2 - \sum_{i=1}^k \chi_i a_i.$$

This shows that x is contained in a ball, thus $\mathcal{H}_L(a)$ is bounded. Furthermore, as the roots of a polynomial depend continuously on the coefficients it is clear that $H_L(a)$ is closed and therefore compact (see Remark 2.6). $\square$

We close this section with a selection of examples of two-dimensional hyperbolic slices which highlight the various mentioned scenarios.

Example 2.24. *Consider $\mathcal{H}_L(a_2, a_4)$, where $a := (a_2, a_4) \in \mathbb{R}^2$ such that $a_2 < 0$ and $a_4 > 0$ and*

$$L: \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ (z_1, z_2, z_3, z_4) & \longmapsto & (z_2, z_4) \end{array}.$$

Then, there are the following three possible situations.
a: *If $a := (a_2, a_4)$ satisfy $a_2^2 - 4a_4 < 0$, the hyperbolic slice $\mathcal{H}_L(a)$ will contain two local extreme points. In particular, $\mathcal{H}_L^2(a) \neq \varnothing$. Furthermore, the local extreme points of $\mathcal{H}_L(a)$ are not global extreme points. Therefore, they are not extreme points of the convex hull of $\mathcal{H}_L(a)$. This is illustrated in Figure 2a.*
b: *For all values $a := (a_2, a_4)$ with $a_2^2 - 4a_4 = 0$, $\mathcal{H}_L(a)$ will contain no local extreme points. But the 2-boundary of $\mathcal{H}_L(a)$ is non-empty. Indeed,*

$$T^4 + a_2 T^2 + a_4 = \left(T - \sqrt{\frac{-a_2}{2}}\right)^2 \left(T + \sqrt{\frac{-a_2}{2}}\right)^2,$$

and thus $(0, a_2, 0, a_4) \in \mathcal{H}_L^2(a)$. This situation is illustrated in Figure 2b.
c: *For the values $a := (a_2, a_4)$ with $a_2^2 - 4a_4 > 0$, $\mathcal{H}_L(a)$ will contain no local extreme point. Moreover, $\mathcal{H}_L^2(a)$ is empty in this case, while $\mathcal{H}_L(a) \neq \varnothing$. This is illustrated in Figure 2c. Indeed, the polynomial $f = T^4 + a_2 T^2 + a_4$ is hyperbolic with the 4 distinct roots*

$$x_{1,2,3,4} := \pm \sqrt{\frac{-a_2 \pm \sqrt{a_2^2 - 4a_4}}{2}}.$$

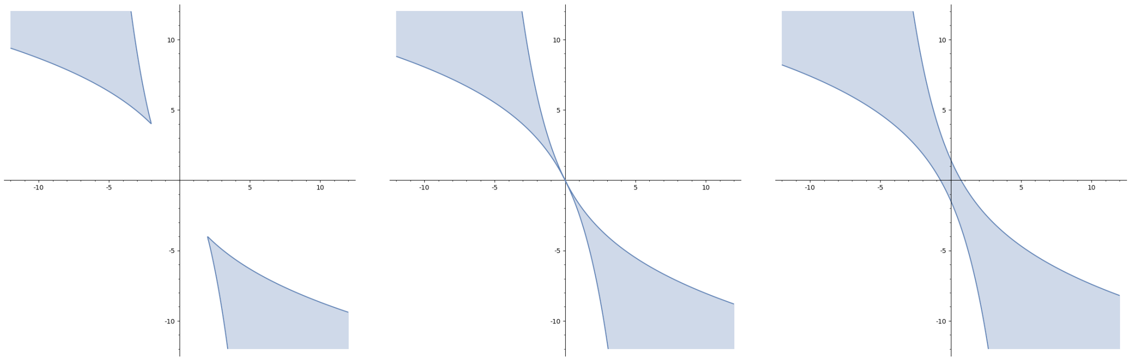
Therefore, the hyperbolic slice $\mathcal{H}_L(a)$ is non-empty. On the other hand, suppose that the 2- boundary $\mathcal{H}_L^2(a)$ is non-empty, i.e., that we can find $(a_1, a_2, a_3, a_4) \in \mathcal{H}_L^2(a)$. This in turn implies that there are $x, y \in \mathbb{R}$ such that the polynomial

$$f_a := T^4 - a_1 T^3 + a_2 T^2 - a_3 T + a_4$$

factors either as

$$f_a = (T - x)^3 (T - y) \text{ or } f_a = (T - x)^2 (T - y)^2.$$

In the first case a comparison of coefficients shows $a_2 = 3xy + 3x^2$ and $a_4 x^3 y$. Since $a_4 > 0$ we must have $x, y \neq 0$ and can solve $y = \frac{a_4}{x^3}$. This implies $a_2 = \frac{3a_4}{x^2} + 3x^2$ and $3x^4 - a_2 x^2 + 3a_4 = 0$. However, since $x \neq 0$, $a_2 < 0$ and $a_4 > 0$ we must have $3x^4 - a_2 x^2 + 3a_4 > 0$, and thus have a contradiction. Analogously, for the second case, comparing coefficients shows $a_2 = 4xy + x^2 + y^2$ and $a_4 = x^2 y^2$. We solve for y and get $y = \pm \frac{\sqrt{a_4}}{x}$ from which we find $a_2 = \frac{a_4}{x^2} + x^2 \pm 4\sqrt{a_4}$. But since $a_2 < 0$, $a_4 > 0$ and $a_2^2 - 4a_4 > 0$ the resulting polynomial equation $x^4 + (\pm 4\sqrt{a_4} - a_2) x^2 + a_4 = 0$ clearly has no real solution.



(A) $\mathcal{H}_L(-3,4)$ contains the local extreme points $(\pm 2,-3,\mp 4,4)$.

(B) $\mathcal{H}_L(-4,4)$ has no local extreme point and $(0,-4,0,4) \in \mathcal{H}^2$.

(C) $\mathcal{H}_L(-5,4)$ has no local extreme point and $\mathcal{H}_L^2(-5,4) = \varnothing$.

3. Positivity of symmetric polynomial functions

In this section we will study real polynomial functions defined by symmetric polynomials. Since every symmetric polynomial can be written in a unique way as a polynomial in elementary symmetric polynomials, we can use the geometric description of hyperbolic slices obtained before to characterize the minimal points of a large class of symmetric polynomial functions which are sparse in an appropriate sense (see Definition 3.5). It had already been observed by various authors that certain symmetric functions attain their minimal values on symmetric points (e.g. [17, 12, 21]). Other authors found that symmetric polynomial functions of a bounded small enough degree attain their minima on points with few distinct coordinates (e.g. [34, 30]). We generalize these results by considering symmetric polynomial functions which are completely characterized through their values on points with at most k distinct coordinates.

3.1. The notions of k-completeness and k-testability.

Definition 3.1. *For $k \in \mathbb{N}$ we consider the set*

$$\mathcal{A}_k := \{x \in \mathbb{R}^n : |\{x_1, \dots, x_n\}| \le k\}$$

of points with at most k different coordinates. Given a symmetric polynomials $f \in \mathbb{R}[\underline{X}]$ and $S \subseteq \mathbb{R}^n$ we say that f is

(1) k-complete *on S if*

$$f(S) = f(S \cap \mathcal{A}_k).$$

(2) k-testable *on S if*

$$\inf_{x \in S} f(x) = \inf_{x \in S \cap \mathcal{A}_k} f(x).$$

In case $S = \mathbb{R}^n$ we may omit it and just speak of k-testable and k-complete polynomials.

The two notions of k-complete and k-testable are very closely connected, but the first one is stronger, while the second one might be interesting in particular in the context of optimization. In order to motivate the study of this class, we exemplify first how algorithmic problems can be substantially simplified for k-complete and k-testable symmetric polynomials.

Definition 3.2. *A decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ which sums up to n is called a* partition of n into k parts. *We will write $\lambda \vdash_k n$ to denote that λ is a partition of n into k parts. Let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial. Then for $\lambda \vdash_k n$ we define*

$$f^\lambda := f(\underbrace{X_1, \dots, X_1}_{\lambda_1-times}, \dots, \underbrace{X_k, \dots, X_k}_{\lambda_k-times}) \in \mathbb{R}[X_1, \dots, X_k].$$

Note that the number of partitions of n into k parts is at most $\binom{n+k}{k}$ and thus polynomial in n for a fixed k. Therefore the above notion allows reducing, for example, the question of whether a symmetric polynomial in n variables is non-negative to a polynomial number of such queries in k variables. It is, for example, known to be NP-hard to decide the non-negativity of a given polynomial of degree 4 (see e.g. [5] or [28]). Clearly, by applying the above procedure, one can obtain algorithmic simplifications which yield polynomial complexity for this kind of problem (see also [11] where this method is applied also for other algorithmic questions). We highlight in particular the following version of Artin's solution to Hilbert's 17th problem for k-complete symmetric polynomials, which is a direct consequence of the sketched procedure of identifying variables.

Proposition 3.3 (Hilbert's 17th problem for k-complete polynomials)**.** *Let $f \in \mathbb{R}[\underline{X}]$ be a symmetric k-testable polynomial. Then f attains only non-negative values on $\mathbb{R}^n$ if and only if for all $\lambda \vdash_k n$ we can find a sum of squares of polynomials $t \in \sum \mathbb{R}[X_1, \dots, X_k]^2$ such that $t \cdot f^\lambda$ is also a sum of squares of polynomials.*

The main interest in the statements presented above is that the reduction of dimension also gives new complexity bounds for the degrees of the polynomials in question. For example, for Hilbert's 17th problem for k-complete polynomials we can adapt the currently known complexity bounds.

Remark 3.4. *Let f be a n-variate k-complete polynomial of degree d. Then f is non-negative if and only if we can write each f^λ as a sum of at most 2^k rational squares by [29]. We can also write each f^λ as a sum of squares of rational functions, where, following [22], we obtain the following degree bounds for the numerators and denominators:*

$$2^{2^{2^{d^{4^k}}}}.$$

3.2. **Sufficient and quasi-sufficient polynomials.** Now, we want to show that it is possible to produce a large class of k-complete symmetric polynomials based on the results on hyperbolic polynomials. Throughout this section we fix $1 \leq k \leq n$ and consider the k linearly independent linear forms $l_1, \dots, l_k \in \mathbb{R}[Z_1, \dots, Z_n]_1$ and the linear map $L\colon\ \mathbb{R}^n \to \mathbb{R}^k,\ z \mapsto (l_1(z), \dots, l_k(z))$. Recall that a symmetric polynomial $f \in \mathbb{R}[\underline{X}]$ can be written uniquely in terms of the elementary symmetric polynomials, say $f = g(e_1, \dots, e_n)$. Now evaluation of f in a point $x \in \mathbb{R}^n$ translates into evaluation of g in a point $z \in \mathcal{H}$ and evaluation on $\mathcal{A}_k$ translates into evaluation of g on $\mathcal{H}^k$. By partitioning

$$\mathcal{H} = \bigcup_{a \in \mathbb{R}^k} \mathcal{H}_L(a) \text{ and } \mathcal{H}^k = \bigcup_{a \in \mathbb{R}^k} \mathcal{H}^k_L(a)$$

for the map L, we can use our previous results to show under some mild conditions that f is k-complete or K-testable if it allows for a special representation in terms of k linear forms of elementary symmetric polynomials. We define these representations in the following.

Definition 3.5. *Let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial and write f in terms of elementary symmetric polynomials, say $f = g(e_1, \dots, e_n)$ for some $g \in \mathbb{R}[Z_1, \dots, Z_n]$.*

*(1) We say that f is $(l_1, \dots, l_k)$-*sufficient *if $g \in \mathbb{R}[l_1, \dots, l_k]$.*
*(2) We say that f is $(l_1, \dots, l_k)$-*quasi-sufficient *if f admits a representation of the form*
$$f = f_0 + f_1 e_1 + \dots + f_n e_n$$
for some $(l_1, \dots, l_k)$-sufficient polynomials $f_0, \dots, f_n$.
*(3) Furthermore, we say that f is $(l_1, \dots, l_k)$-*concave-sufficient *if g is concave on $H_L(a)$ for all $a \in \mathbb{R}^k$.*

*Moreover, we say that a symmetric semi-algebraic set $S \subseteq \mathbb{R}^n$ is $(l_1, \dots, l_k)$-*sufficient*, if it can be described by $(l_1, \dots, l_k)$-sufficient polynomials.*

The following proposition is a direct consequence of the unique representation of a symmetric polynomial of degree d in terms of the elementary symmetric polynomials and may serve as a motivation for the definitions given above.

Proposition 3.6. *Let $f \in \mathbb{R}[\underline{X}]$ be symmetric of degree d. Then f is $(Z_1, \dots, Z_d)$-sufficient and $\left(Z_1, \dots, Z_{\lfloor \frac{d}{2} \rfloor}\right)$-quasi-sufficient.*

Remark 3.7. *The notions defined above are increasingly strict in the following sense: Sufficiency (1) implies quasi-sufficiency (2), which in turn implies concave-sufficiency (3) of both f and $-f$.*

The results on hyperbolic slices now translate to the following statements on symmetric real polynomial functions.

Theorem 3.8. *Let $S \subseteq \mathbb{R}^n$ be a symmetric $(l_1, \ldots, l_k)$-sufficient semi-algebraic set and let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial.*

(1) If f is $(l_1, \ldots, l_k)$-sufficient and if every non-empty hyperbolic slice $\mathcal{H}_L(a)$ contains a local extreme point, then f is k-complete on S.

(2) If f is $(l_1, \ldots, l_k)$-concave-sufficient and $\mathcal{H}_L(a)$ is compact for all $a \in \mathbb{R}^k$, then f is k-testable on S.

(3) If f is $(l_1, \ldots, l_k)$-quasi-sufficient and $\mathcal{H}_L(a)$ is compact for all $a \in \mathbb{R}^k$ and $S \cap \mathcal{A}_k$ is connected, then f is k-complete on S.

(4) If f is $(l_1, \ldots, l_k)$-concave-sufficient and not $(l_1, \ldots, l_k)$-sufficient and

$$\inf_{x \in S} f(x) > -\infty,$$

then f is k-testable on S.

Proof. (1): Let $g \in \mathbb{R}[Z_1, \ldots, Z_n]$ such that $f = g(e_1, \ldots, e_n)$. Let $x \in S$ and consider $z := \Gamma(x)$ and $a := L(z)$. There is $\tilde{z} \in \mathcal{H}_L^k(a)$ by Theorem 2.8 since $\mathcal{H}_L(a)$ admits a local extreme point. So there is $\tilde{x} \in \mathcal{A}_k$ with $\Gamma(\tilde{x}) = \tilde{z}$. Then $f(x) = f(\tilde{x})$ and $\tilde{x} \in S$ since f and S are $(l_1, \ldots, l_k)$-sufficient.

(2): Let $g \in \mathbb{R}[Z_1, \ldots, Z_n]$ such that $f = g(e_1, \ldots, e_n)$. Let $x \in S$ and consider $z := \Gamma(x)$ and $a := L(z)$. Since g is concave on $L^{-1}(a)$ by the concave-sufficiency of f and $\mathcal{H}_L(a)$ is compact we can apply Corollary 2.10 and get that

$$\min_{y \in \mathcal{H}_L(a)} g(y) = \min_{y \in \mathcal{H}_L^k(a)} g(y),$$

i.e., there is $\tilde{z} \in \mathcal{H}_L^k(a)$ with $g(\tilde{z}) \leq g(z)$. Let $\tilde{x} \in \mathcal{A}_k$ with $\Gamma(\tilde{x}) = \tilde{z}$. Then $f(\tilde{x}) \leq f(x)$ and $\tilde{x} \in S$ since S is $(l_1, \ldots, l_k)$-sufficient and we can conclude that f is k-testable on S.

(3): Let $x_0 \in S$. We can apply (2) since f and $-f$ are both $(l_1, \ldots, l_k)$-concave-sufficient by Remark 3.7 and get that

$$\inf_{x \in S} f(x) = \inf_{x \in S \cap \mathcal{A}_k} f(x) \quad \text{and} \quad \sup_{x \in S} f(x) = \sup_{x \in S \cap \mathcal{A}_k} f(x),$$

so there are $x_1, x_2 \in S \cap \mathcal{A}_k$ with $f(x_1) \leq f(x_0)$ and $f(x_2) \geq f(x_0)$. Since $S \cap \mathcal{A}_k$ is connected there is $\tilde{x} \in S \cap \mathcal{A}_k$ with $f(\tilde{x}) = f(x_0)$ by the intermediate value theorem.

(4): Let $g \in \mathbb{R}[Z_1, \ldots, Z_n]$ such that $f = g(e_1, \ldots, e_n)$. There is $x_0 \in S$ with

$$\inf_{x \in S} f(x) = f(x_0)$$

consider $z_0 := \Gamma(x_0)$ and $a := L(z)$. Since g is concave and not constant on $\mathcal{H}_L(a)$, g attains its minimum on an extreme point of $\mathcal{H}_L(a)$, i.e., we can assume that $z_0 \in \mathcal{H}_L^k(a)$ and therefore $x_0 \in \mathcal{A}_k$. □

The existence of local extreme points in Theorem 3.8 (1) is indeed necessary, as in cases without local extreme points it is possible to construct situations where the statement will not hold. We showcase this in the following.

Example 3.9. *Let $K(h) = \mathbb{R}^4$, $l_1 := Z_2$, $l_2 := Z_4$ and $L : \mathbb{R}^4 \to \mathbb{R}^2$, $z \mapsto (l_1(z), l_2(z))$ and consider the (l_1, l_2)-sufficient symmetric polynomial*

$$f = (e_2 + 5)^2 + (e_4 - 4)^2 \in \mathbb{R}[X_1, X_2, X_3, X_4].$$

The 2-boundary $\mathcal{H}_L^2(-5, 4)$ is empty by Example 2.24 (3). So $f(x) > 0$ for all $x \in \mathcal{A}_2$, but $f(1, -1, 2, -2) = 0$.

One can in fact prove that the polynomial f in Example 3.9 is still 3-complete. Indeed, the necessity of the existence of an extreme point in every hyperbolic slice seems to restrict the applications of Theorem 3.8. However, by applying Lemma 2.21 and Lemma 2.23 we can obtain the following version of Theorem 3.8 which avoids this issue at the price of a slightly weaker conclusion.

Corollary 3.10. *Let $S \subseteq \mathbb{R}^n$ be a symmetric $(l_1, \ldots, l_k)$-sufficient semi-algebraic set and let $f \in \mathbb{R}[\underline{X}]$ be a symmetric polynomial.*

(1) If f is $(l_1,\dots,l_k)$-sufficient, then f is $(k+1)$-complete on S.
(2) If f is $(l_1,\dots,l_k)$-concave-sufficient, then f is $(k+2)$-testable on S.
(3) If $f\in\mathbb{R}[\underline{X}]$ is $(l_1,\dots,l_k)$-quasi-sufficient and $S\cap\mathcal{A}_k$ is connected, then f is $(k+2)$-complete on S.

Moreover if $Z_1\in\operatorname{span}(l_1,\dots,l_k)$, then $(k+1)$ in (1) can be replaced by k-complete. If $Z_1,Z_2\in\operatorname{span}(l_1,\dots,l_k)$, then $(k+2)$ in (2) and (3) can be replaced by k.

The results in this section were given entirely for symmetric functions. To conclude this section we remark the following direct translation of the results to even symmetric polynomials or equivalently copositive symmetric polynomials.

Remark 3.11. *The results on symmetric polynomials translate directly to even symmetric polynomials, i.e., polynomials invariant by the natural action of the Hyperoctahedral group $S_2\wr S_n$. Denote by*

$$\begin{aligned}\mathcal{E}:&=\{z\in\mathbb{R}^n \mid T^{2n}-z_1T^{2(n-1)}+\dots+(-1)^n z_n \text{ is hyperbolic}\}\\ &=\{z\in\mathcal{H}\mid T^n-z_1T^{n-1}+\dots+(-1)^n z_n \text{ has only non-negative roots}\}\end{aligned}$$

the set of even hyperbolic polynomials. Furthermore, we define

$$\mathcal{E}^k:=\{z\in\mathcal{E}\mid T^n-z_1T^{n-1}+\dots+(-1)^n z_n \text{ has at most } k \text{ positive roots}\}$$

and $\mathcal{E}_L(a):=\mathcal{E}\cap L^{-1}(a)$ and $\mathcal{E}^k_L(a)$ accordingly. Then the proof of Theorem 2.8 translates to $\operatorname{locextr}(\mathcal{E}_L(a))\subseteq\mathcal{E}^k_L(a)$ and both sets are generically finite. By replacing $\mathcal{A}_k$ by

$$\mathcal{B}_k:=\{x\in\mathbb{R}^n\ :\ |\{x_1^2,\dots,x_n^2\}\smallsetminus\{0\}|\le k\}$$

we can transfer the statements of Theorem 3.8 and Corollary 3.10 about k-completeness and k-testability of (quasi-)sufficient symmetric polynomials to (quasi-)sufficient even symmetric polynomials f, i.e., polynomials that admit a representation of the form

$$f=g(e_1(X_1^2,\dots,X_n^2),\dots,e_n(X_1^2,\dots,X_n^2))$$

with $g\in\mathbb{R}[l_1,\dots,l_k]$. Note that in this case it suffices already to fix the first coefficient in order to obtain compactness, so one can replace $(k+2)$ in Corollary 3.10 (2) and (3) by $(k+1)$.

3.3. **Deciding sufficiency.** Generally the definition of sufficient and quasi-sufficient given above can appear to be not directly verifiable. Especially since mostly one is given a symmetric polynomial without its representation in terms of linear combinations of elementary symmetric polynomials. Therefore, we want to shortly present how to algorithmically approach the question if a given symmetric polynomial is sufficient or quasi-sufficient. In order to decide if a symmetric polynomial $f\in\mathbb{R}[\underline{X}]$ is sufficient for some collection of linear forms $l_1,\dots,l_k$ one has principle two task:

(1) Finding a representation of $f=g(e_1,\dots,e_n)$ in terms of elementary symmetric polynomials: This can be achieved, for example, by using the Gröbner basis $G:=\{g_1,\dots,g_k\}$, where

$$g_k=\sum_{\substack{\alpha\in\mathbb{N}_0^{n-k+1}\\|\alpha|=k}}X_k^{\alpha_1}\cdots X_n^{\alpha_{n-k+1}}+\sum_{i=1}^k(-1)^iY_i\sum_{\substack{\alpha\in\mathbb{N}_0^{n-k+1}\\|\alpha|=k-i}}X_k^{\alpha_1}\cdots X_n^{\alpha_{n-k+1}}$$

of the ideal $I=(e_1-Y_1,\dots,e_n-Y_n)\subseteq\mathbb{R}[\underline{X},Y_1,\dots,Y_n]$ which is independent from f and then by computing the remainder g of f on division by G. One obtains now $f=g(e_1,\dots,e_n)$ (see Proposition 4 and Proposition 5 in §1 of Chapter 7 in [8] for details). Alternatively one can use the algorithm presented in [35].

(2) Once $g\in\mathbb{R}[e_1,\dots,e_n]$ is obtained, one has to decide if there exist $k<n$ linear combinations $l_1,\dots,l_k$ of the $e_1,\dots,e_n$ such that $g\in\mathbb{R}[l_1,\dots,l_k]$. Also this can be accomplished quite concretely, for example, by using the approach outlined by Carlini [7]. As described there, the smallest number k of linear forms $l_1,\dots,l_k$ needed such that $g\in\mathbb{R}[l_1,\dots,l_k]$ is obtained by computing the rank of the Catalectican matrix of g. This matrix is obtained by the coefficients of the partial derivatives

of g. More concretely, one can actually also explicitly construct these linear forms by computing a basis for the vector space of the $(d-1)$-th partial derivatives of g.

The steps described above rely mostly on linear algebra and can be efficiently implemented also for larger numbers of variables.

Remark 3.12. *In the special case when one wants to decide if a symmetric polynomials f is $e_{i_1}, \ldots, e_{i_m}$-quasi-sufficient (where $1 \le i_1 \le \cdots \le i_n \le k$) one can actually proceed with the following examination of the gradient of f without going through the steps above: As a symmetric polynomial f cane be written as $f = g(e_1, \ldots, e_n)$ we have*

$$\nabla f = \nabla g J_{e_1,\ldots,e_n}.$$

Noting that $J_{e_1,\ldots,e_n}$ is invertible over $\mathbb{R}(X_1, \ldots, X_n)$ we get

$$\nabla f J^{-1}_{e_1,\ldots,e_n} = \nabla g.$$

Now, if for $I \subseteq \{1, \ldots, n\}$ the corresponding entries in ∇g are constants, then f is $(e_i)_{\{1,\ldots,n\}\setminus I}$-quasi-sufficient.

We give a short example to illustrate the algorithmic approach.

Example 3.13. *We consider the following toy example of a symmetric polynomial in three variables in order to showcase the methods described above*

$$f = \sum_{\sigma \in S_3} \sigma \Bigg(\frac{1}{2} X_1^3 + X_1^2 X_2^2 + 3X_1^2 X_2 + X_1^3 X_2 + X_1 X_2 X_3 - X_1^2 X_2^2 X_3^2 \\ + \frac{1}{2} X_1^3 X_2^3 X_3^2 - 2X_1^3 X_2^2 X_3 - X_1^3 X_2 X_3 - 2X_1^2 X_2^2 X_3 + \frac{5}{2} X_1^2 X_2 X_3 \Bigg),$$

where S_3 acts on $\mathbb{R}[X_1, X_2, X_3]$ by permutation of variables.

The Gröbner basis corresponding to the ideal

$$I := \langle e_1 - Y_1, e_2 - Y_2, e_3 - Y_3 \rangle$$

is given by

$$G = \{X_1 + X_2 + X_3 - Y_1, X_2^2 + X_2 X_3 - X_2 Y_1 + X_3^2 - X_3 Y_1 + Y_2, X_3^3 - X_3^2 Y_1 + X_3 Y_2 - Y_3\}.$$

By computing the remainder of f on division by G one obtains

$$g = Y_1^3 + Y_1^2 Y_2 - 2Y_1^2 Y_3 - 2Y_1 Y_2 Y_3 + Y_1 Y_3^2 + Y_2 Y_3^2 \in \mathbb{R}[Y_1, Y_2, Y_3]$$

with $f = g(e_1, e_2, e_3)$. In order to compute the Catalactican of g, we fix a monomial basis

$$M = \{M_1, \ldots, M_6\} = \{Y_1^2, Y_1 Y_2, Y_1 Y_3, Y_2^2, Y_2 Y_3, Y_3^2\}$$

for the ternary forms of degree $2 = \deg(g) - 1$. Calculating the partial derivatives

$$\partial_i g = c_{i1} M_1 + \cdots + c_{i6} M_6$$

we obtain he Catalactican C_g of g defined as $(C_g)_{ij} = c_{ij}$, i.e.

$$C_g = \begin{pmatrix} 3 & 2 & -4 & 0 & -2 & 1 \\ 1 & 0 & -2 & 0 & 0 & 1 \\ -2 & -2 & 2 & 0 & 2 & 0 \end{pmatrix}.$$

The number of linear forms needed to express g is then equal to $\operatorname{rank}(C_g) = 2$. In order to find linear forms needed to express g, it suffices to compute a basis for the span of the second partial derivatives of g, we obtain

$$\{Y_1 - Y_3, Y_2 + Y_3\}$$

and indeed

$$g = (Y_2 + Y_3)(Y_1 - Y_3)^2 + (Y_1 - Y_3)^3,$$

i.e. f is $(Y_2 + Y_3, Y_1 - Y_3)$-sufficient and $(Y_1 - Y_3)$-quasi-sufficient.

4. Applications and examples

We will now show some applications of the theory developed here and use it on some concrete examples to underline the potential of the results presented. We begin with examining the following polynomial which was given by Robinson [32] as an example of a non-negative form which is not a sum of squares. Note that this example could also be obtained by a variant of the half degree principle to even symmetric polynomials.

Example 4.1 (Robinson Polynomial)**.** *The non-negativity of the Robinson polynomial*

$$R = X^6 + Y^6 + Z^6 - \left(X^4Y^2 + X^2Y^4 + X^4Z^2 + X^2Z^4 + Y^4Z^2 + Y^2Z^4\right) + 3X^2Y^2Z^2$$

can be easily verified using Remark 3.11. Indeed,

$$R = e_1(X^2,Y^2,Z^2)^3 - 4e_1(X^2,Y^2,Z^2)e_2(X^2,Y^2,Z^2) + 9e_3(X^2,Y^2,Z^2)$$

is a Z_1-quasi-sufficient even symmetric polynomial. Therefore, we only need to examine R on the set

$$\mathcal{B}_1 := \{x \in \mathbb{R}^3 : |\{x_1^2, x_2^2, x_3^2\} \setminus \{0\}| \le 1\}.$$

Since we easily find that the two (dehomogenized) univariate polynomials

$$R_1 = R(1,T,T) = T^4 - 2T^2 + 1 = (T-1)^2(T+1)^2$$
$$R_2 = R(1,T,0) = T^6 - T^4 - T^2 + 1 = (T^2+1)(T-1)^2(T+1)^2.$$

are non-negative, R is indeed non-negative. Moreover, we directly also see that R has at least the 10 projective zeros

$$(1,\pm1,\pm1), (0,\pm1,\pm1), (\pm1,0,\pm1), (\pm1,\pm1,0)$$

which constitute the orbits of $(1,1,1)$ and $(1,1,0)$. One easily checks that these zeros are isolated. From this observation one immediately also obtains that R cannot be a sum of squares. Indeed, since a zero of a sum of squares also has to be a zero of every summand, a sextic which is a sum of squares can have at most 9 isolated zeros.

Furthermore, we will show how our results can be used to verify symmetric inequalities rather easily.

Example 4.2 (AM–GM inequality)**.** *The inequality of arithmetic and geometric means is a standard inequality from analysis, stating that for all $x \in \mathbb{R}^n_{\ge 0}$ we have*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 \cdot x_2 \cdots x_n},$$

or equivalently

$$e_1^n - n^n e_n \ge 0 \text{ on } \mathbb{R}^n_{\ge 0}.$$

By squaring the variables this is equivalent to

$$F = e_1(X_1^2,\ldots,X_n^2)^n - n^n e_n(X_1^2,\ldots,X_n^2)$$

is non-negative, which can be proven by applying again Remark 3.11 similarly to the previous example.

Example 4.3 (Maclaurin's inequality)**.** *More general we have*

$$\sqrt[i]{\frac{e_i(x)}{\binom{n}{i}}} \ge \sqrt[j]{\frac{e_j(x)}{\binom{n}{j}}}$$

for all $x \in \mathbb{R}^n_{\ge 0}$ and $i \le j$ which is equivalent to

$$F = \binom{n}{j}^{2i} e_i(X_1^2,\ldots,X_n^2)^{2j} - \binom{n}{i}^{2j} e_j(X_1^2,\ldots,X_n^2)^{2i}$$

is non-negative. F is (Z_i)-concave-sufficient and even symmetric. First we show that $\inf_{x\in\mathbb{R}^k} f > -\infty$. Since F is in particular (Z_1, Z_i)-concave-sufficient, it suffices to show that

$$F_\lambda := F(\underbrace{X,\ldots,X}_{\lambda_1-times},\underbrace{Y,\ldots,Y}_{\lambda_2-times},\underbrace{0,\ldots,0}_{\lambda_3-times})$$

is bounded from below for all partitions $\lambda_1+\lambda_2+\lambda_3 = n$. Since F_λ is homogeneous it suffices to show that the dehomogenization

$$\tilde{F_\lambda} = F_\lambda(X,1)$$

has positive leading coefficient. It has leading coefficient

$$\binom{n}{j}^{2i}\binom{\lambda_1}{i}^{2j} - \binom{n}{i}^{2j}\binom{\lambda_1}{j}^{2i} > 0$$

for $i \le \lambda_1 < n$ (this can be easily shown by induction on λ_1) and $\tilde{F_\lambda} = 0$ for $\lambda_1 = n$ and for $\lambda_1 < i$. Now we can use Theorem 3.8 (4) and Remark 3.11, so it suffices to check that

$$F_\mu := F(\underbrace{X,\dots,X}_{\mu-times}, \underbrace{0,\dots,0}_{(n-\mu)-times})$$

is non-negative for all partitions $\mu + n - \mu = n$. Since F_μ is homogeneous it suffices to show that the dehomogenization

$$\tilde{F_\mu} = F_\mu(1) = \begin{cases} \binom{n}{j}^{2i}\binom{\mu}{i}^{2j} - \binom{n}{i}^{2j}\binom{\mu}{j}^{2i}, & \text{for } i \le \mu < n \\ 0, & \text{else} \end{cases}$$

is non-negative.

It is interesting to notice that the idea of certifying symmetric inequalities in the way sketched has been done albeit not as general. For example, the main Lemma [26, Lemma 2.4] used to prove some new inequalities between elementary symmetric polynomials can be seen as a special case of Remark 3.11 for Z_1-quasi-sufficient even symmetric polynomials. Indeed our setup also recovers as a special instance of Corollary 3.10 together with Proposition 3.6 the so called Degree and Half-Degree Principle shown in [34].

Corollary 4.4 (Degree Principle)**.** *Let $S \subseteq \mathbb{R}^n$ be a symmetric semi-algebraic set, which can be described by symmetric polynomials of degree at most d. Then S is empty, if and only if $S \cap \mathcal{A}_d$ is empty.*

Corollary 4.5 (Half-Degree Principle)**.** *Let $f \in \mathbb{R}[\underline{X}]$ be symmetric of degree d. Then f is k-complete, where $k := \max\left\{2, \left\lfloor \frac{d}{2} \right\rfloor\right\}$.*

We remark that it is known to be NP-hard already for quartics to decide non-negativity (see e.g. [5] or [28]). However, for univariate polynomials non-negativity can be certified via a sums of squares decomposition. Such a decomposition can be efficiently obtained via semi-definite programming. The feasible region of a semi-definite program is given by a linear matrix inequality (LMI), i.e., an inequality of the form $A_0 + x_1A_1 + x_2A_2 + \ldots + x_nA_n \ge 0$, where $A_0,\dots,A_n$ are real symmetric matrices all of the same size and $x_1,\dots,x_n$ are supposed to be real scalars. Now for a symmetric 1-complete polynomial of degree $2d$ we have that f is non-negative if and only if the univariate polynomial $\tilde{f} := f(T,T,\dots,T)$ of same degree is non-negative. This in turn is the case, if and only if there exists a symmetric matrix $A \in \mathbb{R}^{(d+1)\times(d+1)}$ which is non-negative and for which we have $\tilde{f} = (1,T,T^2,\dots,T^n)\cdot A\cdot(1,T,T^2,\dots,T^n)^t$. Therefore, non-negativity of a 1-complete symmetric polynomial can be decided with semi-definite programming. This motivates the following sufficient criterion for 1-complete polynomials.

Theorem 4.6. *Let $l \in \mathbb{R}[Z_1,\dots,Z_n]_1$ be linear and homogeneous, say $l = \lambda_1Z_1 + \cdots + \lambda_nZ_n$ for some $\lambda_1,\dots,\lambda_n \in \mathbb{R}$. Let f be a l-sufficient symmetric polynomial. Let m denote the largest index i of the non-zero λ_i, i.e., $m := \max\{i \in \{1,\dots,n\} \mid \lambda_i \ne 0\}$. If m is odd, then f is 1-complete.*

Proof. Write f as $f := g(l(e_1\dots,e_n))$ for some univariate polynomial g. Let $x \in \mathbb{R}^n$ and define $a := l(e_1(x),\dots,e_n(x)) \in \mathbb{R}$. We will show that $\mathcal{H}_l^1(a) \ne \varnothing$. Consider the univariate polynomial

$$p := \sum_{i=1}^{m} \lambda_i \binom{n}{i} T^i - a \in \mathbb{R}[T].$$

Since m is odd, p has a real zero $y \in \mathbb{R}$. Consider now $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ defined by $z_i := \binom{n}{i} y^i$. Then $z \in \mathcal{H}_l^1(a)$ by construction. Now

$$f(x) = g(a) = g(l(z_1, \ldots, z_n)) = f(y, \ldots, y).$$

□

Convex sets for which membership can be described via semi-definite programming, i.e., which are projections of feasibility regions of semi-definite programs are called *spectrahedral shadows*. Recently, Scheiderer [33] was able to show that in general the cone of positive semi-definite forms is not in general a spectrahedral shadow. Using Corollary 3.10 and Remark 3.11 we can identify families of convex cones of (even-)symmetric positive semi-definite forms which are spectrahedral shadows, generalizing Theorem 4.29 in [9].

Proposition 4.7. *Let $\mathcal{P}_{2d}$ denote the convex cone of positive semi-definite n-ary forms of degree $2d$ and $2 \leq j \leq n$. Then, the subcones of all (Z_1, Z_j)-sufficient and (Z_1, Z_2)-quasi-sufficient symmetric forms are spectrahedral shadows. Similarly, the subcone of all (Z_1, Z_j)-quasi-sufficient even-symmetric forms is a spectrahedral shadow.*

Proof. All forms in the mentioned subcones are 2-complete by Corollary 3.10 and Remark 3.11. Therefore non-negativity can be decided by restricting to $\mathcal{A}_2$, respectively $\mathcal{B}_2$. Dehomogenizing the resulting binary forms we obtain univariate polynomials, which are non-negative if and only if they are sums of squares. □

5. Conclusion and open questions

We have defined the notion of hyperbolic slices and showed that the local extreme points of such slices correspond to hyperbolic polynomials with few distinct roots. We show that generically these hyperbolic slices contain at most finitely many local extreme points. We expect that this holds generally, i.e., also in those cases when the k-boundary is not finite. In particular, we expect that the convex hull of each connected component of any hyperbolic slice is a polyhedron. Arnold and Giventhal [2, 14] had shown that the hyperbolic slices which are obtained by fixing the first k coefficients are contractible. Our examples show that hyperbolic slices are in general neither connected nor compact and therefore in particular not contractible. It would be very interesting to study the topological properties of these sets. Similarly to the results in [4], an understanding of the topology of these slices might allow for new efficient algorithms to compute the homology of symmetric semi-algebraic sets defined by k-complete polynomials. Furthermore, the definition of hyperbolic slices naturally involved elementary symmetric polynomials. From the viewpoint of symmetric polynomials, it seems interesting to study analogous sets for different choices of n symmetric polynomials which generate all symmetric polynomials. For example, the first author observed in [31] that symmetric polynomials defined by any k Newton sums are at least $(2k+1)$-complete. Finally, a natural question is to explore the connections to invariant polynomials of other groups, most notably finite reflection groups. In [13, 1] the authors showed that the image of polynomial functions invariant by a finite reflection group can be described by the points on flats in the hyperplane arrangement, if the degree is sufficiently small. We expect that the notions and techniques presented here can be transferred also to this more general setup.

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CLASSIFICATION OF $\aleph_0$-CATEGORICAL C-MINIMAL PURE C-SETS

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Abstract

We classify all $\aleph_0$-categorical and C-minimal C-sets up to elementary equivalence. As usual the Ryll-Nardzewski Theorem makes the classification of indiscernible $\aleph_0$-categorical C-minimal sets as a first step. We first define *solvable* good trees, via a finite induction. The trees involved in initial and induction steps have a set of nodes, either consisting of a singleton, or having dense branches without endpoints and the same number of branches at each node. The class of *colored* good trees is the elementary class of solvable good trees. We show that a pure C-set M is indiscernible, finite or $\aleph_0$-categorical and C-minimal iff its canonical tree $T(M)$ is a colored good tree. The classification of general $\aleph_0$-categorical and C-minimal C-sets is done via finite trees with labeled vertices and edges, where labels are natural numbers, or infinity and complete theories of indiscernible, $\aleph_0$-categorical or finite, and C-minimal C-sets.

Keywords: C-minimality, $\aleph_0$-categoricity, trees, first-order theories.
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1. Introduction

C-sets are sets equipped with a C-relation. They can be understood as a slight weakening of ultrametric structures. They generalize in particular linear orders and allow rich combinatorics. They are therefore not classifiable, unless you restrict their class. It is what we do here: we consider $\aleph_0$-categorical and C-minimal C-sets. C-minimality is the minimality notion fitting in this context: any definable subset in one variable is quantifier free definable using the C-relation alone. In the case of ultrametric structures this corresponds to finite Boolean combinations of closed or open balls. We classify here all $\aleph_0$-categorical and C-minimal C-sets up to elementary equivalence (in other words we classify all finite or countable such structures). Although C-minimality is a generalization of o-minimality, our result does not generalize Pillay and Steinhorn's result: they classify (Theorem 6.1 in [P–S]) all $\aleph_0$-categorical and o-minimal linearly ordered structures while we only classify $\aleph_0$-categorical and C-minimal pure C-sets.

To state our result let us introduce some material. A C-set M has a canonical tree, $T(M)$, in which M appears as the set of leaves, with the C-relation defined as follows : for $\alpha \in M$, call $br(\alpha) := \{x \in T(M); x \leq \alpha\}$ the branch α defines in $T(M)$; then for α, β and γ in M, $M \models C(\alpha, \beta, \gamma)$ iff in $T(M)$, $br(\beta) \cap br(\gamma)$ strictly contains $br(\alpha) \cap br(\beta)$ (which then must be equal to $br(\alpha) \cap br(\gamma)$). Let us give a very simple example: call *trivial* a C-relation satisfying $C(\alpha, \beta, \gamma)$ iff $\alpha \neq \beta = \gamma$ and suppose M is not a singleton; then C is trivial on M iff $T(M)$ consists of a root, say r, and the elements of M as leaves, all having r as a predecessor. The C-set (M, C) and the tree $(T(M), <)$ are uniformly biinterpretable. As usual the Ryll-Nardzewski Theorem makes the classification of indiscernible $\aleph_0$-categorical C-minimal sets as a first step in our work. Recall that a structure is said to be indiscernible iff all its elements have the same complete type We characterize indiscernible, $\aleph_0$-categorical and C-minimal C-sets by their canonical tree. First we define by induction solvable trees. Consider on leaves above a node a the equivalence relation "$br(\alpha) \cap br(\beta)$ contains nodes strictly bigger than a". An equivalence class is called a *cone* at a. So, the number of cones at a coincides with the intuitive notion of the number of branches. A 0-solvable good tree is a singleton (with the only possible C-relation: the empty relation). There are three types of 1-solvable good trees. Either the tree T consists of a unique node with at least two leaves immediately above. Or for any leaf α of T, $br(\alpha)$ consists of a dense linear order and its leaf α, and at each node there is the same number (a natural number greater than 2 or infinity) of cones. Or each $br(\alpha)$ consists of a dense linear order, α and a predecessor of α, and there are two numbers m and μ (natural numbers greater than 1, or infinity) such that at each node of T there are exactly μ infinite cones and m cones which consist of a single leaf. An $(n+1)$-solvable good tree is an n-solvable good tree in which each leaf is substituted with a copy of a 1-colored good tree, the same at each leaf, with some constraints on the parameters m and μ occurring on both sides of the construction. A solvable good tree is an n-solvable good tree for some integer n. And a colored good tree is a tree elementary equivalent to a solvable one. We prove that a pure C-set M is indiscernible, finite or $\aleph_0$-categorical and C-minimal iff its canonical tree $T(M)$ is a colored good tree.

The reduction of the general classification to that of indiscernible structures uses a very precise description of definable subsets in one variable. $\aleph_0$-categoricity is combined with the classical description coming from C-minimality to produce a "canonical partition" of the structure in finitely many definable subsets, each of them maximal indiscernible. The characterization of $\aleph_0$-categorical and C-minimal C-sets is done via finite trees with labeled vertices and edges, where labels are natural numbers, or infinity, and complete theories of indiscernible, $\aleph_0$-categorical or finite C-minimal C-sets. The reconstruction of the structure from such a finite labeled tree uses again an induction on the depth of the tree.

Chapter 2 lists some preliminaries. In Chapter 3 we draw a certain amount of consequences of indiscernibility, $\aleph_0$-categoricity and C-minimality of a C-structure, which leads to the notion of precolored good tree (no inductive definition this time). Chapters 4 to 6 are dedicated to colored good trees. Chapter 4 presents 1-colored good trees, which in fact are the same thing as precolored good trees of depth 1. In Chapter 5 we define the extension of a colored good tree by a 1-colored good tree, construction which is the core of the inductive definition of $(n+1)$-colored good trees from n-colored good trees. General colored good trees are defined and completely axiomatized in Chapter 6. In Chapter 7 we show that the classes of precolored good trees, of colored good trees as well as of canonical trees of indiscernible, finite or $\aleph_0$-categorical and C-minimal C-sets do in fact coincide. Chapter 8 gives a complete classification of $\aleph_0$-categorical and C-minimal C-sets.

Notice that, if M is indiscernible the set of leaves is indiscernible in $T(M)$ but the tree $T(M)$, except the singleton, never is. Its set of nodes may be indiscernible. It is the case for 1-colored good trees described in Section 4 and no other colored good tree. This is why our classification has almost nothing to do with the classification of countable 1-transitive trees given by Barbina, Chicot and Truss (see [CT] and [BC]). The only common point is as follows : 1-colored good trees of type (0) are trivially 2-transitive and a 1-colored good tree of type (1) deprived of its leaves is 2-transitive too. Notice also that the colors" we considere here are <u>not</u>, as often in model theory, unary predicates. Our classification extend probably, with no other complications than technical, to structures in a language consisting of C and finitely many unary predicates.

2. Preliminaries

2.1. C-sets and good trees

Definition 2.1. *A C-*relation *is a ternary relation, usually called C, satisfying the four axioms:*
1. $C(x,y,z) \to C(x,z,y)$
2. $C(x,y,z) \to \neg C(y,x,z)$
3. $C(x,y,z) \to [C(x,y,w) \vee C(w,y,z)]$
4. $x \neq y \to C(x,y,y)$.
*A C-*set *is a set equipped with a C-relation.*

C-relations appear in [AN], [M–S] or [H–M], where they satisfy additional axioms. Our present definition comes from [D]. As already mentioned in the introduction, a C-set M has a canonical tree, which is in fact bi-interpretable with M, as we explain now.

Definition 2.2. *We call* tree *an order in which for any element x the set $\{y; y \leq x\}$ is linearly ordered.*
Call a tree good *if :*
- it is a meet semi-lattice (i.e. any two elements x and y have an infimum, or meet*, $x \wedge y$, which means: $x \wedge y \leq x, y$ and $(z \leq x, y) \to z \leq x \wedge y$),*
- it has maximal elements, or leaves*, everywhere (i.e. $\forall x, \exists y\ (y \geq x \wedge \neg \exists z > y)$)*
- and any of its elements is a leaf or a node (i.e. of form $x \wedge y$ for some distinct x and y).

Let T be a good tree. It is convenient to consider T in the language $\{<, \wedge, L\}$ where $\wedge$ is the function $T \times T \to T$ defined above and L a unary predicate for the set of leaves (cf. Definition 2.2).

Proposition 2.3. *C-sets and good trees are bi-interpretable classes.*

Let us explain these two interpretations in a few words. More details can be found in [D].
Call branch of a tree any maximal subchain. The set of branches of T carries a canonical C-relation: $C(\alpha,\beta,\gamma)$ iff $\alpha \cap \beta = \alpha \cap \gamma \subsetneq \beta \cap \gamma$. Now, leaves of T may be identified to branches via the map $\alpha \mapsto br(\alpha) := \{\beta \in T; \beta \leq \alpha\}$. Thus, if $Br_l(T)$ denotes the set of branches with a leaf of T, the two-sorted structure $(T, <, Br_l(T), \in)$ is definable in $(T, <)$, and the canonical C-relation on $Br_l(T)$ also. **We denote this C-set** $M(T)$. This gives the definition of a C-set in a good tree. The canonical tree of a C-set provides the reverse construction. It is (almost) the representation theorem of Adeleke and Neumann ([AN], 12.4), slightly modified according to [D]. Let us describe their construction. Given a C-set (M, C), define on M^2 binary relations

$$(\alpha,\beta) \preccurlyeq (\gamma,\delta) :\Leftrightarrow \neg C(\gamma,\alpha,\beta) \& \neg C(\delta,\alpha,\beta)$$

$$(\alpha,\beta)R(\gamma,\delta) :\Leftrightarrow \neg C(\alpha,\gamma,\delta)\&\neg C(\beta,\gamma,\delta)\&\neg C(\gamma,\alpha,\beta)\&\neg C(\delta,\alpha,\beta).$$

Then the relation $\preccurlyeq$ is a pre-order, R is the corresponding equivalence relation and the quotient $T := M^2/R$ is a good tree. [1]

Proposition 2.4 summarizes these facts in a more precise way than Proposition 2.3 did.

Proposition 2.4. *Given a C-set M, there is a unique good tree such that M is isomorphic to its set of branches with leaf, equipped with the canonical C-relation. This tree is called* the canonical tree of M *and is denoted $T(M)$.*
Let L be the set of leaves of $T(M)$. Then $\langle M, C\rangle$ and $\langle T(M), <, \wedge, L\rangle$ are first-order bi-interpretable, quantifier free and without parameters, and M and $L(T(M))$ are definably isomorphic. Therefore an embedding $M \subseteq N$ induces an embedding $T(M) \subseteq T(N)$. Moreover, given a good tree T, $T(M(T))$ and T are definably isomorphic.

2.2. C-structures and C-minimality

Definition 2.5. *A C-*structure *is a C-set possibly equipped with additional structure.*
*A C-structure $\mathcal{M}$ is called C-*minimal *iff for any structure $\mathcal{N} \equiv \mathcal{M}$ any definable subset of N is definable by a quantifier free formula in the pure language $\{C\}$.*

Remark 2.6. *Any finite C-structure is C-minimal.*

C-minimality has been introduced by Deirdre Haskell, Dugald Macpherson and Charlie Steinhorn as the minimality notion suitable to C-relations ([H–M], [M–S]). We define now some particular definable subsets of $\mathcal{M}$ which, due to C-minimality, generate by Boolean combination all definable subsets of $\mathcal{M}$. If we want to distinguish between nodes and leaves of the tree $T(M)$, we will use Latin letters $x, y, etc...$ to denote nodes and Greek letters $\alpha, \beta, etc...$ for leaves (cf. Definition 2.2). According to the representation theorem, elements of M are also represented by Greek letters.

Definition 2.7.
- *For α and β two distinct elements of M, the subset of M: $\mathcal{C}(\alpha\wedge\beta, \beta) := \{\gamma \in M; C(\alpha,\gamma,\beta)\}$ is called the* cone *of β at $\alpha\wedge\beta$; $\alpha\wedge\beta$ is called its* basis*. We also use the notation, for elements $y > x$ from $T(M)$, $\mathcal{C}(x,y) := \mathcal{C}(x,\alpha)$ for any (or some) $\alpha \in M$ such that $br(\alpha)$ contains y, and we say that $\mathcal{C}(x,y)$ is the cone of y at x.*
- *For α and β in M, the subset of M: $\mathcal{C}(\alpha\wedge\beta) := \{\gamma \in M; \neg C(\gamma,\alpha,\beta)\} = \{\gamma \in M; \alpha\wedge\beta \leq \gamma\}$ is called the* thick cone *at $\alpha\wedge\beta$; $\alpha\wedge\beta$ is its* basis*. Note that, if*

[1] Adeleke and Neumann work in fact with the set of pairs of distinct elements of M, instead of M^2 as we do (and reverse order). It is the reason why we get maximal elements everywhere in the tree, meanwhile they did not get any. In the other direction also, $Br_l(T)$ is interpretable in T meanwhile the "covering set of branches" considered by Adeleke and Neumann is not determined by T.

$\alpha \neq \beta$, the thick cone at $\alpha \wedge \beta$ is the disjoint union of all cones at $\alpha \wedge \beta$[2].

- *For $x < y \in T(M)$ the* pruned cone *at x of y is the cone at x of y minus the thick cone at y, in other words the set $\mathcal{C}(]x, y[) = \{\gamma \in M; x < (\gamma \wedge y) < y\}$. The interval $]x, y[$ is called the* axis *of the pruned cone, x its* basis.

Note that the word "cone" follows the terminology of Haskell, Macpherson and Steinhorn while our "thick cone" replace their "0-levelled set" (with the motivation that we do not use here n-levelled sets for $n \neq 0$). We also replace "interval" by "pruned cone" with the intention that an "interval" always lives in a linear order.

It is easy to see that the subsets of M definable by an atomic formula of the language $\{C\}$ are M, $\emptyset$, singletons, cones and complements of thick cones. We can therefore rephrase the above definition of C-minimality as follows: A C-structure $\mathcal{M}$ is C-minimal iff for any structure $\mathcal{N} \equiv \mathcal{M}$ any definable subset of N is a Boolean combination of cones and thick cones.

Proposition 2.8. *Let $\mathcal{M}$ be a C-minimal C-set and A a cone, thick cone or pruned cone with a dense axis in M. Then, considered as a pure C-set, A is C-minimal too.*

Proof: The trace of a cone on a cone, say A, is a (relative) cone: this means that this trace can be described as $\{x \in A; C(\alpha, \beta, x)\}$ for two parameters α and β from A. More generally the trace of a possibly thick cone on a possibly thick cone is a possibly thick cone. Thus the above statement is trivial for cones. For a pruned cone, C-minimality is ensured by the axis density, see [D], p. 70, Example and Lemma 3.12 (the C-minimality considered there is in some sense "external" and a priori stronger than the "internal" one considered in the above statement). $\square$

We explain now how the biinterpretation we have seen between M and $T(M)$ remains valid in the expanded context of C-minimality. Given a C-structure $\mathcal{M}$ consider M as the set of leaves of $T(M)$ and add to the tree structure of $T(M)$ all subsets of some cartesian power $T(M)^n$ which are $\emptyset$-definable in $\mathcal{M}$ as $\emptyset$-definable sets. The structure obtained is called the *structure induced by $\mathcal{M}$ on $T(M)$*. The reverse construction is a bit more subtle:

Definition 2.9. *Let $\mathcal{N}$ be a structure and A a $\emptyset$-definable subset of N. By definition the language of the* structure induced *by $\mathcal{N}$ on A consists of all subsets of some A^n which are definable in $\mathcal{N}$ without parameters.*
We say that A is stably embedded *in $\mathcal{N}$ if for all integer n every subset of A^n which is definable in $\mathcal{N}$ with parameters, is definable with parameters from A.*

[2] In the particular case of ultrametric spaces the C-relation is defined as follows: $C(x, y, z)$ iff $d(x, y) = d(x, z) < d(y, z)$. The thick cones are the closed balls and cones are the open balls. Some balls may be open and closed. In the same way as a closed ball, say of radius $r \neq 0$, is partitioned into open balls of radius r, a thick cone at a node n is partitioned in cones at n.

In this case the subsets of some A^n definable in $\mathcal{N}$ or in the structure induced by $\mathcal{N}$ on A are the same.

Proposition 2.10. *Whatever additional structure we consider on $T(M)$, M is stably embedded in $T(M)$.*

Proof: Consider φ a formula without parameters of the expanded tree $T(M)$ with $n+m$ variables, parameters $c = (c_1, \dots, c_m)$ from $T(M)$ and the set $D := \{x = (x_1, \dots, x_n) \in M^n; T(M) \models \varphi(c, x)\}$. Each c_i is of the form $c_i = \alpha_i \wedge \beta_i$ for some $\alpha_i, \beta_i \in M$ hence $D := \{x \in M^n; T(M) \models \varphi(\alpha_1 \wedge \beta_1, \dots, \alpha_m \wedge \beta_m, x)\}$, a set which is definable with parameters from M. $\square$

Proposition 2.11. *Let $\mathcal{M}$ be a C-minimal C-structure and T its canonical tree with the structure induced by $\mathcal{M}$. Then:*

1. *Each branch $br(\alpha)$ of $Br_l(T)$ is o-minimal in T, in the sense that, any subset of $br(\alpha)$ definable in T is a finite union of intervals with bounds in $br(\alpha) \cup \{-\infty\}$.*
2. *Any node c of T is strongly minimal in the sense that, any definable set of cones at c is finite or cofinite (in the set of all cones at c).*

Proof: Haskell and Macpherson [H–M] Lemma 2.7 (i) and (ii). $\square$

Remark 2.12. *Using "rosy theories" and a result of Pillay (Theorem 1.4 in [P]) we see that any branch $br(\alpha)$ of T is in fact stably embedded in (T, α) and o-minimal for the induced structure.*

2.3. Some definability properties in the canonical tree

We have defined (possibly thick or pruned)(Definition 2.7) cones as subsets of M. But they have their counterparts in the canonical tree that we define below. So cones are subsets of M as well as of $T(M)$, we hope the context and the distinct notation $\mathcal{C}$ or Γ will make the choice clear.
As previously, when we want to make a difference, Latin letters $x, y, etc...$ denote nodes of $T(M)$ which are not leaves and Greek letters $\alpha, \beta, etc...$ leaves.

Definition 2.13.
- *For α and β two distinct elements of M, the subset of $T(M)$: $\Gamma(\alpha \wedge \beta, \beta) := \{t \in T(M); \alpha \wedge \beta < t \wedge \beta\}$ is called the* cone *of β at $\alpha \wedge \beta$*[3]*. Note that it is the canonical tree of $\mathcal{C}(\alpha \wedge \beta, \beta)$.*
As for cones in M, we also use the notation, for elements $y > x$ from T, $\Gamma(x, y) := \Gamma(x, \alpha)$ for any (or some) $\alpha \in M$ such that $br(\alpha)$ contains y and we say that $\Gamma(x, y)$ is the cone of y at x.

[3] Be aware that in [H–M] a cone of nodes always contains its basis, in other words a cone at a is the union of a and what we call here a cone.

- *For α and β in M, the subset of $T(M)$: $\Gamma(\alpha \wedge \beta) = \{t \in T(M); \alpha \wedge \beta \leq t\}$ is called the* thick cone *at $\alpha \wedge \beta$. Note that it is the canonical tree of $\mathcal{C}(\alpha \wedge \beta)$. Let x be a node of $T(M)$, note that $\Gamma(x) = \bigcup_{\substack{\alpha \in M \\ x \in br(\alpha)}} \Gamma(x, \alpha) \cup \{x\}$.*

- *For $x < y \in T(M)$, the* pruned cone *at x of y is the set $\Gamma(]x, y[) = \{t \in T(M); x < (t \wedge x) < t \wedge y\} := \Gamma(x, \beta) \setminus \Gamma(y)$ where β is any branch containing y. It is the canonical tree of $\mathcal{C}(]x, y[)$. The interval $]x, y[$ is called the* axis *of the pruned cone.*

The basis *of a (possibly thick or pruned) cone is defined analogously to what is done for subsets of M.*

Definition 2.14. *We say that a leaf α of T is* isolated if there exists a node x in T such that $x < \alpha$ and there is no node between x and α, in other words, α gets a predecessor in T. If α is an isolated leaf, then its unique predecessor is denoted by $p(\alpha)$.

Definition 2.15. *Let x be a node of T. We say that a cone Γ at x is an* inner cone *if the two following conditions are realized:*

1. *x has no successor on any branch $br(\alpha)$ where α is a leaf and $\alpha \in \Gamma$. Note that, x has a successor (say x^+) on $br(\alpha)$ for some $\alpha \in \Gamma$, iff Γ is a thick cone (the thick cone at x^+).*

2. *There exists $t \in \Gamma$ such that, for any $t' \in T$ with $x < t' < t$, t' is of same tree-type as x.*

Otherwise, we say that Γ is a border cone.

Remark 2.16. *An inner cone is always infinite. The cone $\Gamma(p(\alpha), \alpha)$ at the predecessor $p(\alpha)$ of an isolated leaf α is a border cone which consists only of that leaf.*

Definition 2.17. *The* color *of a node x of a tree T is the couple $(m, \mu) \in (\mathbb{N} \cup \{\infty\})^2$ where m is the number of border cones at x and μ the number of inner cones at x.*

Lemma 2.18. *Suppose the C-set $\mathcal{M}$ is finite or $\aleph_0$-categorical.*

1. *Then the color of a node of $T(M)$ is $\emptyset$-definable in the pure order of $T(M)$, which means that there are unary formulas φ_k and ψ_k, $k \in \mathbb{N} \cup \{\infty\}$, of the language $\{<\}$ such that, for any node x of $T(M)$ and k,*

$$T(M) \models \varphi_k(x) \text{ iff there are exactly } k \text{ border cones at } x,$$

$$T(M) \models \psi_k(x) \text{ iff there are exactly } k \text{ inner cones at } x.$$

2. *If $\mathcal{M}$ is furthermore C-minimal then there is no node of color (∞, ∞).*

Proof: 1. By the Ryll-Nardzewski Theorem, or finitness, Condition 2 of Definition 2.15 is first-order.
2. At a node of color (∞, ∞) the set of cones is partitioned in two infinite definable sets. This contradicts strong minimality (see Proposition 2.11, 2) $\square$

3. Canonical trees of indiscernible finite or $\aleph_0$-categorical C-minimal C-sets

We say that a structure is *indiscernible* if it realizes only one complete 1-type over $\emptyset$.

3.1. Indiscernible finite or $\aleph_0$-categorical C-structures with o-minimal branches in their canonical trees

Definition 3.1. *A* basic *interval of a linear ordered set O will mean a singleton or a dense (non empty and infinite) convex subset with bounds in $O \cup \{-\infty\}$.*

For T a good tree and α a leaf of T the set $br(\alpha)$ is a chain of T with maximal element α.

Definition 3.2. *A basic* one-typed *interval of T is a basic interval, say I, of $br(\alpha) \setminus \{\alpha\}$ for some leaf α of T such that all elements of I have same tree-type over $\emptyset$.*

Theorem 3.3. *Let $\mathcal{M}$ be an indiscernible finite or $\aleph_0$-categorical C-structure. Let T be its canonical good tree. Assume that for each leaf α of T, any subset of the chain $br(\alpha)$ definable in T is a finite union of basic intervals with bounds in $br(\alpha) \cup \{-\infty\}$. Then there exists an integer $n \geq 1$ such that for any leaf α of T, the branch $br(\alpha)$ can be written as a disjoint union of its leaf and n basic one-typed intervals, $br(\alpha) = \bigcup_{j=1}^{n} I_j(\alpha) \cup \{\alpha\}$ with $I_j(\alpha) < I_{j+1}(\alpha)$. This decomposition is unique if we assume that the $I_j(\alpha)$ are maximal one-typed, that is, $I_j(\alpha) \cup I_{j+1}(\alpha)$ is not a one-typed basic interval. Possible forms of each $I_j(\alpha)$ are $\{x\}$, $]x, y[$ and $]x, y]$. The decomposition is independent of the leaf α, that is, the form (a singleton or not, open or closed on the right) of $I_j(\alpha)$ for a fixed j as well as the tree-type of its element do not depend on the leaf α.*

Remark 3.4. *Remember (Proposition 2.11) that Haskell and Macpherson have shown that, if $\mathcal{M}$ is C-minimal, then for each leaf α, any subset of $br(\alpha)$ definable in T is a finite union of intervals with bounds in $br(\alpha) \cup \{-\infty\}$. Thus the conclusion of the above theorem remains the same if we add the hypothesis that $\mathcal{M}$ is C-minimal and remove the condition on $Br_l(T)$.*

Proof of Theorem 3.3. In the following, a "branch of T" will always mean a branch with a leaf, i.e. an element of $Br_l(T)$. By Ryll-Nardzewski Theorem the $\aleph_0$-categoricity of $\mathcal{M}$ implies that for any integer p there is a finite number of p-types over $\emptyset$. Now T is interpretable without parameters in $\mathcal{M}$ where it appears as a definable quotient of M^2. Since there is a finite number of $2p$-types over $\emptyset$ in M, there is a finite number of p-types

in T. Hence, T is finite or $\aleph_0$-categorical. Thus we can partition the tree T into finitely many sets S such that two nodes in T have the same complete type over $\emptyset$ iff they are in the same set S. The trace on any branch $br(\alpha)$ of such a set S is definable and thus, by o-minimality, a finite union of intervals. In fact it consists of a unique interval: if a node x belongs to the left first interval of $S \cap br(\alpha)$, then by definition of the sets S any other element of $S \cap br(\alpha)$ will too (look at the formula without parameter $\exists \beta \in L$ (x belongs to the first interval of $br(\beta)$)). For the same reason, if $S \cap br(\alpha)$ has a first element, then this interval is in fact a singleton.
Hence, for a given leaf α, $br(\alpha)$ is the order sum of finitely many maximal one-typed intervals. Using indiscernibility, the number of such basic intervals, the form (singleton, open or closed on the right) of each of them, and the tree-type of its elements, depend only on its index and not on the branch. □

Lemma 3.5. *Let α, β be two distinct leaves of T. Let $j^\star$ be the unique index such that $\alpha \wedge \beta \in I_{j^\star}(\alpha)$. Then, $\forall j < j^\star$, $I_j(\alpha) = I_j(\beta)$. Moreover, $I_{j^\star}(\alpha) \cap I_{j^\star}(\beta)$ is an initial segment of both $I_{j^\star}(\alpha)$ and $I_{j^\star}(\beta)$.*

Proof: By definition, $br(\alpha) \cap br(\beta) = I_1(\alpha) \cup \cdots \cup I_{j^\star-1}(\alpha) \cup \{t \in I_{j^\star}(\alpha); t \leq \alpha \wedge \beta\}$ (or $\{t \in I_{j^\star}(\alpha); t \leq \alpha \wedge \beta\}$ if $j^\star = 1$). The same is true with β instead of α.
Therefore, by definition and uniqueness of the partition of each branch into maximal basic one-typed intervals, we get $\forall j < j^\star$, $I_j(\alpha) = I_j(\beta)$. Moreover, $\{t \in I_{j^\star}(\alpha); t \leq \alpha \wedge \beta\} = \{t \in I_{j^\star}(\beta); t \leq \alpha \wedge \beta\} = I_{j^\star}(\alpha) \cap I_{j^\star}(\beta)$. □

3.2. Precolored good trees

By Lemma 2.18 all nodes of a one-typed basic interval are of same color. In order to describe the theory of the canonical tree of an indiscernible $\aleph_0$-categorical or finite C-minimal C-structure, we define now precolored good trees which are constructed from the conclusion of Theorem 3.3, replacing "one-typed basic interval" by the (in general different) notion of "one-colored basic interval".

In this subsection, T will be a good tree, L its set of leaves and N its set of nodes.

Definition 3.6. *One-colored basic interval*
We say that a basic interval I of $br(\alpha) \setminus \{\alpha\}$ for some leaf α of T is one-colored *if I satisfies one of the following conditions:*

- *(0) I is a singleton $\{x\}$ and the color of x is $(k, 0)$, for k a natural number greater that 2 or infinity, that is, there are exactly k distinct cones at x, all border cones. We say that I is of* color $(k, 0)$.
- *(1.a) I is open on both left and right sides: $I =]x, y[$. Any element of I is of color $(0, k)$, for k an integer greater that 2 or infinity, that is, there are exactly k distinct cones at any element of I, and all are inner cones. We say that the basic interval I is of* color $(0, k)$.

(1.b) *I is open on the left side and closed on the right side: $I =]x, y]$ and any element of I is of color (m, μ), for $m, \mu \in \mathbb{N}^* \cup \{\infty\}$, that is, there are exactly m border cones (i.e. m distinct leaves) and μ inner cones at any point of I. We say that the basic interval I is of* color (m, μ).

Definition 3.7. *We say that T is a* precolored good tree *if there is no node of color (∞, ∞) and there exists an integer n, such that for all $\alpha \in L$:*

(1) *the branch $br(\alpha)$ can be written as a disjoint union of its leaf and n basic one-colored intervals $br(\alpha) = \cup_{j=1}^{n} I_j(\alpha) \cup \{\alpha\}$, with $I_j(\alpha) < I_{j+1}(\alpha)$.*

(2) *The $I_j(\alpha)$ are maximal one-colored, that is, $I_j(\alpha) \cup I_{j+1}(\alpha)$ is not a one-colored basic interval, and for all $j \in \{1, \cdots, n\}$, the color of $I_j(\alpha)$ is independent of α.*

(3) *For any $\alpha, \beta \in L$ and $j \in \{1, \cdots, n\}$, if $\alpha \wedge \beta \in I_j(\alpha)$, then $\alpha \wedge \beta \in I_j(\beta)$, $I_j(\alpha) \cap I_j(\beta)$ is an initial segment of both $I_j(\alpha)$ and $I_j(\beta)$; and for any $i < j$, $I_i(\alpha) = I_i(\beta)$.*

The integer n, which is unique by maximality of the basic one-colored intervals, is called the depth *of the precolored good tree T.*

Corollary 3.8. *Let M be a finite or $\aleph_0$-categorical, indiscernible and C-minimal C-set. Then $T(M)$ is a precolored good tree.*

Proof: The result follows directly from Theorem 3.3, Lemma 2.18 and Lemma 3.5. □

Proposition 3.9. *Let T be a precolored good tree, then all leaves of T are isolated or all leaves of T are non isolated.*

Proof: Let α be a leaf of T. Assume that α has a predecessor $p(\alpha)$, then the last interval $I_n(\alpha)$ is closed on the right, that is either $I_n(\alpha) = \{p(\alpha)\}$ of color $(k, 0)$, or $I_n(\alpha) =]x, p(\alpha)]$ of color (m, μ) with $m \neq 0$. By definition of precolored good trees, either for any leaf β, the last interval of $br(\beta)$ is of color $(k, 0)$, or for any leaf β, the last interval of $br(\beta)$ is of color (m, μ), with $m \neq 0$. In both cases, β has a predecessor. □

Definition 3.10. *Definition of functions $e_1, \ldots, e_{n-1}$ on leaves.*
Let T be a precolored good tree of depth $n > 1$. For any leaf α and for $1 \leq j < n$, we denote $e_j(\alpha)$ the lower bound of $I_{j+1}(\alpha)$ and E_j the range of the function e_j.

Proposition 3.11. *Let T be a precolored good tree of depth $n > 1$. Let α, β be two leaves of T. For $1 \leq j < n$, if $e_j(\alpha)$, $e_j(\beta) \leq \alpha \wedge \beta$, then $e_j(\alpha) = e_j(\beta)$. Hence, we can extend the functions e_j to partial functions from T to N in the following way:*

$Dom(e_j) = \bigcup_{\alpha \in L} (\{e_j(\alpha)\} \cup I_{j+1}(\alpha) \cup \cdots \cup I_n(\alpha) \cup \{\alpha\})$, *and,*

$\forall \alpha \in L, \forall x \in br(\alpha) \cap Dom(e_j)$, $e_j(x) = e_j(\alpha)$.

The range of e_j is still E_j. The partial functions e_j are definable in the pure order.

Proof: Let α, β be two leaves and j an index such that $e_j(\alpha)$, $e_j(\beta) \leq \alpha \wedge \beta$. We can assume without loss of generality that $e_j(\alpha) \leq e_j(\beta) \leq \alpha \wedge \beta$. Let $j^\star$ be the unique index such that $\alpha \wedge \beta \in I_{j^\star}(\alpha)$. By definition of e_j, $j+1 \leq j^\star$. Either, $j+1 < j^\star$ and by Definition 3.7 (3), $I_{j+1}(\alpha) = I_{j+1}(\beta)$, hence $e_j(\alpha) = e_j(\beta)$; or $j+1 = j^\star$, and by 3.7 (3) again, $I_{j+1}(\alpha) \cap I_{j+1}(\beta)$ is an initial segment of both $I_{j+1}(\alpha)$ and $I_{j+1}(\beta)$, hence $e_j(\alpha) = e_j(\beta)$.
By Lemma 2.18, the color of a node is definable in the pure order. Now, all nodes of $I_j(\alpha)$ have the same color, $I_j(\alpha)$ is a maximal interval of $br(\alpha)$ with this property, and there are only finitely many such maximal intervals in $br(\alpha)$. This shows that the bounds of $I_j(\alpha)$ are $\{\alpha\}$-definable in the pure order. □

The next proposition describes the form of maximal basic one-colored intervals in terms of the functions e_j. By convention, when a basic interval is denoted $]a, b[$, b has no predecessor. **We extend the definition of** p: for any $c \in T$ having a predecessor, this predecessor is denoted $p(c)$.

Proposition 3.12. *Let T be a precolored good tree of depth n.*
Assume first $n = 1$. Then, uniformly in α, $I_1(\alpha)$ is of the form, either (0): $\{r\} = \{p(\alpha)\}$ where r is the root, or (1.a): $]-\infty, \alpha[$, or (1.b): $]-\infty, p(\alpha)]$.
Assume now $n > 1$. Then, uniformly in α,
- $I_1(\alpha)$ is of the form, either (0): $\{r\}$ (and $r = e_1(\alpha)$ or $r = p(e_1(\alpha))$), or (1.a): $]-\infty, e_1(\alpha)[$, or (1.b): ($]-\infty, e_1(\alpha)]$ or $]-\infty, p(e_1(\alpha))]$);
- for $2 \leq j \leq n-1$, $I_j(\alpha)$ is of the form, either (0): $\{e_{j-1}(\alpha)\}$, or (1.a): $]e_{j-1}(\alpha), e_j(\alpha)[$, or (1.b): ($]e_{j-1}(\alpha), e_j(\alpha)]$ or $]e_{j-1}(\alpha), p(e_j(\alpha))]$);
- $I_n(\alpha)$ is of the form, either (0): $\{e_{n-1}(\alpha)\} = \{p(\alpha)\}$, or (1.a): $]e_{n-1}(\alpha), \alpha[$, or (1.b): $]e_{n-1}(\alpha), p(\alpha)]$.
Moreover, for $j < n$, if $I_j(\alpha)$ is open on the right, then $I_{j+1}(\alpha)$ is a singleton.
Finally T has isolated leaves iff $I_n(\alpha)$ is of form (0) or (1.b).

Proof: Note first that $I_1(\alpha)$ is a singleton iff T has a root and in this case the unique element of $I_1(\alpha)$ must be this root.
Case $n = 1$. Then, for any leaf α, $br(\alpha) = I_1(\alpha) \cup \{\alpha\}$, so, by definition of one-colored basic intervals, the assertion is clear.
Case $n > 1$. For $j < n$, recall that $e_j(\alpha)$ is the lower bound of $I_{j+1}(\alpha)$. If $I_{j+1}(\alpha)$ is a singleton, then its unique element must be $e_j(\alpha)$. If $I_{j+1}(\alpha)$ is not a singleton, it is open on the left, hence $e_j(\alpha)$ is in $I_j(\alpha)$.
If $I_1(\alpha) = \{r\}$, then $r = e_1(\alpha)$ if $I_2(\alpha)$ is not a singleton, and $r = p(e_1(\alpha))$ otherwise. If $I_1(\alpha)$ is open on the right, it must be case $(1.a)$. If it is closed right, either $I_2(\alpha)$ is the singleton $\{e_1(\alpha)\}$, hence $I_1(\alpha) =]-\infty, p(e_1(\alpha))]$, or $I_2(\alpha)$ is open on the left with lower bound $e_1(\alpha)$, hence $I_1(\alpha) =]-\infty, e_1(\alpha)]$.
For, $2 \leq j \leq n-1$, it runs similarly. The case $j = n$ is similar to the case $n = 1$.

The other assertions are trivial. □

Proposition 3.13. *Let T be a precolored good tree of depth n with isolated leaves. If $I_n(\alpha) = \{p(\alpha)\}$, for any $\alpha \in L$, then the set $p(L) := \{p(\alpha); \alpha \in L\}$ is a maximal antichain of T. If $I_n(\alpha) =]e_{n-1}(\alpha), p(\alpha)]$, then $p(L) = \bigcup_{\alpha \in L} I_n(\alpha)$.*

Proof: If $I_n(\alpha) = \{p(\alpha)\}$ for any $\alpha \in L$, let α and β be two distinct leaves such that $p(\alpha) \leq p(\beta)$. Then $\alpha \wedge \beta = p(\alpha)$. Hence, by Lemma 3.5, $p(\alpha) = p(\beta)$. This shows that $p(L)$ is an antichain of T. To prove it is maximal, let $t \in T$; either t is a leaf and $t > p(t)$, or t is a node, hence there exists a leaf α such that $t < \alpha$, thus $t \leq p(\alpha)$.
Assume now $I_n(\alpha) =]e_{n-1}(\alpha), p(\alpha)]$ (in other words $I_n(\alpha)$ is of type $(1.b)$) and let $x \in I_n(\alpha)$. Suppose that $x < p(\alpha)$, then $\Gamma(x, \alpha)$ is an inner border cone at x, by definition of inner cones. By Definition 3.6 $(1.b)$, there exists a border cone at x, say $\Gamma(x, \beta)$, hence $x = p(\beta) \in p(L)$. □

4. 1-colored good trees

In Section 6 we will introduce a very concrete class, the class of colored good trees, which will turn out to be the same thing as precolored good trees. Its definition is inductive. The present section defines 1-colored good trees. Section (5) will present a construction which gives the induction step.

4.1. Definition

Definition 4.1. *Let T be a good tree. We say that T is a* 1-colored good tree *if T satisfies one of the following group of properties.*

- *(0) T consists of a unique node and m leaves, where m is a natural number greater than 2 or infinity.*
- *(1.a) There exists μ, a natural number greater than 2 or infinity, such that for any leaf α of T, $]-\infty, \alpha[$ is densely ordered and at each node of T there are exactly μ cones, all infinite.*
- *(1.b) There exists (m, μ), where m and μ are natural numbers greater than 1 or infinity, $(m, \mu) \neq (\infty, \infty)$, such that for any leaf α of T, α has a predecessor, the node $p(\alpha)$, $]-\infty, p(\alpha)]$ is densely ordered and at each node of T there are exactly m leaves and μ infinite cones.*

We will say that (0), (1.a) or (1.b) is the type *of the* 1*-colored good tree and $(m, 0)$, $(0, \mu)$, or (m, μ) its* branching color*.*

Remark 4.2. *By Corollary 3.12 a precolored good tree T of depth* 1 *is a* 1*-colored good tree of branching color (m, μ) where (m, μ) is the color of any node of T.*

4.2. Examples

In the following pictures, a continous line means a dense order and a dashed line means that there is no node between its two extremities.

(0) Trees of form (0) are canonical trees of C-sets equipped with the trivial C-relations ($C(\alpha, \beta, \gamma)$ iff $\alpha \neq \beta = \gamma$), in other words of pure sets.

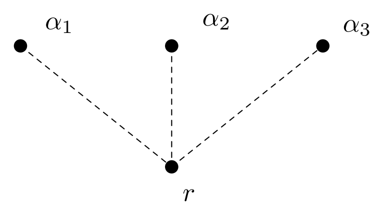

Fig.1 *Type* (0) $m = 3, \mu = 0$

(1.a) Example of color $(0, \mu)$.

Let $\mathbb{Q}$ be the set of rational numbers and μ an integer ≥ 2 or $\aleph_0$. Let $\mathcal{M}$ be the set of applications with finite support from $\mathbb{Q}$ to μ, equipped with the C-relation: $C(\alpha, \beta, \gamma)$ iff the maximal initial segment of $\mathbb{Q}$ where β and γ coincide (as functions) strictly contains the maximal initial segment where α and β coincide.

The thick cone at $\alpha \wedge \beta$ is the set $\{\gamma \in M; \gamma$ coincide with α and β on the maximal initial segment where α and β coincide $\}$. If α and β are different and q is the first rational number where $\alpha(q) \neq \beta(q)$, then there are μ possible values for $\gamma(q)$, in other words there are μ different cones at $\alpha \wedge \beta$. So $\mathcal{M}$ is 1-colored of type $(0, \mu)$.

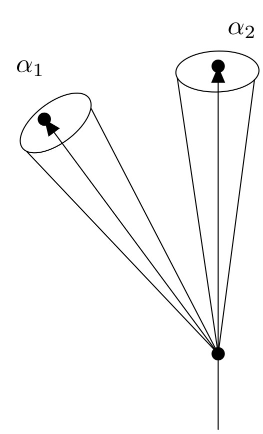

Fig.2 *Type* $(1.a)$ $m = 0, \mu = 2$

(1.b) Example of color (m, μ), $m \geq 1$ and $\mu \geq 2$.

Consider a tree T of type $(1.a)$ of color $(0, \mu)$. Decompose it in nodes and leaves as

$N \cup L$. For any $m \geq 1$ consider now the tree $N \cup (N \times m)$ with the order extending the one of N, elements in $N \times m$ all incomparable and $a < (b, r)$ iff $a \leq b$ for $a, b \in N$ and $r < m$ (in other words: we remove the leaves of T and add m new leaves at each node; so, the set of nodes remains the same). This tree is of type $(1.b)$ of color (m, μ).

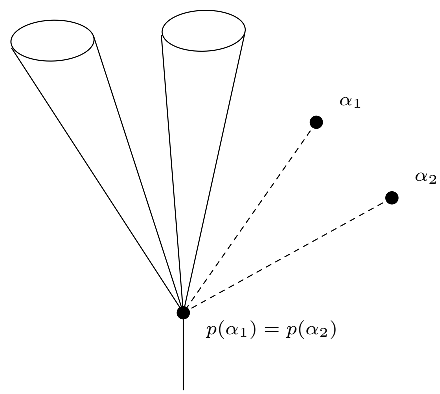

Fig. 3 *Type* $(1.b)$ $m = 2, \mu = 2$

Example of color (m, μ), $m \geq 1$ and $\mu = 1$.
The construction is similar to the previous one: for O a dense linear order without endpoints and m a natural number greater than 1 or infinity, consider the tree $T = O \cup (O \times m)$ with the order extending the one of O, elements in $O \times m$ all incomparable and $a < (b, r)$ iff $a \leq b$ for $a, b \in O$ and $r < m$. The set of nodes of T is O, the vertical line in the picture below. It is a branch without leaf, i.e. a maximal chain of T without greatest element, the unique one in T. Note that O is definable in T. Furthermore O and T are bi-interpretable (for $m = \infty$ we have to assume T and O countable).

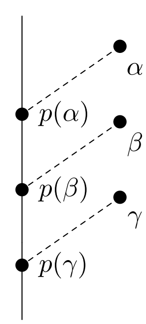

Fig. 4 *Type* $(1.b)$ $m = 1, \mu = 1$

4.3. Axiomatisation and quantifier elimination

Definition 4.3. *For m and μ in $\mathbb{N} \cup \{\infty\}$ such that $m + \mu \geq 2$, we denote $\Sigma_{(m,\mu)}$ the set of axioms in the language $\mathcal{L}_1 := \{L, N, \leq, \wedge\}$ describing 1-colored good trees of* *branching color (m,μ), and S_1 the set of all these $\mathcal{L}_1$-theories, $S_1 := \{\Sigma_{(m,\mu)}; (m,\mu) \in (\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\})$ with $m+\mu \geq 2\}$.*

When dealing with models of $\Sigma_{(m,\mu)}$, $\mu \neq 0$, we want to have the predecessor function in the language. For this reason we introduce $D_p := \{x; \{y; y < x\}$ has a maximal element $\}$, p the function equal to the predecessor function on D_p and the identity on its complement, and $F_p = p(D_p)$. Note that these definitions make sense in any tree and in a model of $\Sigma_{(m,\mu)}$, $m \neq 0$, we have $D_p = L$ and $F_p = N$.

Definition 4.4. $\mathcal{L}_1 := \{L, N, \leq, \wedge\}$ *and* $\mathcal{L}_1^+ := \mathcal{L}_1 \cup \{p, D_p, F_p\}$.

Proposition 4.5. *Any theory in S_1 is $\aleph_0$-categorical, hence complete. Moreover, it admits quantifier elimination in a natural language, $\Sigma_{(m,0)}$ in $\{L,N\}$, $\Sigma_{(0,\mu)}$ in $\mathcal{L}_1$ and $\Sigma_{(m,\mu)}$ with $m,\mu \neq 0$ in $\mathcal{L}_1^+$ (namely in $\{L,N,\leq,\wedge,p\}$).*

Proof: Trees of form (0) consist of one node and leaves. They are clearly $\aleph_0$-categorical and eliminate quantifiers in the language $\{L,N\}$.
So from now on, we assume that $\Sigma = \Sigma_{m,\mu}$, where $\mu \neq 0$. Note that in this case, a model of Σ has no root. We will prove $\aleph_0$-categoricity and quantifier elimination using a back and forth between finite $\mathcal{L}_1$-substructures in the case where $m = 0$ (and $\mathcal{L}_1^+$-substructures in the case where $m \neq 0$) of any two countable models of Σ, say T and T'. We will use the following facts.

Fact 0: 1. Assume first $m = 0$. Then all leaves (respectively all nodes) of T and T' have same quantifier free $\mathcal{L}_1$-type. Any singleton is an $\mathcal{L}_1$-substructure.
2. Assume now $m \neq 0$. Then all leaves (respectively all nodes) of T and T' have same quantifier free $\mathcal{L}_1^+$-type. Any node is an $\mathcal{L}_1^+$-substructure. If α is a leaf, then $\{\alpha, p(\alpha)\}$ is an $\mathcal{L}_1^+$-substructure.
Proof: Completness of quantifier free types '$t \in N$' and '$t \in L$' is proven by inspection of quantifier free formulas. What regards substructures is clear. $\dashv$

In what follows A is a finite subset of T which is a substructure in the language $\mathcal{L}_1$ if $m = 0$ (resp. $\mathcal{L}_1^+$ if $m \neq 0$), hence closed under $\wedge$ (resp. $\wedge$ and p), and φ is a partial $\mathcal{L}_1$-isomorphism (resp. $\mathcal{L}_1^+$-isomorphism) from T to T' with domain A.

Fact 1: Let t be an element of T, $t \notin A$. Then there exists a unique node n_t of T such that n_t is less or equal to an element of A, and for any $a \in A$, $t \wedge a = n_t \wedge a$.
Proof: The set $B = \{t \wedge a; a \in A\}$ is a linearly ordered finite set (of nodes since t is not in A). Let n_t be its greatest element. So, there exists $y \in A$ such that $n_t = t \wedge y$, and therefore $n_t \leq y$. Moreover, it is easy to see that, since n_t is the greatest element of B, for any $a \in A$, $t \wedge z = n_t \wedge z$. Unicity is clear. $\dashv$

Note that, $n_t \leq t$ and ($n_t = t$ iff t is a node smaller than an element of A).

Fact 2: Assume first that $m = 0$. Let $t \in T \setminus A$. Then the $\mathcal{L}_1$-substructure $\langle A \cup \{t\}\rangle$

generated by A and t is the minimal subset containing A, t, n_t (*id est* $A \cup \{t, n_t\}$ if $n_t \neq t$ and $A \cup \{t\}$ if $n_t = t$).
Assume now that $m \neq 0$. Let x be a node of $T \setminus A$. Then the $\mathcal{L}_1^+$-substructure $\langle A \cup \{x\}\rangle$ generated by $A \cup \{x\}$ is the minimal subset containing A, x, n_x. If α is a leaf of $T \setminus A$, the $\mathcal{L}_1^+$-substructure $\langle A \cup \{\alpha\}\rangle$ generated by $A \cup \{\alpha\}$ is the minimal subset containing A, α, n_α, and $p(\alpha)$.
Proof: Assume first that x is a node of $T \setminus A$. Then for any $a \in A$, $x \wedge a = n_x \wedge a$. By definition, there is $z \in A$ such that $n_x \leq z$, so for any $a \in A$, $n_x \wedge a = n_x$ or $n_x \wedge a = z \wedge a \in A$. Now $p(x) = x$. Thus $\langle A \cup \{x\}\rangle = A \cup \{x, n_x\}$ (or $A \cup \{x\}$ if $n_x = x$). Assume now that α is a leaf of $T \setminus A$. If α is non isolated the same argument applies. If α is isolated then for any $a \in A$, $p(\alpha) \wedge a = \alpha \wedge a = n_\alpha \wedge a$. And as above, the minimal subset containing A, α, n_α and $p(\alpha)$ is closed under p and $\wedge$. $\dashv$

Fact 3: Let Γ be a cone at $a \in A$, such that $\Gamma \cap A = \emptyset$. Then there exists a cone Γ' of T' at $\varphi(a)$ such that $\Gamma' \cap \varphi(A) = \emptyset$. Moreover, if Γ is infinite, resp. consists of a single leaf, then there is such a Γ' infinite, resp. consisting of a single leaf.
Proof: If Γ is an infinite cone and μ is infinite, resp. $\Gamma = \{\alpha\}$ and m is infinite, the result is obvious since A is finite.
If now Γ is infinite and μ is finite, there are exactly μ infinite cones at both a and $\varphi(a)$; since $A_{>a} := \{x \in A; x > a\}$ and $A'_{>\varphi(a)} := \{x \in A'; x > \varphi(a)\}$ have same quantifier free type, one of the cones at $\varphi(a)$, say Γ', must be such that $\Gamma' \cap \varphi(A) = \emptyset$. If $\Gamma = \{\alpha\}$ and $m \neq 0$ is finite, then, $a = p(\alpha)$ and there are exactly m leaves above both a and $\varphi(a)$. We consider again $A_{>a}$ and $A'_{>\varphi(a)}$; since $\alpha \notin A$, there exists $\alpha' \notin \varphi(A)$ above $\varphi(a)$. $\dashv$

Fact 4: Let $x \in T \setminus A$ such that $n_x = x$. Then, x is a node and φ can be extended to a partial $\mathcal{L}_1$-isomorphism if $m = 0$ (resp. $\mathcal{L}_1^+$-isomorphism if $m \neq 0$) with domain $\langle A \cup \{x\}\rangle = A \cup \{x\}$.
Proof: Since $n_x = x$, $\langle A \cup \{x\}\rangle$ is equal to $A \cup \{x\}$. Since A is finite and closed under $\wedge$ it contains a smallest element, say a, bigger than x. If the set $\{y \in A; y < x\}$ is not empty, set $b := Max\{y \in A; y < x\}$ and $I :=]\varphi(b), \varphi(a)[$; set $I :=]-\infty, \varphi(a)[$ otherwise. If $m = 0$, I is dense. If $m \neq 0$, since A is closed under p, a is not a leaf, neither is $\varphi(a)$, so in this case too, I is dense. So in both cases, there is x' in I. For such an x', $A \cup \{x\}$ and $\varphi(A) \cup \{x'\}$ are isomorphic trees, closed under p and $\wedge$. $\dashv$

Fact 5: Let $t \in T \setminus A$. Then φ can be extended to a partial $\mathcal{L}_1$-isomorphism (resp. a partial $\mathcal{L}_1^+$-isomorphism) with domain $\langle A \cup \{n_t\}\rangle$.
Proof: By Fact 4. $\dashv$

Fact 6: Let $t \in T \setminus A$. Then φ can be extended to a partial $\mathcal{L}_1$-isomorphism (resp. a partial $\mathcal{L}_1^+$-isomorphism) with domain $\langle A \cup \{t\}\rangle$.
Proof: By Fact 5, we can assume that $t \neq n_t$ and $n_t \in A$. Let Γ be the cone of t at n_t,

then by definition of n_t, $\Gamma \cap A = \emptyset$. Assume first that $m = 0$. Since Γ is infinite, there exists by Fact 3, an infinite cone Γ' at $\varphi(n_t)$ such that $\Gamma' \cap \varphi(A) = \emptyset$. Then we can extend φ to $\langle A \cup \{t\}\rangle$, by setting $\varphi(t) = t'$, where t' is any node of Γ' if t is a node, or any leaf of Γ' is t is a leaf. Such a leaf exists in Γ' since T, as a good tree, has leaves everywhere".
Assume now that $m \neq 0$. If Γ consists of a leaf, id est, t is a leaf and $n_t = p(t)$, then, by fact 3, there exists a cone Γ' at $\varphi(n_t)$ which consists only of a leaf α'. Then, we can extend φ to $\langle A \cup \{t\}\rangle$, by setting $\varphi(t) = \alpha'$. If Γ is infinite then, by fact 3, there exists an infinite cone Γ' at $\varphi(n_t)$ in T' such that $\Gamma' \cap \varphi(A) = \emptyset$. If t is a node, we can extend φ to $\langle A \cup \{t\}\rangle$, by setting $\varphi(t) = t'$, where t' is any node of Γ'. If t is a leaf, $p(t) \in \Gamma$ and we can extend φ to $\langle A \cup \{t\}\rangle$, by setting $\varphi(t) = t'$, and $\varphi(p(t)) = p(t')$, where t' is any leaf of Γ'. $\dashv$

By Facts 1 to 6, the family of partial isomorphisms between finite subsructures of T and T' respectively has the forth (and back) property, which shows quantifier elimination. By Fact 0 this family is not empty whatever T and T' are. If they are countable, Facts 1 to 6 allow us to extend any of these partial isomorphisms to an isomorphism between T and T', which shows $\aleph_0$-categoricity. □

Theorem 4.6. *1. Precolored good trees of depth 1 are exactly the 1-colored good trees. For such a tree its color is its branching color.*

2. If T is such a tree, $M(T)$ is C-minimal, indiscernible and $\aleph_0$-categorical (or finite).

Proof: 1. Let T be a 1-colored good tree of color (m, μ). By quantifier elimination (in the language $\{L, N\}$, $\mathcal{L}_1$ or $\mathcal{L}_1^+$, see Proposition 4.5) all nodes of T have same tree-type. Singletons consisting of a leaf (in case $m \neq 0$) are the border cones and the infinite cones (in case $\mu \neq 0$) are the inner cones. Moreover all leaves have same type. So, any branch of T is the union of its leaf and a one-colored basic interval of color (m, μ) and T is a precolored good tree.
Conversely, it has already been noticed in Remark 4.2 that 1-precolored good trees are 1-colored good trees.
2. The unique atomic formula where a leaf variable α is involved and which does not define in any C-set a finite or cofinite union of cones at a same node, is of the form $p(\alpha) = y$, y a node; but it does in $M(T)$ since the color (∞, ∞) is forbidden at y. By quantifier elimination, this gives C-minimality. Proposition 4.5 has proven $\aleph_0$-categoricity. Indiscernability has be proven in the part 1. of the proof. □

Corollary 4.7. *In a 1-colored good tree T of type $(1.a)$ any cone is elementary equivalent to T. If T of type $(1.b)$ any infinite cone is elementary equivalent to T. If c is a node of T the pruned cone $]-\infty, c[$ is elementary equivalent to T.*

Proof: In all cases the subtree we consider is a 1-colored good tree of same type and same color as T . □

5. Extension of trees

5.1. General construction

Let T and T_0 be two trees. We define $T \rtimes T_0$, the "extension of T by T_0", as the tree consisting of T in which each leaf is replaced by a copy of T_0. More formally, let L_T and N_T be respectively the set of leaves and nodes of T, L_0 and N_0 the set of leaves and nodes of T_0. As a set, $T \rtimes T_0$ is the disjoint union of N_T and $L_T \times T_0$. The order on $T \rtimes T_0$ is defined as follows:

$\forall x, x' \in N_T$, $T \rtimes T_0 \models x \leq x'$ iff $T \models x \leq x'$;

$\forall (\alpha, t), (\alpha', t') \in L_T \times T_0$,

$$T \rtimes T_0 \models (\alpha, t) \leq (\alpha', t') \text{ iff } T \models \alpha = \alpha' \text{ and } T_0 \models t \leq t';$$

$\forall x \in N_T,\ (\alpha, t) \in L_T \times T_0$, $T \rtimes T_0 \models x \leq (\alpha, t)$ iff $T \models x \leq \alpha$.

Note that, by construction, N_T embeds canonically in $T \rtimes T_0$ as an initial subtree of $N_{T \rtimes T_0}$.

Some illustrations will be given at the end of next subsection.

Lemma 5.1. *$T \rtimes T_0$ is a tree.*
If T is a singleton, $T \rtimes T_0$ is the same thing as T_0. If T_0 is a singleton, $T \rtimes T_0$ is the same thing as T.
The set of nodes of $T \rtimes T_0$ is the disjoint union $N_T \cup L_T \times N_0$, its set of leaves is $L_T \times L_0$.
$T \rtimes T_0$ is good if T and T_0 are.
For trees T_1, T_2 and T_3, $(T_1 \rtimes T_2) \rtimes T_3$ and $T_1 \rtimes (T_2 \rtimes T_3)$ are canonically isomorphic trees.

Proof: Clear from the definition. Associativity comes essentially from the associativity of Cartesian product and Boolean union. □

Definition 5.2. *We define the equivalence relation $\sim$ corresponding to the construction of $T \rtimes T_0$:*
- $\sim$ is the equality on N_T;
- on $L_T \times T_0$ equivalence classes are the copies of T_0, id est the subsets $\{\alpha\} \times T_0$ for $\alpha \in L_T$.

Lemma 5.3. *Distinct equivalence classes a, b satisfy: $\exists u \in a, \exists v \in b,\ u < v$ iff $\forall u \in a, \forall v \in b,\ u < v$. Consequently the quotient $T \times T_0/\sim$ inherits the tree structure of $T \times T_0$ and $T \times T_0/\sim$ and T are isomorphic trees.*
The $\sim$-class of any element of N_T is a singleton. Consequently the embedding $N_T \subseteq T \times T_0$ gives when taking $\sim$-classes the embedding $N_T \subseteq T$.

Proof: Clear from definition of the equivalence relation. □

5.2. Extension of good trees

Recall that p denotes the predecessor (partial) function.
From now on T and T_0 are good trees, no singletons, and we require furthermore three conditions.

Definition 5.4. *We define Conditions* $(\star)$, $(\star\star)$ *and* $(\star\star\star)$*:*
$(\star)$ *Either all leaves of* T *are isolated or all leaves of* T *are non isolated.*
$(\star\star)$ *If* T *has non isolated leaves,* T_0 *should have a root.*
$(\star\star\star)$ *If* T *has isolated leaves, then* $p(L_T)$ *is convex, id est* $\forall x, y, z \in T,\ (x, z \in p(L_T) \wedge x < y < z) \to y \in p(L_T)$.

Lemma 5.5. *All 1-colored good trees satisfy Conditions* $(\star)$ *and* $(\star\star\star)$.

Proof: If T is 1-colored of type (0) all leaves of T are isolated and $p(L_T)$ consists of the root. If T is of type $(1.a)$ all leaves are non isolated. If T is of type $(1.b)$, all leaves are isolated and $p(L_T)$ is equal to the set of nodes of T, which is convex. $\square$

As already noticed, $T \rtimes T_0$ is a good tree with set of leaves $L_T \times L_0$ and set of nodes $N_T \cup L_T \times N_0$.
Let us call σ the canonical embedding of N_T in $T_0 \rtimes T$ and, for each $\alpha \in L_T$, τ_α the embedding of T_0 in $T \rtimes T_0$, $x \mapsto (\alpha, x)$.
In the case where T_0 has a root, L_T also embeds in $T \rtimes T_0$ by the map $\rho : \alpha \mapsto (\alpha, r_0)$, where r_0 is the root of T_0. Via σ and ρ, T embeds as an initial subtree of $T \rtimes T_0$ and $\tau_\alpha(T_0)$ is the thick cone at $\rho(\alpha)$.
If T_0 has no root, the embedding of N_T does not extend naturally to an embedding of T into $T \rtimes T_0$ but T will appear as a quotient of $T \rtimes T_0$. Define in this case $\rho : L_T \to T \rtimes T_0$ as the (non injective) map $\alpha \mapsto \sigma \circ p(\alpha)$. Note that by $(\star\star)$, T has isolated leaves hence $p(\alpha)$ is defined and is a node of N_T thus $\sigma \circ p(\alpha)$ is well defined. In this case, $\tau_\alpha(T_0)$ is a cone at $\rho(\alpha)$.
In both cases, $\rho(\alpha) = \inf \tau_\alpha(T_0)$.

From now on we will consider σ as the identity and not write it.

Lemma 5.6. *For any* $(\alpha, t) \in L_T \times T_0$*, if* $cl(\alpha, t)$ *denotes the equivalence class of* (α, t)*, we have:*
- $cl(\alpha, t) = \tau_\alpha(T_0)$.
- *If* T_0 *has a root, say* r_0*,* $cl(\alpha, t)$ *is the thick cone at* $\rho(\alpha)$*; so* $cl(\alpha, t) = cl(\alpha, r_0)$.
- *If* T_0 *has no root,* $cl(\alpha, t)$ *is the cone of* t *at* $\rho(\alpha)$. $\square$

Definition 5.7. *The partial function* $e : T \rtimes T_0 \to: T \rtimes T_0$ *is defined as follows:*
- $Dom(e) = L_T \times T_0$ *if* T_0 *has a root and* $Dom(e) = (L_T \times T_0) \cup p(L_T)$ *if* T_0 *has no root;*
- $\forall (\alpha, t) \in L_T \times T_0$, $e((\alpha, t)) = \rho(\alpha)$*, and if* T_0 *has no root, for any* $\alpha \in L_T$, $e(p(\alpha)) = p(\alpha)$.
We set $E := \rho(L_T)$, $E_{\geq} := \{x; \exists y \in E, y \leq x\}$, $E_{>} := E_{\geq} \setminus E$, $E_{<}$ *the complement of* $E_{\geq}$ *in* $T \rtimes T_0$ *and* $E_{\leq} := E_{<} \cup E$.

Proposition 5.8. *1. If T_0 has a root, then E is an antichain and $x \sim y$ iff ($x = y$ or ($x, y \in Dom(e)$ and $e(x) = e(y)$)).*

2. If T_0 has no root, then $x \sim y$ iff ($x = y$ or ($x, y \in Dom(e)$ and $e(x) = e(y) < x \wedge y$)).

3. In both cases, $\forall \alpha \in L, E \cap br(\alpha)$ has $e(\alpha)$ as a greatest element.

Proof: Assume first that T_0 has a root, say r_0. By definition, for any $(\alpha, t) \in L_T \times T_0 = Dom(e)$, $e((\alpha, t)) = \rho(\alpha) = (\alpha, r_0)$. Hence, E is an antichain. And, for all $(\alpha, \beta) \in L$, $E \cap br((\alpha, \beta)) = \{e((\alpha, \beta))\}$.
Moreover, the equivalence class of (α, t) is the thick cone at $(\alpha, r_0) = e((\alpha, t))$. Therefore, $(\alpha, t) \sim (\alpha', t')$ iff $e((\alpha, t)) = e((\alpha', t'))$.
Assume now that T_0 has no root. Then T has isolated leaves and for any $(\alpha, t) \in L_T \times T_0$, $e((\alpha, t)) = \rho(\alpha) = p(\alpha) = e(p(\alpha))$. By definition, the equivalence class of (α, t) is the cone of t at $\rho(\alpha)$, so $(\alpha, t) \sim (\alpha', t')$ iff $\rho(\alpha) = \rho(\alpha')$ and $(\alpha, t) \wedge (\alpha', t') > \rho(\alpha)$. In other words, $(\alpha, t) \sim (\alpha', t')$ iff $e((\alpha, t)) = e((\alpha', t')) < (\alpha, t) \wedge (\alpha', t')$. This prove the second assertion.
If T_0 has no root, $E = \{p(\alpha); \alpha \in L_T\}$. So let (α, β) be a leaf of $T \rtimes T_0$ and α' be a leaf of T, such that $p(\alpha') \in br((\alpha, \beta))$. Then, $p(\alpha') \leq \alpha$ in T. So, $p(\alpha') \leq p(\alpha) = e(\alpha, \beta)$. Hence, $e(\alpha, \beta)$ is the greatest element of $E \cap br((\alpha, \beta))$. □
The following pictures illustrate extensions $T \rtimes T_0$ with T_0 a 1-colored good tree. They are organized in two groups, the first group has two pictures, the second one three. On the left of both groups is the tree T. On the right the possible kinds of extensions it gives rise to. On the first pair of pictures T has non isolated leaves. So T_0 must have a root, hence be of type (0). On the second group of pictures T has isolated leaves. So T_0 may have or not a root.
As previously, a continuous line means a dense linear order and a dashed line means a gap.

1. T with non isolated leaves.

We have represented only two branches of T. The picture is drawn with T_0 of color $(3, 0)$.

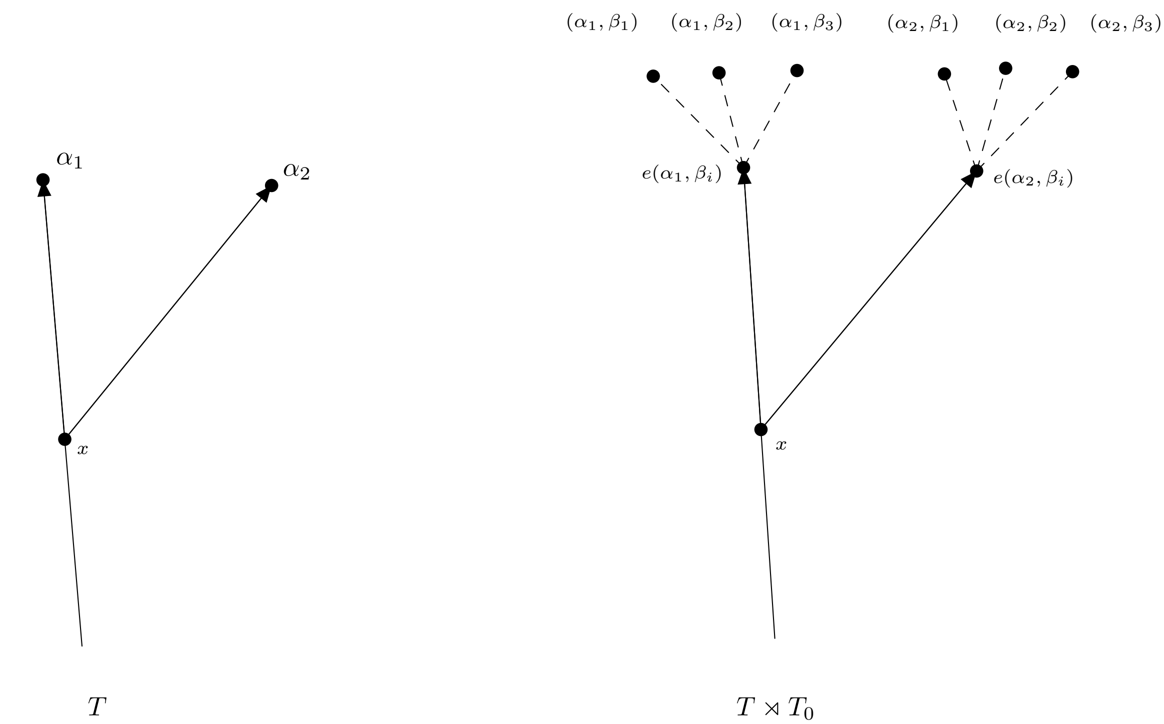

Fig.5

2. T with isolated leaves.

Triangles to the right represent infinite cones, triangles to the left represent unions of cones (finite or infinite cones, depending on trees colors). On the first picture right T_0 has no root, on the last picture it has color $(3, 0)$.

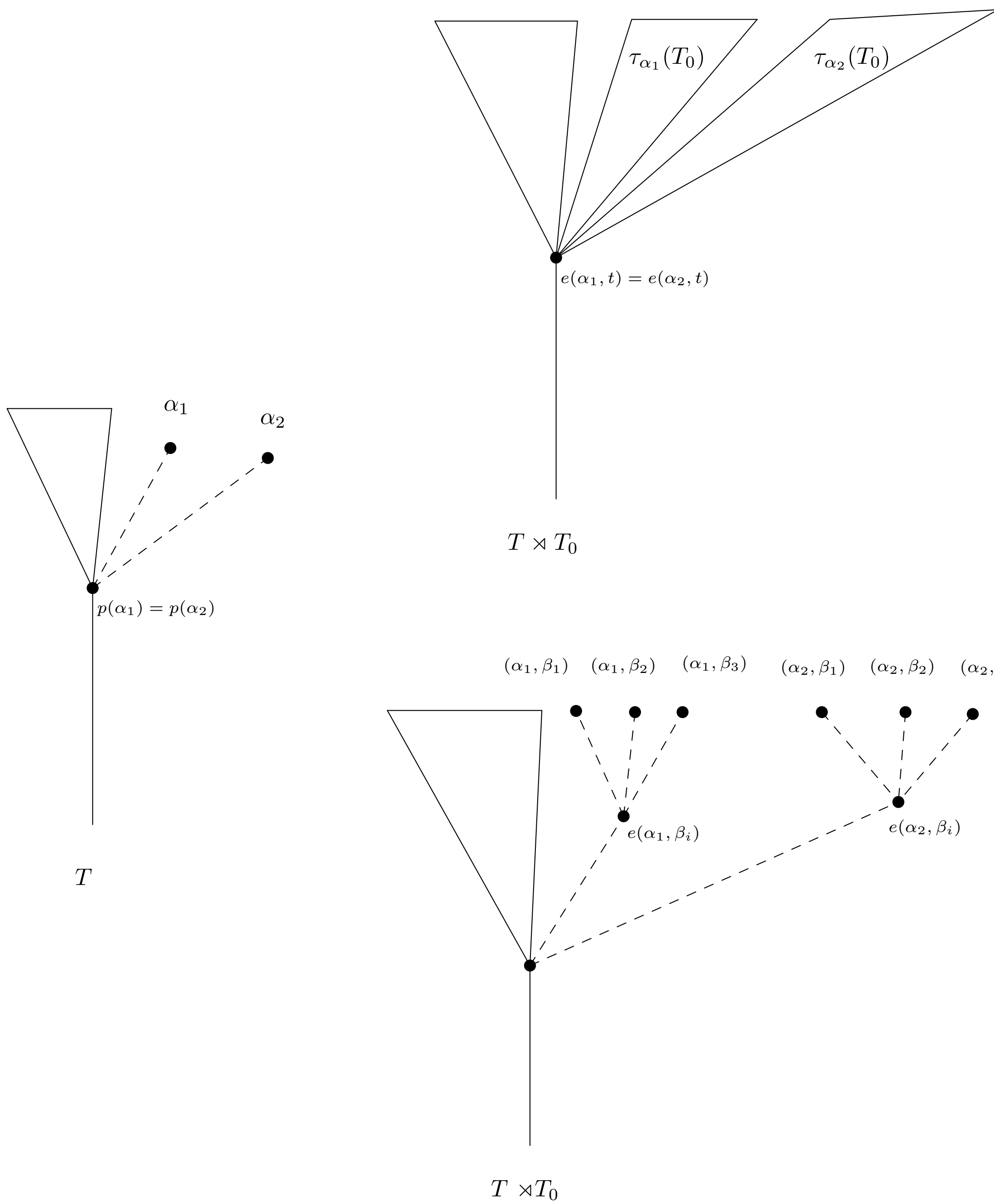

Fig. 6

The tree $T \rtimes T_0$ equipped with E does not know about T and T_0 as shows example below. But it almost does as we will see in Corollary 5.12, first two items.

Example 5.9. *Let* $\cdot\!:$ *and* $\triangleleft$ *be 1-colored good trees of color* $(2,0)$ *and* $(0,2)$ *respectively. Then* $\cdot\!: \rtimes (\cdot\!: \rtimes \triangleleft) = (\cdot\!: \rtimes \cdot\!:) \rtimes \triangleleft$*. Consider on both side of the identity the final extension, namely on the left side the extension with factors* $\cdot\!:$ *and* $\cdot\!: \rtimes \triangleleft$*, and on right side the extension with factors* $\cdot\!: \rtimes \cdot\!:$ *and* $\triangleleft$*. Then, on both sides,* E *consists of the successors of the root But* $\cdot\!: \rtimes \triangleleft$ *has a root while* $\triangleleft$ *has not. Hence, the tree* $T \rtimes T_0$ *equipped with* E *does not even know whether* T_0 *has a root or not.*

5.3. Language and theory of $T \rtimes T_0$

As previously defined $\mathcal{L}_1 = \{\leq, \wedge, N, L\}$. Let $\mathcal{L}_2 := \mathcal{L}_1 \cup \{e, E, E_{\geq}\}$.

We will have to consider on the tree T **some additional structure given by additional unary functions**. As they naturally appear these functions are partial but, again, in model theoretical framework, they have to be defined everywhere. So each such function f appears together with two unary predicates D_f and F_f for the domain and the range of the original f. In this way, **let $\mathcal{F}$ be a finite set of of unary functions and** $\mathcal{P} = \{D_f, F_f; f \in \mathcal{F}\}$ **a set of of unary predicates**. They will be required to satisfy:

Conditions $(4\star)$: for any $f \in \mathcal{F}$,
. $L \subseteq D_f$, $D_f = \{x\,; \exists y \in F_f, y \leq x\}$ and $F_f \cap L = \emptyset$,
. $\forall t \notin D_f,\ f(t) = t$, and $f(D_f) = F_f$,
. $\forall t \in D_f,\ f(t) \leq t$,
. $\forall t \in F_f,\ f(t) = t$.

We define $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{F} \cup \mathcal{P}$ and $\mathcal{L}' = \mathcal{L}_2 \cup \mathcal{F} \cup \mathcal{P}$. Note that Conditions $(4\star)$ are first order in $\mathcal{L}$. We interpret $\mathcal{L}'$ on $T \rtimes T_0$ as follows:
- we have already defined the $\mathcal{L}_2$-structure;
- for f a function in $\mathcal{F}$:
. $F_f^{T \rtimes T_0} = F_f^T$ and $D_f^{T \rtimes T_0} = (D_f^T \cap N_T) \dot\cup L_T \times T_0$ (recall that N_T embeds as an initial subtree in $T \rtimes T_0$);
. $\forall x \in (D_f^T \cap N_T)$, $f^{T \rtimes T_0}(x) = f^T(x)$ and
$\forall (\alpha, t) \in L_T \times T_0$, $f^{T \rtimes T_0}(\alpha, t) = f^T(\alpha)$ (which belongs to N_T since $L_T \subseteq D_f^T$ and $f(D_f^T) \cap L_T = \emptyset$ (both conditions due to $(4\star)$) hence to $N_{T \rtimes T_0}$).
Conditions $(4\star)$ are true on $T \rtimes T_0$ for the set of functions $\mathcal{F} \cup \{e\}$, $D_e = E_{\geq}$ and $F_e = E$.

We will see (in Corollary 5.12) how the construction of $T \rtimes T_0$ can be retraced in its $\mathcal{L}_2$-theory up to the phenomenon pointed out in Example 5.9, and also that the definition of its $\mathcal{L}'$-structure is canonical (in Lemma 5.13).

Definition 5.10. *Let* Σ'' *be the following theory in the language* $\mathcal{L}_2$*:*
- $(\leq, \wedge)$ *is a good tree;*

- *E is convex:* $\forall x, y, z,\ (x, z \in E\ \wedge\ x < y < z) \to y \in E$;
- $D_e = E_{\geq}$;
- $E = e(D_e) = e(L)$ *and* $\forall x \notin D_e, e(x) = x$;
- $L \subseteq D_e$ *and* $E \cap L = \emptyset$;
- $\forall x, e(x) \leq x$;
- $\forall x \in D_e,\ E \cap br(x)$ *has* $e(x)$ *as a greatest element, where* $br(x) := \{y; y \leq x\}$.

In models of Σ'', $E_{\geq}$ is the same thing as D_e and is therefore quantifier free definable. This allows us to use freely notations $E_{\geq}$, $E_{<}$, $E_{\leq}$ or $E_{>}$.
In the following statement cases (1) and (2) correspond to the two possible extensions producing a same model of Σ'', as seen in Example 5.9.

Lemma 5.11. *Let Λ be a model of Σ''. Consider on Λ a binary relation $\sim$ such that:*
- either E is an antichain and
either (1): $x \sim y$ iff $(x, y \in E_{<}$ and $x = y)$ or $(x, y \in E_{\geq}$ and $e(x) = e(y))$,
or (2): $x \sim y$ iff $(x, y \in E_{<}$ and $x = y)$ or $(x, y \in E_{\geq}$ and $e(x) = e(y) < x \wedge y)$,
- or E is not an antichain and (2).
Then $\sim$ is an equivalence relation compatible with the order in the sense of Lemma 5.3. More precisely, for $x \in \Lambda$ such that the class $\bar{x}$ of x is not a singleton, then $\bar{x} = \Gamma(e(x))$ in case (1) and $\bar{x} = \Gamma(e(x), x)$ in case (2).

Proof: Let $x \in \Lambda$ such that $\bar{x}$ is not a singleton.
Let $y \in \bar{x}$, then $x, y \in E_{\geq}$ and by definition of $\sim$, $e(y) = e(x)$. Since $e(y) \leq y$, $y \in \Gamma(e(x))$. If we are in case (2), $e(x) = e(y) < x \wedge y$, thus $y \in \Gamma(e(x), x)$.
Conversely, let $y \in \Gamma(e(x))$, then $y \in E_{\geq}$ and $e(x) \leq x \wedge y$. Since $e(x) \leq y$ and $e(y) \leq y$, $e(x)$ and $e(y)$ are comparable. In case (1), E is an antichain, thus $e(x) = e(y)$. Assume now $y \in \Gamma(e(x), x)$, so $x \wedge y > e(x)$. Then, $e(x) \in br(y) \cap E$, hence $e(x) \leq e(y)$. If $x \wedge y \leq e(y)$, then by convexity of E, $x \wedge y \in E$ so $x \wedge y \leq e(x)$ which gives a contradiction. Thus, $e(y) < x \wedge y$ therefore, $e(y) \leq e(x)$. Finally, $e(x) = e(y) < x \wedge y$. □

Notations Let Λ be a model of Σ'', $\sim$ as above. We denote $\bar{\Lambda}$ the good tree $\bar{\Lambda} := \Lambda / \sim$; and, for $x \in \Lambda$, $\bar{x}$ the equivalence class of x in $\bar{\Lambda}$.

Corollary 5.12. *Let Λ and cases (1) and (2) be as in Lemma 5.11.*

1. *In case (1), Λ is the disjoint union $E_{<} \dot{\cup} \dot{\bigcup}_{x \in E} \Gamma(x)$, where $E_{\leq}$ is an initial subtree, E is an antichain and $\sim$ is the identity on $E_{<}$. Hence $\bar{\Lambda}$ is a tree canonically isomorphic to $E_{\leq}$ with E its set of leaves. If all thick cones $\Gamma(x)$, $x \in E$ are isomorphic trees, say all isomorphic to Γ_0, then $\Lambda = \bar{\Lambda} \rtimes \Gamma_0$.*

2. *In case (2), $\Lambda = E_{\leq} \dot{\cup} \dot{\bigcup}_{x \in E_{>}} \Gamma(e(x); x)$ with $E_{\leq}$ an initial subtree and $\sim$ the equality on $E_{\leq}$; $E_{\leq}$ embeds canonically in the tree of nodes of $\bar{\Lambda}$. If all cones $\Gamma(e(x), x)$, $x \in E_{>}$, are isomorphic trees, say all isomorphic to Γ_0, then $\Lambda = \bar{\Lambda} \rtimes \Gamma_0$.*

3. *In both cases, $E_{\leq}$ can be identified with $\bar{E}_{\leq} := \{\bar{x}; x \in E_{\leq}\}$ and E with $\bar{E} := \{\bar{x}; x \in E\}$ and considered as living in $\bar{\Lambda}$.*

Proof: 1. In this case E is an antichain and by definition of the relation $\sim$, Λ is the disjoint union of an initial tree with the union of disjoint final trees indexed by points from E, namely $\Lambda = E_{<}\dot{\cup}\dot{\bigcup}_{x\in E}\Gamma(x)$ which is also $E_{\leq}\dot{\cup}E_{>}$, with $\sim$ the equality on $E_{\leq}$ and $\bar{x} = \overline{e(x)}$ for $x \in E_{>}$. Thus the inclusion $E_{\leq} \subseteq \Lambda$ induces the equality $E_{\leq} = \bar{\Lambda}$ where more precisely $E_{<}$ is identified with the set of nodes of $\bar{\Lambda}$ and E with its set of leaves.
2. By definition of $\sim$ in case (2), Λ has the form indicated. Hence the inclusion $E_{\leq} \subseteq \Lambda$ induces an inclusion $E_{\leq} \subseteq \bar{\Lambda}$. Take any $c \in E$. By axioms of Σ'', $c = e(\alpha)$ for some leaf $\alpha \geq c$. Since $E \cap L = \emptyset$, $\alpha > c$ and $c = \alpha \wedge \beta$ for another leaf $\beta \neq \alpha$. If $e(\beta) \neq e(\alpha)$ then $\bar{\beta} \neq \bar{\alpha}$ hence $\bar{c}$ is a node of $\bar{\Lambda}$. If $e(x) = e(\alpha)$ for any leaf x such that $c = \alpha \wedge x$, then any cone at c is an equivalence class; now there are at least two different cones, hence, again, $\bar{c}$ is a node of $\bar{\Lambda}$. Thus $E_{\leq}$ is contained in the set of nodes of $\bar{\Lambda}$.
3. Follows directly from 1. and 2. □

Lemma 5.13. *Suppose furthermore $\bar{\Lambda}$ equipped with an $\mathcal{L}$-structure model of $(4\star)$. For $f \in \mathcal{F}$ we note $\bar{f}$ the interpretation in $\bar{\Lambda}$ of the symbol f from $\mathcal{L}$. Then there is exactly one $\mathcal{L}'$-structure on Λ defined as follows: for each function $f \in \mathcal{F}$:*

1. *For $x \in E_{\leq}$, $x \in D_f$ iff, in $\bar{\Lambda}$, $\bar{x} \in D_{\bar{f}}$ and in this case $f(x)$ is the unique $y \in E_{\leq}$ such that $\bar{y} = \bar{f}(\bar{x})$ in $\bar{\Lambda}$.*

2. *For $x \in E_{\geq}$, $f(x) = f(e(x))$.*

This $\mathcal{L}'$-structure on Λ satisfies conditions $(4*)$ for the set of functions $\mathcal{F}\cup\{e\}$ with $F_e = E$ and $F_f = F_{\bar{f}}$ (following the identification stated in Corollary 5.12, (3)) for $f \in \mathcal{F}$.

Proof: The uniqueness of y in 1 is given by Corollary 5.12 and 1 and 2 are compatible since e is the identity on E. For $f \in \mathcal{F}$ and $x \in E_{\geq}$, $f(x) \in E_{\leq}$; now $e(x) := max(E\cap br(x))$ hence $f(x) \leq e(x) \leq x$; for $x \in E_{\leq}$, "$f(x) = \overline{f(x)} \leq \bar{x} = x$". Other Conditions $(4\star)$ for f on Λ follow from $E \cap L = \emptyset$ and Conditions $(4\star)$ for $\bar{f}$ on $\bar{\Lambda}$. □

Definition 5.14. *Let T and T_0 be good trees satisfying conditions $(\star)$, $(\star\star)$ and $(\star\star\star)$. Assume T furthermore equipped with an $\mathcal{L}$-structure model of $(4\star)$. We introduce the theory Σ' in the language $\mathcal{L}'$ consisting of Σ'' strengthened as follows. Let $\sim$ be the relation defined as in Lemma 5.11, (1) if T_0 has a root and (2) if it has not. Then we add the axioms and axiom schemes:*
- for any $f \in \mathcal{F}$, conditions 1 and 2 of Lemma 5.13;
- for all $x \in E_{\geq}$ if T_0 has a root or $x \in E_{>}$ if T_0 has no root, the $\sim$-class of x is elementary equivalent to T_0 (as a pure tree);
- the quotient modulo $\sim$ and T are elementary equivalent $\mathcal{L}$-structures;

- if T_0 has no root then by Condition $(\star\star)$ leaves of the quotient modulo $\sim$ have a predecessor and, interpreted in the quotient modulo $\sim$, $\bar{E} = \bar{p}(\bar{L})$, where $\bar{L}$ and $\bar{p}$ denote the interpretation in $\bar{\Lambda}$ of the symbols L and p.

Proposition 5.15. *Σ' is a complete axiomatization of $T \rtimes T_0$. If $(T, \mathcal{L})$ and T_0 are $\aleph_0$-categorical or finite then Σ' has a unique finite or countable model.*

Proof: We note first that $T \rtimes T_0$ is a model of Σ'. We prove now the completion of this theory.
Assume first that T_0 has a root. Take $\Lambda \models \Sigma'$. Assume CH for short and Λ as well as T and T_0 saturated of cardinality finite or $\aleph_1$. As an $\mathcal{L}_2$-structure, Λ must be the extension $T \rtimes T_0$ described in Corollary 5.12, case (1). By Lemma 5.13 the rest of the $\mathcal{L}$-structure on Λ as well is determined by its restriction to $E_{\leq}$ id est by the $\mathcal{L}$-structure T. So Σ' has a unique saturated model of cardinality finite or $\aleph_1$. This shows the completeness of Σ'.
We consider now the case where T_0 has no root and suppose as previously that Λ, T and T_0 are saturated of cardinality finite or $\aleph_1$. This time $\Lambda = E_{\leq} \dot{\cup} \dot{\bigcup}_{x \in E_>} \Gamma(e(x); x)$ and $N_T = E_{\leq}$ (recall that, by Lemma 5.12 (3), N_T lives also in Λ). By the third axiom scheme, $\bar{E} = \bar{p}(L_T)$ hence the $\mathcal{L}_2$-structure on Λ must be the extension $T \rtimes T_0$ described in Corollary 5.12, case (2). By Lemma 5.13 again the rest of the $\mathcal{L}$-structure on Λ is determined by the $\mathcal{L}$-structure T. This shows the uniqueness of the saturated model of cardinality finite or $\aleph_1$ and the completeness of Σ'.
If T and T_0 are the unique finite or countable models of their respective theory, we show in the same way as above that $T \rtimes T_0$ is the unique finite or countable model of Σ', which shows that this theory is $\aleph_0$-categorical too. $\square$

Definition 5.16. *If Σ is a complete axiomatization of T as an $\mathcal{L}$-structure and Σ_0 is a complete axiomatization of T_0 (in $\mathcal{L}_1$), $\Sigma \rtimes \Sigma_0$ will denote the theory Σ' (of $\mathcal{L}'$).*

5.4. When T_0 is 1-colored

In this section we work under the additional assumption that T_0 is 1-colored. We show that, in this case the properties we are interested in transfer from T to $T \rtimes T_0$.
Let us recall (see Section 2) that $M(T)$ and $M(T \rtimes T_0)$ denote the C-structures with canonical trees T and $T \rtimes T_0$ respectively.

Proposition 5.17. *If T eliminates quantifiers in $\mathcal{L} \cup \{p, D_p, F_p\}$ (as defined in 4.4), then $\Sigma \rtimes \Sigma_0$ eliminates quantifiers in $\mathcal{L}' \cup \{p, D_p, F_p\}$.*

Proof: We keep notations of the proof of Proposition 5.15. So T, T_0 and $\Lambda = T \rtimes T_0$ are the finite or $\aleph_1$-saturated models of Σ, Σ_0 and $\Sigma \rtimes \Sigma_0$ respectively. Take any finite tuple from Λ. Close this tuple under e. Write it in the form $(x, y_1, \ldots, y_m)$ where x is a tuple from $E_{\leq}$, $y_1, \ldots, y_m$ tuples from $E_>$ such that all components of each y_i have same image under e, call it $e(y_i)$ (thus, $e(y_1), \ldots, e(y_m)$ are among coordinates of x), and $e(y_i) \neq e(y_j)$

for $i \neq j$. Take $(x', y'_1, \dots, y'_m) \in \Lambda$ having same quantifier free $(\mathcal{L}' \cup \{p, D_p, F_p\})$-type than $(x, y_1, \dots, y_m)$. Thus $x' \in E_{\leq}$ and $(y'_1, \dots, y'_m) \in E_{>}$. Since $E_{\leq}$ embeds canonically in T, we may see x and x' as living in T, where they have same complete type if $\mathcal{L} \cup \{p, D_p, F_p\}$ eliminates quantifier of T. Thus there is an automorphism σ of T sending x to x'. Any automorphism, say f, of Λ extending $\sigma \restriction E_{\leq}$ will send for each i, $e(y_i)$ to $\sigma(e(y_i))$. Hence $f(y_i)$ and y'_i are in the same copy of T_0, say T_0^i.
Assume first that T_0 is of type (0). Since T_0 consists of one root and leaves and $f(y_i)$ as well as y'_i consists of distinct leaves, there is an automorphism σ_i of T_0^i sending $f(y_i)$ to y'_i. The union of σ, the σ_i and the identity on other copies of T_0 is an automorphism of Λ sending $(x, y_1, \dots, y_m)$ to $(x', y'_1, \dots, y'_m)$.
We consider now the case where T_0 has no root thus $E_{\leq} = N_T$ and
$\Lambda = E_{\leq} \,\dot\cup\, \dot\bigcup\{\Gamma(e(z)\,;z)\,; z \in E_{>}\}$.
- If T_0 is of type $(1.a)$, it eliminates quantifier in $\mathcal{L}_1$ which gives σ_i as above.
- If T_0 is of type $(1.b)$, it eliminates quantifiers in $\mathcal{L}_1 \cup \{p\}$ and in T_0 the interpretation of D_p is L_{T_0}. For each embedding of T_0 in Λ as a cone $\Gamma(e(x); x)$ we have the inclusions $L_{T_0} \subseteq L_\Lambda \subseteq D_{p_\Lambda}$ and for any leaf α of (this) T_0, $p_{T_0}(\alpha) = p_\Lambda(\alpha)$. Thus, for each i, $f(y_i)$ and y'_i have same type in T_0, which gives σ_i as previously.
In all cases, the automorphism of Λ we have constructed respects the language $\mathcal{L}' \cup \{p_L, p_L(L)\}$ where p_L is the restriction of the predecessor function to the set of leaves and $p_L(L)$ its image. Thus we have shown that Λ eliminates quantifier in this language. Now adding p_L to $\mathcal{L}'$ is quantifier free equivalent to adding p: $D_p = (D_{\bar{p}} \cap E_{\leq}) \cup L$ and p coincides with $\bar{p}$ on $D_{\bar{p}} \cap E_{\leq}$ and with p_L on L. □

Proposition 5.18. *Consider on $M(T \rtimes T_0)$ and $M(T)$ the structure induced by their canonical tree, respectively $(T \rtimes T_0, \mathcal{L}')$ and $(T, \mathcal{L})$. Then $M(T \rtimes T_0)$ is C-minimal iff $M(T)$ is.*

Proof: Let $\Lambda \models \Sigma \rtimes \Sigma_0$. For $A \subseteq L(\bar{\Lambda})$, $A_\Lambda := \{\alpha \in L(\Lambda); \bar{\alpha} \in A\}$ is a cone in $\bar{\Lambda}$ iff A is a cone in Λ, of same type (thick or not) except when A consists of a non isolated leaf (in $\bar{\Lambda}$) and A_Λ a is a cone. This proves two things. First $\bar{\Lambda}$ is C-minimal if Λ is. Secondly if $\bar{\Lambda}$ is C-minimal any subset of Λ of the form A_Λ is a Boolean combination of cones and thick cones. The general case is processed by hand.

Fact: For x a leaf of Λ, a composition of functions from $\mathcal{F} \cup \{p, e\}$ applied to x is, up to equality, a constant or of the form $x, p(x)$ (necessary only if T_0 is of type $(1.b)$) or $t(e(x))$ where t is a composition of functions from $\mathcal{F} \cup \{p\}$ (hence a term of $\mathcal{L} \cup \{p\}$).

Assume the first function right in the term is p. If T_0 is of type (0) we replace p with e. If T_0 is of type $(1.b)$, $p(x) \notin D_p$. Conclusion: at most one p right. If a term t is a composition of functions from $\mathcal{F} \cup \{e\}$, then $t(p(x)) = t(x)$. Indeed, $e(x) < x$ if $x \in L$ hence $e(x) = e(p(x))$ (by definition $e(x) = max(E \cap br_x)$), and $f(x) = f(e(x))$. Conclusion: in composition no p right needed. Finally, for $f \in \mathcal{F} \cup \{e\}$, $f(x) = f(e(x))$. So, if a term is neither x nor $p(x)$, we may assume it begins right with the function e. $\dashv$

So non constant terms in x are all smaller that x, thus linearly ordered. Consequently, up to a definable partition of $L(\Lambda)$ (namely into the two sets $\{x; t(x) \geq t'(x)\}$ and $\{x; t(x) < t'(x)\}$), terms of the form $t(x) \wedge t'(x)$ are not to be considered. To summarize, it is enough to consider subsets definable by formulas $t(x) \leq t'(x)$, $t(x) = t'(x)$, $t(x) \leq a$, $t(x) = a$, $t(x) \in E, D_e, F_f$ or D_f, and $(t(x) \wedge a) = b$ where t and t' are of the form described in the above fact. To φ a one variable formula from $\mathcal{L}$ without constant associate a formula φ_Λ (also from $\mathcal{L}$, one variable and without constant) such that $\Lambda \models \varphi_\Lambda(x)$ iff $\bar{\Lambda} \models \varphi(\bar{x})$. Then $\varphi(e(x))$ is equivalent to:
- $\varphi_\Lambda(x)$ when T_0 has a root, and
- $\psi_\Lambda(x)$ with $\psi(y) = \varphi(\bar{p}(y))$ when T_0 has no root,
both already handled. Are left to be considered:
- $t(e(x)) < x$ and, if T_0 is of type $(1.b)$, $t(e(x)) < p(x) < x$ are always true,
- $x \in E, F_f$ always wrong, as $p(x) \in E, F_f$ are since $p(x)$ occurs only if T_0 is if type $(1.b)$,
- $x, p(x) \in E_{\geq}, D_f$ always true,
- $t(x)\Box b$ and $(t(x) \wedge a)\Box b$ with $\Box \in \{<, =, >\}$, formulas that we treat now.
For $b \in E_>$, $t(e(x)) \geq b$ is always wrong and $t(e(x)) < b$ is equivalent to $t(e(x)) < e(b)$. For $b \in E_{\leq}$, $\Lambda \models t(e(x))\Box b$ iff $\bar{\Lambda} \models t(e(x))\Box\bar{b}$. For $b \in E_>$, $(t(e(x)) \wedge a) \geq b$ is always wrong and $(t(e(x)) \wedge a) < b$ iff $(t(e(x)) \wedge a) < e(b)$. For $a \in E_>$, $(t(e(x)) \wedge a) = (t(e(x)) \wedge e(a))$. Finally, for a and b in $E_{\leq}$, $\Lambda \models (t(e(x)) \wedge a)\Box b$ iff $\bar{\Lambda} \models (t(e(x)) \wedge a)\Box\bar{b}$. We are left with formulas $x\Box b$, $p(x)\Box b$, $(x \wedge a)\Box b$ and $(p(x) \wedge a)\Box b$ which are routine. $\Box$

Proposition 5.19. *As previously consider on $M(T \rtimes T_0)$ and $M(T)$ the structure induced by their canonical tree. Then $M(T \rtimes T_0)$ is indiscernible iff $M(T)$ is.*

Proof: The right-to-left implication follows clearly from our proof of C-minimality transfer from $L(T)$ to $L(T \rtimes T_0)$. The other direction is trivial since T is a definable quotient of $L(T \rtimes T_0)$ (and leaves are sent to leaves in the quotient). $\Box$

We conclude this section with an uniformity result:

Proposition 5.20. *1. The tree T_0 has a root iff the $\mathcal{L}_2$-structure $T \rtimes T_0$ satisfies both sentences " $\forall x \in L$, $x \in D_p$ and: $\forall x, y \in L, \neg(p(x) < p(y))$.*

2. The equivalence relation $\sim$ is $\mathcal{L}_2$-definable, uniformly in T and uniformly in T_0. This makes T uniformly $\mathcal{L}_2$-interpretable in $T \rtimes T_0$.

Proof: 1. Note that an element (α, β) of $L_T \times T_0 = L_{T \rtimes T_0}$ belongs to D_p iff β has a predecessor in T_0 and in this case $p((\alpha, \beta)) = (\alpha, p(\beta))$. So $D_p \supseteq L$ iff T_0 is of type (0) or $(1.b)$. In this case $p(\alpha, \beta) = (\alpha, p(\beta))$.
Assume first that T_0 has a root, say r_0.
Let (α, β), (α', β') be two leaves of $T \rtimes T_0$ such that $p(\alpha, \beta) \leq p(\alpha', \beta')$. By definition of the order in $T \rtimes T_0$ and the remark three lines above, $\alpha = \alpha'$ and $p(\beta) \leq p(\beta')$. But

$p(\beta) = p(\beta') = r_0$, thus $p(\alpha, \beta) = p(\alpha', \beta')$. So the second sentence of (1) is satisfied. Assume now that T_0 is of type $(1.b)$. Let (α, β) be a leaf of $T \rtimes T_0$, then by definition of such a 1-colored good tree, in T_0 any element of $]-\infty, p(\beta)[$ is the predecessor of a leaf, say $p(\beta')$. So we have in $T \rtimes T_0$, $p(\alpha, \beta) < p(\alpha, \beta')$.
2. The first item above allows us to first order distinguish whether T_0 has a root or not. Items 1 and 2 of Lemma 5.11 give the fitting definitions for both cases. $\square$

To rethink of Example 5.9, the previous proposition tells us that, if $T \rtimes T_0$ knows that T_0 is 1-colored, then it knows also whether or not T_0 has a root.

6. Solvable and general colored good trees

Lemma 6.1. *Let T_0 be a 1-colored good tree and T a good tree such that $T \rtimes T_0$ is well defined. Then:*
- T_0 satisfies Condition $(\star\star\star)$;
- leaves of $T \rtimes T_0$ are isolated iff leaves of T_0 are, and $T \rtimes T_0$ satisfies Condition $(\star)$;
- $T \rtimes T_0$ satisfies Condition $(\star\star\star)$.

Proof: The first assertion comes from Lemma 5.5. The second one is clear. Let us prove the third one. If $T \rtimes T_0$ has isolated leaves then T_0 is of type (0) or $(1.b)$ and for all $(\alpha, \beta) \in L_{T \rtimes T_0}$, $p(\alpha, \beta) = (\alpha, p(\beta)) \in L_T \times T_0$. If T_0 is of type (0) T embeds canonically in $T \rtimes T_0$ and, via this embedding, $p(L_{T \rtimes T_0}) = L_T$, an antichain in $T \rtimes T_0$ hence convex. If T_0 is of type $(1.b)$ then $p(L_{T \rtimes T_0}) = (T \rtimes T_0) \setminus (L_{T \rtimes T_0} \cup N_T)$ which is clearly convex. $\square$

Definition 6.2. *A* solvable good tree *is either a singleton or a tree of the form* $(\dots(T_1 \rtimes T_2) \rtimes \cdots) \rtimes T_n$ *for some integer* $n \geq 1$*, where* $T_1, \cdots, T_n$ *are* 1*-colored good trees such that, for each* i*,* $1 \leq i \leq n-1$*, if* T_i *is of type* $(1.a)$ *then* T_{i+1} *is of type* (0).

Remark 6.3. *- By Lemma 6.1 and an easy induction on n, $(\dots(T_1 \rtimes T_2) \rtimes \cdots) \rtimes T_n$ is a well defined good tree.*
- Taking into account extension associativity proven in Lemma 5.1 we will allow ourselves to write simply $T_1 \rtimes \cdots \rtimes T_n$ instead of $(\dots(T_1 \rtimes T_2) \rtimes \cdots) \rtimes T_n$.
- If $T_1 \rtimes \cdots \rtimes T_n$ is a solvable good tree as in Definition 6.2 then for any $k \leq n$, $T_1 \rtimes \cdots \rtimes T_k$ and $T_{k+1} \rtimes \cdots \rtimes T_n$ are solvable good trees.
- Conversely, let $T' = T_1 \rtimes \cdots \rtimes T_n$ and $T'' = T_{n+1} \rtimes \cdots \rtimes T_{n+m}$ be solvable good trees as in Definition 6.2 and such that, if T_n is of type $(1.a)$ *then T_{n+1} is of type* (0). *Then $T' \rtimes T'' = T_1 \rtimes \cdots \rtimes T_{n+m}$ and $T' \rtimes T''$ is a solvable good tree.*
- T is a solvable good tree iff it is either a singleton or a 1*-colored good tree or of the form $T = T' \rtimes T_n$ for T' a solvable good tree which is not a singleton and T_n a* 1*-colored good tree.*

One difficulty is that a solvable good tree may have decompositions into iterated extensions of 1-colored good trees of different lengths.

Example 6.4. *1. Consider the extension* $T = T_1 \rtimes T_2$ *where* T_1 *and* T_2 *are 1-colored. If* T_1 *is of type (1.b) of color, say* $(1,1)$ *and* T_2 *is of type (1.a) of color* $(0,2)$*, then* $T_1 \rtimes T_2$ *is still 1-colored of type (1.a) of color* $(0,2)$.
2. Consider now the extension $T_1 \rtimes T_2 \rtimes T_3$ *where* T_1, T_2 *and* T_3 *are 1-colored. If* T_1 *and* T_3 *are of type (1.a) of color* $(0,m)$ *and* T_2 *is of type (0) of color* $(m,0)$*, then* $T_1 \rtimes T_2 \rtimes T_3$ *is again of type (1.a) of color* $(0,m)$.

We will now do two things: introduce technical tools in order to characterize decompositions of minimal length and find all exceptional situations where two or more terms of the decomposition "collapse".

Definition 6.5. *Let* T *be a good tree and* x *a node of* T*. Extending Definition 4.1, we call* branching color *of* x *and we note b-col*$_T(x)$ *the couple* $(m_T(x), \mu_T(x))$, $m_T(x), \mu_T(x) \in \mathbb{N} \cup \{\infty\}$*, where* $m_T(x)$ *is the number of cones at* x *which are also thick cones (in other words the number of elements of* T *which have* x *as a predecessor) and* $\mu_T(x)$ *is the number of cones at* x *which are not thick cones.*

In a pure solvable tree $T = T' \rtimes T_n$ in all "non exceptional situations" we will be able to define in terms of change of branching color the function e associated to the extension $T' \rtimes T_n$.

Remark 6.6. *- Branching color is definable in the pure order of* T *in the sense of Lemma 2.14 (no* $\aleph_0$*-categoricity needed now).*
- Let T *be a* 1*-colored good tree. Then the branching color of any of its nodes is its color in the sense of Definition 2.17 (so the same for any node of* T*).*

Lemma 6.7. *Let* T' *be a solvable good tree, not a singleton, and* T_n *a 1-colored good tree such that* $T := T' \rtimes T_n$ *is well defined. Let* $E, E_>$ *and* $E_<$ *be as in Definition 5.7. Then for any* $x \in N_T$,
- if $x \in E_<$*, then b-col*$_T(x) =$ *b-col*$_{T'}(x)$*;*
- if $x \in E$ *and* T_n *has a root, then b-col*$_T(x)$ *is the branching color (in* T_n*) of the root of* T_n *(hence of the form* $(m,0)$*);*
- if $x \in E$ *and* T_n *has no root, then b-col*$_T(x) = (0, \mu_{T'}(x) + m_{T'}(x))$*;*
- if $x \in E_>$*, then b-col*$_T(x)$ *is the color of any node of* T_n*.*

Proof: Clear by construction of $T' \rtimes T_n$. □

Proposition 6.8. *Let* $T = T' \rtimes T_n$ *as in Lemma 6.7 and* e *be as Definition 5.7. Then the function* e *is definable in the pure order except when* T_n *is of type* $(1.a)$ *of color* $(0, \mu_n)$ *and, if* $T' = T_{n-1}$ *or* $T' = T^- \rtimes T_{n-1}$ *for* T_{n-1} *a 1-colored good tree as given by Remark 6.3, then, either:*
Exception 1: T_{n-1} *is* 1*-colored of type* $(1.b)$ *of color* (m_{n-1}, μ_{n-1}) *and* $\mu_n = m_{n-1} + \mu_{n-1}$ *or,*

Exception 2: T_{n-1} *is* 1*-colored of type* (0) *and, if* $T^- = T_{n-2}$ *or* $T^- = T^= \rtimes T_{n-2}$ *for* T_{n-2} *a 1-colored good tree, then* T_{n-2} *is of type* $(1.a)$ *of color* $(0, \mu_{n-2})$ *and* $\mu_{n-2} = m_{n-1} = \mu_n$*.*

Proof: In this proof "definable" means "definable in the pure order". Note that if the restriction of e to L_T is definable, then $E_{\geq} = \{x \in T; \exists \alpha \in L_T,\ x \geq e(\alpha)\}$ is definable and for all $x \in E_{\geq}$, $e(x) = e(\alpha)$ for any $\alpha \in L_T$, $\alpha \geq x$, so e is definable.
If T_n has a root, then $e(\alpha) = p(\alpha)$, for any $\alpha \in L_T$, so e is definable.
If T_n is of type $(1.b)$, then by Lemma 6.7, the color of any element of $E_>$ is (m_n, μ_n), with $m_n \neq 0$, while, if $x \in E$, $b\text{-}col_T(x) = (0, \mu_{T'}(x) + m_{T'}(x))$. Therefore, for any $\alpha \in L_T$, $e(\alpha) = max\ (br(\alpha) \cap \{x \in N; b\text{-}col(x) = (0, \mu)\})$, so e is definable.
So from now on T_n is of type $(1.a)$ hence, by Condition $(\star\star)$, T_{n-1} is of type (0) or $(1.b)$. We will prove that if T satifies neither conditions of Exception 1 nor conditions of Exception 2, then e is definable.
Again by Lemma 6.7, the branching color of any element of $E_>$ is $(0, \mu_n)$, and if $x \in E$, then $b\text{-}col_T(x) = (0, \mu_{T'}(x) + m_{T'}(x))$. We are going to apply one more time Lemma 6.7, this time to the extension $T' = T^- \rtimes T_{n-1}$ and its corresponding subsets $E'_<$, E' and $E'_>$.
If T_{n-1} is of type $(1.b)$, then $E \subset E'_>$, therefore for any $x \in E$, the branching color of x in T' is its branching color in T_{n-1}. If the first Exception is not realized, then $\mu_n \neq m_{n-1} + \mu_{n-1}$ and e is definable as follows: for any $\alpha \in L_T$, $e(\alpha) = max\ (br(\alpha) \cap \{x \in N_T; b\text{-}col(x) = (0, m_{n-1} + \mu_{n-1})\})$.
If T_{n-1} is of type (0), $E = E'$, hence for any $x \in E$, $b\text{-}col_{T'}(x) = (m_{n-1}, 0)$, so $b\text{-}col_T(x) = (0, m_{n-1})$. Therefore if $\mu_n \neq m_{n-1}$, e is definable as above. Now, if $\mu_n = m_{n-1}$, we must consider the branching colors of nodes of $E'_<$ thus we must look down at the tree $T^- = T^= \rtimes T_{n-2}$ and its corresponding subsets $E^-, E^-_<$ and $E^-_>$. If T_{n-2} is of type (0), or $(1.b)$, by the previous discussion E^- is definable in the pure order and $E' = E$ is the subset of all successors of nodes of E^-, hence definable in the pure order too. If T_{n-2} is of type $(1.a)$, then the branching color of the nodes of $E^-_>$ is $(0, \mu_{n-2})$. If the second Exception is not realized, $\mu_{n-2} \neq m_{n-1}$, so as previously, the function e is definable. $\square$

Definition 6.9. *We define* n-solvable good trees *by induction on* $n \in \mathbb{N}$*:*
- a 0*-solvable good tree is a singleton;*
- a 1*-solvable good tree is the same thing as a* 1*-colored good tree;*
- an $(n+1)$*-solvable good tree is a tree of the form* $T \rtimes T_{n+1}$ *with* T *an* n*-solvable good tree and* T_{n+1} *a* 1*-colored good tree, which is not a* k*-solvable good tree for any* $k \leq n$*.*

Proposition 6.10. *An* n*-solvable good tree* T *with* $n > 1$ *has a unique decomposition* $T' \rtimes T_n$ *with* T' *an* $(n-1)$*-solvable good tree and* T_n *a* 1*-colored good tree. If* $n > 0$ *it has a unique decomposition* $T_1 \rtimes \cdots \rtimes T_n$ *such that each* T_i *is a* 1*-colored good tree. In this decomposition, no two consecutive factors realize Exception 1 and no three consecutive factors realize Exception 2. If* $T_1 \rtimes \cdots \rtimes T_n$ *is* n*-solvable and such that each* T_i *is* 1*-colored, then for any* k *and* ℓ*,* $1 \leq k \leq \ell \leq n$*,* $T_k \rtimes \cdots \rtimes T_\ell$ *is* $(\ell - k + 1)$*-solvable.*

Proof: By definition, if $n > 1$, there exist an $(n-1)$-solvable good tree T' and a 1-colored good tree T_n such that $T' \rtimes T_n$. Since T is an n-solvable good tree then, T neither realizes Exception 1 nor Exception 2. Hence, by Proposition 6.8, the function e is definable in T and $T_n = E_{\leq}$ if $\forall \alpha \in L$, $e(\alpha) = p(\alpha)$, $T_n = E_{<}$ otherwise. This gives the unicity of T', and unicity of T_n as well since $\sim$ (defined in 5.2) is definable from e and $\sim$-classes are subtrees isomorphic to T_n.
Unicity of the decomposition $T_1 \rtimes T_2 \rtimes \cdots \rtimes T_n$ follows by induction on $n > 0$. The last assertion is now clear. □

Corollary 6.11. *Let T be a solvable good tree, then there exists a unique $n \in \mathbb{N}$ such that T is an n-solvable good tree.* □

From now on n is supposed to be positive.

Definition 6.12. *We first define and interpret by induction the language $\mathcal{L}_n$ on n-solvable good trees.*
The language $\mathcal{L}_1 = \{\leq, \wedge, N, L\}$ has already been defined and $\mathcal{L}_{n+1} := \mathcal{L}_n \cup \{e_n, E_n, E_{\geq,n}\}$ where e_n is a symbol for a unary function and E_n and $E_{\geq,n}$ are unary predicate symbols. The language $\mathcal{L}_1$ is interpreted naturally as in any good tree.
If T' is an $(n+1)$-solvable good tree, it has a unique decomposition $T' = T \rtimes T_{n+1}$ with T an n-solvable good tree and T_{n+1} a 1-colored good tree. We refer now to subsection 5.3 with the following adaptations: $\mathcal{F} := \{e_1, \ldots, e_{n-1}\}$, the language denoted $\mathcal{L}$ in 5.3 becomes now language $\mathcal{L}_n$ and $\mathcal{L}'$ becomes now $\mathcal{L}_{n+1}$. By induction hypothesis $\mathcal{L}_n$ is interpreted on T and satisfies $(4*)$. This gives the interpretation of $\mathcal{L}_{n+1}$ on T' and shows it satisfies $(4*)$.
Next we define $\mathcal{L}_n^+ := \mathcal{L}_n \cup \{p, D_p, F_p\}$ (for p, D_p and F_p as defined before Definition 4.4). In any n-solvable good tree $\mathcal{L}_n^+$ is an extension by definition of $\mathcal{L}_n$.

Proposition 6.13. *Let T be an n-solvable good tree, Σ its complete theory in the language $\mathcal{L}_n$ and T_0 a 1-colored good tree, Σ_0 its complete theory in the language $\mathcal{L}_1$. Then $\Sigma \rtimes \Sigma_0$ (as defined in Definition 5.16) is the complete theory of $T \rtimes T_0$ in the language $\mathcal{L}_{n+1}$.*

Proof: We proceed by induction on n. Case $n = 1$ is given by Proposition 4.5 and the induction step by Proposition 5.15. □

Proposition 6.14. *Let T be an n-solvable good tree. Then*
- T eliminates quantifiers in the language $\mathcal{L}_n^+$,
- functions and predicates of $\mathcal{L}_n$ are definable in the pure order,
- T is finite or $\aleph_0$-categorical,
- $M(T)$ is indiscernible and C-minimal.

Proof: The proof runs by induction on n. The first item follows from Propositions 4.5 (case $n = 1$) and 5.17 (induction step) and the second one from Propositions 6.10 and 6.8

(induction step, nothing to prove here when $n = 1$). The third one from Propositions 4.5 for the case $n = 1$ and 5.13 for the induction step. The fourth one from Proposition 4.6 for the case $n = 1$ and Propositions 5.17 and 5.18 for the induction step. □

Definition 6.15. *Let $T_1, T_2, \cdots, T_n$ be 1-colored good trees neither realizing Exception 1 nor 2. Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_n$ be their theories in the language $\mathcal{L}_1$ and $\Sigma_1 \rtimes \cdots \rtimes \Sigma_n$ the $\mathcal{L}_n$-theory defined by induction using Proposition 5.15, Definition 5.16 and extension associativity. By Proposition 6.14, $\Sigma_1 \rtimes \cdots \rtimes \Sigma_n$ is an extension by definition of its restriction to $\mathcal{L}_1$ and we will also consider it as a theory in the language $\mathcal{L}_1$. We denote S_n, $n \geq 1$, the set of all theories $\Sigma_1 \rtimes \Sigma_2 \rtimes \cdots \rtimes \Sigma_n$ in the language $\mathcal{L}_1$ and S_0 the $\mathcal{L}_1$-theory of the singleton.*

Definition 6.16. *For $n \in \mathbb{N} \cup \{\infty\}$, we call n-*colored *any model of S_n.*

Corollary 6.17. *For any $n \in \mathbb{N} \cup \{\infty\}$, any finite or countable n-colored good tree is n-solvable.*

Proof: By Proposition 6.14 any theory in S_n is $\aleph_0$-categorical. □

Remark 6.18. *The class of n-colored good trees, n at least two, is not elementary as shows the following example (but the class of all i-colored good trees, for some $i \leq n$, is). Take 1-colored good trees, T of color $(0, \infty)$ and for each $n \in \mathbb{N}^{\geq 1} \cup \{\infty\}$, T_n of color $(1, n)$. By Proposition 6.8, for $n \in \mathbb{N}^{\geq 1}$, all $T_n \rtimes T$ are 2-colored. But any non trivial ultraproduct of them is 1-colored as it is equivalent to $T_\infty \rtimes T$ which realizes Exception 1.*

The following theorem summarizes much of what has been proven in this section.

Theorem 6.19. *For any integer n any theory in S_n is complete and admits quantifier elimination in the language $\mathcal{L}_n^+$. Furthermore S_n is the set of all complete theories of n-colored good trees.* □

7. Classification of indiscernible $\aleph_0$-categorical C-minimal pure C-sets

Theorem 7.1. *Let M be a pure C-set. Then the following assertions are equivalent:*

(i) M is finite or $\aleph_0$-categorical, C-minimal and indiscernible

(ii) $T(M)$ is a precolored good tree.

(iii) $T(M)$ is a colored good tree.

Proof: $(i) \Rightarrow (ii)$: This is Corollary 3.8.
$(iii) \Rightarrow (i)$: This is Theorem 6.19.
$(ii) \Rightarrow (iii)$
We will prove the result by induction on the depth n of $T(M)$.

The case of depth 1 is given by Remark 4.2.

Asume that any precolored good tree of depth n is a colored good tree. Let T be a precolored good tree of depth $n+1$. By Corollary 3.12, for any leaf α, the latest one-colored interval $I_{n+1}(\alpha)$ of the branch $br(\alpha)$ is either $\{p(\alpha)\}$, case (0), or $]e_n(\alpha), \alpha[$, case (1.a), or $]e_n(\alpha), p(\alpha)]$, case (1.b).

In case (0) the thick cone T_α at $p(\alpha)$ is a 1-colored good tree of type (0), and in case (1.a) or (1.b), the cone T_α of α at $e_n(\alpha)$ is a 1-colored good tree of type (1.a) or (1.b) respectively. Let us call (m_{n+1}, μ_{n+1}) the color (independent of α) of the 1-colored good tree T_α. Thus by Proposition 4.5, for any α, $T_\alpha \models \Sigma_{m_{n+1},\mu_{n+1}}$. Let T_{n+1} be the countable or finite 1-colored good tree model of $\Sigma_{m_{n+1},\mu_{n+1}}$.

Now, T is an $\mathcal{L}_2$-structure when interpreting e by e_n, $E = Im(e_n)$, $E_{\geq} = Dom(e_n)$ and, as such, a model of Σ'' (cf 5.10). Let us consider on T the equivalence relation $\sim$ associated to e_n, as defined in 5.11, (1) if T_{n+1} is of type (0) and (2) otherwise, and $\overline{T} := T/\sim$. Suppose T is countable or finite. So, by categoricity of 1-colored good trees and (5.12), $T \equiv \overline{T} \rtimes T_{n+1}$.

By induction hypothesis and Corollary 6.17 there are 1-colored good trees $T_1, \ldots, T_k$ such that $\overline{T} = T_1 \rtimes T_2 \rtimes \cdots \rtimes T_k$, hence, $T = T_1 \rtimes T_2 \rtimes \cdots \rtimes T_k \rtimes T_{n+1}$. Hence T is a colored good tree. This remains true for any $T' \equiv T$ by definition of colored good trees. This allows us to remove the temporary assumption that T is countable or finite. □

Remark 7.2. *Since a tree of the form $T_1 \rtimes T_2 \rtimes \cdots \rtimes T_n$ where the T_i are 1-colored is always an m-colored good tree for some $m \leq n$, the proof of (ii) $\Rightarrow$ (i) above shows that a precolored good tree of depth n is an m-colored good tree, with $m \leq n$.*

Corollary 7.3. *A good tree is precolored of depth n iff it is n-colored.*

Proof: We proceed again by induction on n. The case $n = 1$ is Theorem 4.6.

Let now T be a precolored good tree of depth $n+1$, then by the remark above, T is m-colored with $m \leq n+1$. Assume for a contradiction that $m < n+1$, then by induction hypothesis, T is precolored of depth m, which contradicts the unicity of the depth of a precolored good tree (see Definition 3.7), hence $m = n+1$. Conversely, if T is n-colored, then T is a precolored good tree whose depth must therefore be n. □

We make now completely precise the equivalence between colored and precolored good trees. In what follows, the E_i and the $E_{\geq,i}$ are predicates of the language $\mathcal{L}_n$ as in Definition 6.12, $E_{<,i} := N_T \setminus E_{\geq,i}$ and $E_{\leq,i} := E_{<,i} \cup E_i$.

Definition 7.4. *Let $T \equiv T_1 \rtimes T_2 \rtimes \cdots \rtimes T_n$ be an n-colored good tree, $n \geq 1$, where each T_i is 1-colored. For $n = 1$ we set $I_1 := N_T$. For $n \geq 2$ and i, $1 \leq i \leq n$, we define by induction on i the subset I_i of N_T as follows:*

- $I_1 := E_1 = E_{\leq,1}$ if T_1 is of type (0) or (1.b), and $I_1 := E_{<,1}$ if T_1 is of type (1.a);

- for i, $1 < i < n$, $I_i := E_i = E_{\leq,i} \setminus \bigcup_{1\leq j\leq i-1} I_j$ if T_i is of type (0) or (1.b), and

$I_i := E_{<,i} \setminus \bigcup_{1 \le j \le i-1} I_j$ *if* T_i *is of type* (1.a)*;*
- $I_n := E_{n-1}$ *if* T_n *is of type* (0)*, and* $I_n := E_{>,n-1} \cap N_T$ *otherwise.*

Proposition 7.5. *Let* $T \equiv \ldots T_1 \rtimes T_2 \rtimes \cdots \rtimes T_n$ *be an* n*-colored good tree,* $n \ge 1$*, where each* T_i *is* 1*-colored. Then, for each* i*,* $1 \le i \le n$*, and each leaf* α *of* T*, the set* $I_i \cap br(\alpha)$ *is the* 1*-colored basic interval* $I_i(\alpha)$ *of* T *seen as a precolored good tree of depth* n *(as in Definition 3.7).*

Proof: It is clear from their definition that the I_i cover N_T. Thus it is enough to prove that all nodes of each I_i have same tree-type in T. It will follow from quantifier elimination (given in Theorem 6.19). Up to logical equivalence, in $\mathcal{L}_n$ atomic formulas in the single variable x are either tautologies, or always false, or of the form E_i or $E_{\ge,i}$ applied to x or $e_j(x)$, or an equality between two such terms. Indeed, since $e_i \circ e_j = e_{min\{i,j\}}$ there is no need to consider terms in x with more than one function e_i; since $e_i(x) \le x$ and, $e_i(x) < e_j(x)$ if $i < j$, there is no need of $\wedge$ and $<$ either. Now, an equality $e_j(x) = x$ occurs iff $E_j(x)$, and $E_j(x)$ depends only on the types of T_j and T_{j+1} if $x \in I_j$. We have still to deal with the function p. Now, p coincides always with either the identity or some function e_j; moreover an equality $p(y) = y$ or $p(y) = e_j(y)$ is determined by the formula $I_j(y)$ and the type of T_j. More precisely, $I_j \cap D_p = \emptyset$ if T_j is of type (1); if T_j is of type (0) then $I_j \subseteq D_p$ and p and e_j coincide on I_j. This achieves the proof. $\square$

Proposition 7.6. *Let* $T \equiv T_1 \rtimes T_2 \rtimes \cdots \rtimes T_n$ *be an* n*-colored good tree,* $n \ge 1$*, where each* T_i *is* 1*-colored of color* (m_i, μ_i) *and* c *a node of* T*. Then the color of* c *is* (m_i, μ_i) *where* i *is the unique index such that* $c \in I_i$*. Inner cones at* c *have same theory as* $T_i \rtimes \cdots \rtimes T_n$ *and border cones have same theory as* $T_{i+1} \rtimes \cdots \rtimes T_n$*. If* $I_i(\alpha)$ *is dense for* $\alpha \in L$ *then* $(T \setminus \Gamma(c)) \equiv T$*.*

Proof: All nodes in I_i have same tree type which is different from the tree type of any node of I_{i+1}. Thus, let Γ be a cone at c. Either Γ contains a non empty dense interval $]c, d[$ included in I_i, then Γ is inner, by definition. Now, there are μ_i such cones. Or, $\Gamma = \Gamma(c, \alpha)$ for some leaf α and either $I_i(\alpha) =]e_{i-1}(\alpha), c]$ or $I_i(\alpha) = \{c\}$. There are m_i such cones. To determine the theory of these cones, we can without loss of generality argue in the countable model. There is a copy of $T_i \rtimes T_{i+1} \rtimes \cdots \rtimes T_n$ containing c. In this copy c can be identified with a node d of T_i. Any cone Γ at c is canonically isomorphic to $\mathcal{C} \rtimes T_{i+1} \rtimes \cdots \rtimes T_n$, where $\mathcal{C}$ is a cone of T_i at d. If Γ is inner then $\mathcal{C}$ is inner in T_i, hence isomorphic to T_i (see Corollary 4.7). Thus, inner cones of T at c are isomorphic to $T_i \rtimes T_{i+1} \rtimes \cdots \rtimes T_n$. If Γ is border then $\mathcal{C}$ is a leaf of T_i. Thus, border cones of T at c are isomorphic to $\bullet \rtimes T_{i+1} \rtimes \cdots \rtimes T_n$ with $\bullet$ a singleton hence to $T_{i+1} \rtimes \cdots \rtimes T_n$.
In the same way, if $I_i(\alpha)$ is dense for $\alpha \in L$, then $(T_i \setminus \Gamma(d)) \equiv T_i$, hence $(T \setminus \Gamma(c)) = T_1 \rtimes \cdots \rtimes T_{i-1} \rtimes (T_i \setminus \Gamma(d)) \rtimes T_{i+1} \rtimes \cdots \rtimes T_n$. $\square$

8. General classification

In this section we reduce the general classification of finite or $\aleph_0$-categorical and C-minimal C-sets to the classification of indiscernible ones, previously achieved in section 7. By the Ryll-Nardzewski Theorem, any $\aleph_0$-categorical structure is a finite union of indiscernible subsets. In a C-minimal structure $\mathcal{M}$ these subsets have a very particular form. Let us give an idea: there exists a finite subtree Θ of $T := T(M)$, closed under $\wedge$ and $\emptyset$-algebraic with the following properties:

- any $a \in \Theta$, except its root, has a predecessor in Θ since Θ is finite, call it a^-; now, in T, $]a^-, a[$ is either empty or not a singleton and dense, and in the second case, the pruned cone $\mathcal{C}(]a^-, a[)$ is indiscernible in M,
- for a as above and $b \in \Theta$, $b > a$, then $\mathcal{C}(]a^-, b[)$ is not indiscernible,
- and more...

An equivalence relation is defined over Θ which identifies for example points a and b such that none of $\mathcal{C}(]a^-, a[)$ and $\mathcal{C}(]b^-, b[)$ is empty and $\mathcal{C}(]a^-, a[) \cup \mathcal{C}(]b^-, b[)$ is indiscernible (this is only an example; there are other elements to be identified). We call vertices the elements of the quotient $\bar{\Theta}$ of Θ. They are finite antichains of T. We consider on $\bar{\Theta}$ the order induced by the order of Θ (it is the classical order on antichains); it makes $\bar{\Theta}$ a finite tree. An (oriented) edge links vertices A and $B > A$ iff A is the predecessor of B in $\bar{\Theta}$, $A = B^-$. Vertices and edges of $\bar{\Theta}$ are labeled. As an example, on a vertex A, a first label gives the (finite) cardinality of A seen as a subset of T, and a label on the edge (A^-, A) says whether, for any $a \in A$, $]a^-, a[$ is empty or not: this second label exists iff this interval is not empty and it gives the complete theory of the indiscernible C-set $\mathcal{C}(]a^-, a[)$.
There are other labels on vertices which are also either cardinals in $\mathbb{N} \cup \{\infty\}$ or complete theories of indiscernible finite or $\aleph_0$-categorical and C-minimal C-sets. Conversely, we have isolated eleven properties which are true in $\bar{\Theta}$ and such that, given a labeled graph Ξ sharing these eleven properties, there is a finite or $\aleph_0$-categorical and C-minimal C-set M such that $\bar{\Theta}(M) = \Xi$. In this sense, the classification of finite or $\aleph_0$-categorical and C-minimal C-sets is reduced to that of indiscernible ones.

8.1. The canonical partition

Proposition 8.1. *Let $\mathcal{M}$ be a finite or $\aleph_0$-categorical structure, then there exists a unique partition of M into a finite number of $\emptyset$-definable subsets which are maximal indiscernible.*

Proof: By $\aleph_0$-categoricity, there is a finite number of 1-types over $\emptyset$. By compacity, each of these types is consequence of one of its formulas. $\square$

Definition 8.2. *We call this partition the* canonical partition. *Thereafter it will be denoted $(M_1, \cdots, M_r)$.*

We reformulate here for convenience the description given in $[D]$ in the proof of Proposition 3.7, with a small difference: instead of working with $T(M)$ we will work with $T(M)^*$

defined as follows: $T^* := T$ if T has a root and $T^* := T \cup \{-\infty\}$ otherwise, with $-\infty < T$. In the last case, we say that "$-\infty$ exists". Note that the tree T^* has always a root, which is either the root of T or $-\infty$. By C-minimality each M_i of the canonical decomposition is a finite boolean combination of cones and thick cones. We will be more precise. Let D be the set of bases of cones and thick cones appearing in these combinations.

Definition 8.3. *We define* $\Theta_0 := \{x \in T(M)^*; \textit{for some } c \in D, x \leq c\}$ *and* $\Theta_1 := \{x \in \Theta_0; \exists i \neq j, \alpha \in M_i, \beta \in M_j, x \in br(\alpha) \cap br(\beta)\}$. *We define:*

$U := \{\textit{suprema of branches from } \Theta_1\}$

$B := \{\textit{branching points of } \Theta_1\}$

$S := \{c \in \Theta_1 \setminus (U \cup B)$; *the thick cone at* c *without the cone of* <u>*the*</u> *branch of* Θ_1 *intersects non trivially both* M_i *and* M_j *for a couple* (i, j), $i \neq j$ $\}$

$I := \big\{$*infima* $\in \Theta_1 \setminus (U \cup B \cup S)$ *of intervals on branches of* Θ_1 *which are maximal for being contained in* $\{c \in \Theta_1 \setminus (U \cup B \cup S)$; *the thick cone at* c *without the cone of* <u>*the*</u> *branch of* Θ_1 *is entirely contained in a same* M_i $\big\}$

$\Theta := U \cup B \cup S \cup I$.

Remark 8.4. *- Since* D *is finite,* Θ_0 *and* Θ_1 *are trees with finitely many branches, which implies that* U *and* B *are finite;* S *is finite since it is contained in* D*;* I *is finite by o-minimality of branches of* Θ_1*. Hence* Θ *is finite.*
- Θ_1, U, B, S, I *and* Θ *are all definable from the* M_i*, hence* $\emptyset$*-definable since the* M_i *are. As* Θ *is finite, it is contained in the algebraic closure of the empty set.*
- Θ *is a subtree of* $T(M)^*$ *closed under* $\wedge$*. Because it is finite each element of* Θ *has a predecessor in* Θ*. Elements of* Θ *which are nodes (or leaves) in* $T(M)$ *may not be nodes (or leaves) in* Θ*. So, to avoid confusion we will use the words* vertices *and* edges *for the tree* Θ*.*
- We have the equivalence: $\mathcal{M}$ *is not indiscernible iff* Θ *is not empty iff the root of* $T(M)^*$ *belongs to* Θ*.*

Proposition 8.5. *Let* $\mathcal{M}$ *be a* C*-minimal,* $\aleph_0$*-categorical structure. Then the subsets* $M_1, \cdots, M_r$ *of the canonical partition are the orbits over* $\emptyset$ *of* $acl(\emptyset)$*-definable subsets of the form:*

- *cones*
- *almost thick cones (i.e. cofinite unions of cones at a same basis)*
- *pruned cones* $\mathcal{C}(]b, a[)$ *where* $b < a$ *and* $]b, a[$ *is a dense interval without extremities,*

all these cones having their basis in Θ *as well as the other extremity (namely* a*) of the axis in case of pruned cones.*

Proof: By definition of Θ, any M_i is a finite union of pruned cones $\mathcal{C}(]b, a[)$, cones and thick cones at a, with $a, b \in \Theta$ and a the predecessor of b in Θ. By $\emptyset$-definability, M_i is the union of the orbits over $\emptyset$ of these sets (for more details, see [D], Proposition 3.7). This gives the proposition except the fact that $]b, a[$ is a dense interval without extremity. This result follows from $\aleph_0$-categoricity using the following facts.

Fact 8.6. *Assume some subset of the canonical partition is of the form* $M_j = \bigcup_{i=1}^{n} \mathcal{C}(]b_i, a_i[)$. *Let* (b, a) *be one of the couples* (b_i, a_i). *Then all the elements of the pruned cone* $\mathcal{C}(]b, a[)$ *have same type over* (b, a) *in* $\mathcal{M}$.

Proof: Assume $\mathcal{M}$ ω-homogeneous. Then, for $x, y \in \mathcal{C}(]b, a[)$ there exists an automorphism of $\mathcal{M}$ sending x to y. Such an automorphism preserves M_j hence preserves a and b. Therefore x and y have the same type over (b, a). $\dashv$

Fact 8.7. *All nodes of* $]b, a[$ *have same type over* (b, a).

Proof: This is a direct consequence of the preceding Fact, since any node of $]b, a[$ is of the form $b \wedge x$, where $x \in \mathcal{C}(]b, a[)$. $\dashv$

Now, since all the nodes of $]b, a[$ have same type over $\emptyset$, either $]b, a[$ is dense or consists of a unique node, or contains an infinite discrete order which is not possible by $\aleph_0$-categoricity. In the case where $]b, a[$ consists of a single node, say c, $\mathcal{C}(]b, a[)$ is an almost thick cone, namely the thick cone at c without $\mathcal{C}(c, b)$. So, $\mathcal{C}(]b, a[)$ changes from the third category to the second category of subsets. $\square$

In particular, Fact 8.6 has the following consequence.

Fact 8.8. *If* $a \in \Theta$ *has a predecessor in* $T(M)^*$, *then this predecessor belongs also to* Θ. $\dashv$

Notations. Our aim is now to understand the structure induced by $\mathcal{M}$ on a pruned cone $\mathcal{C}(]b, a[)$ of the canonical partition as in Proposition 8.5. It is in general not a pure C-set but we know by Proposition 2.8 that, as a pure C-set it is C-minimal. So what we have done in the previous sections applies to the C-minimal pure C-set $\mathcal{C}(]b, a[)$. This means that its canonical tree $\Gamma(]b, a[)$ is a colored good tree, say an n-colored good tree for some integer n, which must be greater than 1 since $]b, a[$ contains at least one node. Thus $\Gamma(]b, a[) =: T \equiv T_1 \rtimes \cdots \rtimes T_n$ for 1-colored good trees $T_1, \cdots, T_n$. Recall (from Section 5.3) that T_1 may be taken a definable quotient of T. We call this T_1 the *first level* of T. Since $]b, a[$ is dense, T_1 is infinite, of type $(1.a)$ or $(1.b)$. Its set of nodes, N_1, embeds definably in T, as the set I_1 defined in Definition 7.4. Note that, when M is countable, then the elementary equivalence becomes an isomorphism: $\Gamma(]b, a[) = T_1 \rtimes \cdots \rtimes T_n$.
If Σ is the complete theory of the pure tree T, Σ_1 will denote the theory of its first level T_1 and $\Sigma_{>1}$ the theory of the $(n-1)$-colored good tree $T_2 \rtimes \cdots \rtimes T_n$ or, to understand it in definable terms from T, the theory of each non trivial $\sim_1$-equivalence class for $\sim_1$ the relation corresponding to the extension $T_1 \rtimes (T_2 \rtimes \cdots \rtimes T_n)$ (see Section 5.2). For $i \in \{1, \cdots, n\}$, (m_i, μ_i) will denote the color of the 1-colored good tree T_i.

Lemma 8.9. *Let $\mathcal{C}(]b,a[)$ be a pruned cone as in Fact 8.6. Then $]b,a[$ is included in the set of nodes of the first level of $\Gamma(]b,a[)$, the colored good tree associated to $\mathcal{C}(]b,a[)$.*

Proof: Any $\alpha \in \mathcal{C}(]b,a[)$ satisfies $(\alpha \wedge a) > b$ hence $I_1(\alpha)$ (considered in $\Gamma(]b,a[)$) intersects $]b,a[$ non trivially. Take any $c \in I_1 \cap]b,a[$. Then the formula "x belongs to I_1 (taken in the tree $\Gamma(]b,a[)$)" is true for $x = c$. By Fact 8.7 it should be true for any $x \in]b,a[$. $\square$

Till now we have exploited that each set M_i of the canonical partition is indiscernible. We use now that it is *maximal* indiscernible, i.e. if $i \neq j$, there are no $\alpha \in M_i$ and $\beta \in M_j$ with same type.

Lemma 8.10. *Let $a \in \Theta$ be maximal in Θ, a not the root of Θ. Let a^- be its predecessor in Θ. If the interval $]a^-,a[$ is empty, then a is not a leaf of $T(M)$ and there exist at least two cones at a with different complete theories as colored good trees.*

Proof: Since a is maximal, following the notation of Definition 8.3, a is in U, i.e. a is the supremum of some branch from Θ_1. If $]a^-,a[$ is empty, a is in Θ_1, hence a belongs to at least two branches of different type in M. In particular a is not a leaf. $\square$

Lemma 8.11. *Let M be a C-minimal C-set. Let $a, b \in T(M)$, with $b < a$ and such that the interval $]b,a[$ is not empty, not a singleton and is dense. Assume that the canonical tree $\Gamma(]b,a[)$ of the pruned cone $\mathcal{C}(]b,a[)$ is an n-colored good tree and let $\Sigma(]b,a[)$ be its complete theory. Assume furthermore that $]b,a[$ is contained in the set of nodes of the first level of $\Gamma(]b,a[)$. Let $\mathcal{C}$ be the union of at least two cones at a, such that each of these cones is indiscernible. Then, $T(\mathcal{C}(]b,a[) \cup \mathcal{C})$ is a model of $\Sigma(]b,a[)$ if and only if one of the following cases appears (where we follow the conventions preceding Lemma 8.9):*

(a) *$m_1 = 0$, $n \geq 2$, and $T(\mathcal{C})$ is an $(n-1)$-colored good tree model of $\Sigma(]b,a[)_{>1}$.*

(b) *$m_1 = 0$, and $\mathcal{C}$ is the union of exactly μ_1 cones at a, all with canonical tree model of $\Sigma(]b,a[)$.*

(c) *$m_1 \neq 0$ and,*
- if $n = 1$, then $\mathcal{C}$ is the union of exactly m_1 cones which consist of a leaf, and μ_1 cones with canonical tree model of $\Sigma(]b,a[)$.
- if $n \geq 2$, then $\mathcal{C}$ is the union of exactly m_1 cones with canonical tree model of $\Sigma(]b,a[)_{>1}$ and exactly μ_1 cones with canonical tree model of $\Sigma(]b,a[)$.

Proof: By hypothesis, $]b,a[$ is contained in the first level of $\Gamma(]b,a[)$ and $\mu_1 \neq 0$ since $]b,a[$ is dense. Note that $\mathcal{C}$ becomes the thick cone at a in the C-set $\mathcal{C}(]b,a[) \cup \mathcal{C} =: \mathcal{H}$.

We prove first the "if" direction.
- Assume (a). Then, T_1 is of type (1.a) and, in $T(\mathcal{H})$, a is the root of an $(n-1)$-colored good tree model of $\Sigma(]b,a[)_{>1}$. Let T_1' be the first level of $\Gamma(]b,a[)$ plus the additional element a which is now the leaf of the branch $]b,a[$. Then, T_1' is a model of $\Sigma(]b,a[)_1$.

If M is countable, by $\aleph_0$-categoricity, the $(n-1)$-colored good tree $T(\mathcal{C})$ is isomorphic to $\Gamma(]b,a[)_{>1}$. Hence, $T(\mathcal{H}) = T_1' \rtimes \Gamma(]b,a[)_{>1}$. In general, due to Proposition 5.15, $T(\mathcal{H}) \equiv T_1' \rtimes \Gamma(]b,a[)_{>1}$. Hence $T(\mathcal{H})$ is a model of $\Sigma(]b,a[)$.

- Assume (b). Take any model $\mathcal{G}$ of $\Sigma(]b,a[)$ and d any node in the first level of $\mathcal{G}$. So $\mathcal{G}$ appears as the disjoint union of the pruned cone $\Gamma(]-\infty,d[)$ (considered in $\mathcal{G}$), $\{d\}$ and μ_1 cones at d, which are all models of $\Sigma(]b,a[)$ by Proposition 7.6. By Proposition 7.6 again, $\Gamma(]-\infty,d[)$ is a model of $\Sigma(]b,a[)$. By hypothesis (b) $T(\mathcal{H})$ admits a similar decomposition with a instead of d. Since $\Sigma(]b,a[)$ is complete, we are able to carry on an infinite back and forth between $\mathcal{G}$ and $T(\mathcal{H})$. Hence $T(\mathcal{H})$ is a model of $\Sigma(]b,a[)$.

- Assume $n=1$, so $\Sigma(]b,a[) = \Sigma_{m_1,\mu_1}$, and (c). We argue similarly to Case (b). Take $\mathcal{G}$ any model of $\Sigma(]b,a[)$ and d any node of $\mathcal{G}$. So $\mathcal{G}$ is the disjoint union of its pruned cone $\Gamma(]-\infty,d[)$, $\{d\}$, m_1 leaves immediately above d and μ_1 inner cones at d. By Proposition 7.6 these μ_1 cones at d are all models of Σ_{m_1,μ_1} and $\Gamma(]-\infty,d[)$ is a model of $\Sigma(]b,a[)$. By hypothesis (c) $T(\mathcal{H})$ admits a similar decomposition with a instead of d. Thus $\mathcal{G} \equiv T(\mathcal{H})$.

- Finally, assume $n \geq 2$ and (c). As above, take any model $\mathcal{G}$ of $\Sigma(]b,a[)$ and d a node in the first level of $T(\mathcal{G})$. So $\mathcal{G}$ is the disjoint union of its pruned cone $\Gamma(]-\infty,d[)$, $\{d\}$, m_1 border cones at d and μ_1 inner cones at d. By Proposition 7.6, $\Gamma(]-\infty,d[)$ and inner cone at d are models of $\Sigma_{(]b,a[)}$ and border cones at d are models of $\Sigma(]b,a[)_{>1}$. Again, by hypothesis (c), $T(\mathcal{H})$ admits a similar decomposition with a instead of d, thus $\mathcal{G} \equiv T(\mathcal{H})$.

Conversely, assume $T(\mathcal{H})$ is an n-colored good tree model of $\Sigma(]b,a[)$. Since $]b,a[$ belongs to the first level of $T(\mathcal{H})$, the color of a is (m_1,μ_1) or (m_2,μ_2).

Assume first that the color of a is (m_1,μ_1). Let $\Gamma(a,\alpha)$ be a cone at a, then either $\Gamma(a,\alpha)$ is an inner cone and its theory is $\Sigma(]b,a[)$, or $\Gamma(a,\alpha)$ is a border cone, model of $\Sigma(]b,a[)_{>1}$ if $n>1$ and consisting of a leaf otherwise (by Proposition 7.6 again).

If $m_1 = 0$, then there are only inner cones at a, all models of $\Sigma(]b,a[)$, and we are in case (b).

If $m_1 \neq 0$, and $n=1$, the assertion is clear.

If $m_1 \neq 0$ and $n \geq 2$, then there are m_1 border cones at a all models of $\Sigma(]b,a[)_{>1}$, μ_1 inner cones at a all models of $\Sigma(]b,a[)$ and we are in case (c).

Assume now that the color of a is (m_2,μ_2). Then, necessarily, for any leaf α of $T(\mathcal{H})$ greater than a, $I_1(\alpha)$ is open on the right with upper bound a, hence the first level of $T(\mathcal{H})$ is of type $(1.a)$. So, $m_1 = 0$, and a is the root of an $(n-1)$-colored good tree model of $\Sigma(]b,a[)_{>1}$. So we are in case (a). $\square$

Lemma 8.12. *Let $\Sigma \in S_n$ be a complete theory of n-colored good trees without root and V a new unary predicate. Let $\mathcal{L}_1^V$ be the language $\mathcal{L}_1 \cup \{V\}$ and Σ^V be the $\mathcal{L}_1^V$-theory which consists of Σ together with the axiom $\mathcal{V}$: V is a "branch" (i.e. a maximal chain) in the first level of any (some) model of Σ and V has no leaf. Let $\wedge_V$ be the function $\wedge_V : x \mapsto x \wedge V$. Then the theory Σ^V is complete, admits quantifier elimination in the language $\mathcal{L}_n^{V+} := \mathcal{L}_n^+ \cup \{V, \wedge_V\}$, and is $\aleph_0$-categorical. Its models have an indiscernible* *and C-minimal set of leaves.*

Proof: Consistency of Σ^V: consider a tree $T = T_1 \rtimes T_2 \rtimes \cdots \rtimes T_n$ model of Σ with T_1 countable or finite. Since T has no root, T_1 not only is infinite but has $2^{\aleph_0}$ branches. Hence $2^{\aleph_0}$ many of them have no leaf, which shows Σ^V to be consistent.

We first prove the Lemma for $n = 1$.

Let $\Sigma = \Sigma_{m,\mu} \in S_1$, $\mu \neq 0$, be a complete theory of 1-colored good tree without root. We will use a back and forth argument between finite $\mathcal{L}_1^{V+}$-substructures of any two countable models T and T' of Σ^V as in the proof of Proposition 4.5. In what follows, Facts 1 to 6 refers to this proof.

Fact: If $m = 0$ complete quantifier free $\mathcal{L}_1^{V+}$-types of Σ are: $x \in L$, $x \in V$, $x \in N \setminus V$. If $m \neq 0$ complete quantifier free $\mathcal{L}_1^{V+}$-types of Σ are: $x \in L$ and $p(x) \in V$, $x \in L$ and $p(x) \notin V$, $x \in V$, $x \in N \setminus V$. In both cases the $\mathcal{L}_1^{V+}$-substructure generated by a singleton x is the smallest subset containing x, $p(x)$ and $x \wedge V$.

Proof: If $x \notin L$, then $p(x) = x$. If $x \in L$, then $x \notin V$ and $p(x) \wedge V = x \wedge V$. Moreover, for all $n \in \mathbb{N}$, $p^n(x) = x$ or $p^n(x) = p(x)$. The fact is now clear. $\dashv$

This fact shows that the family of partial isomorphisms between finite substructures of T and T' is not empty. We show now it has the back and forth property. Let A be a finite $\mathcal{L}_1^{V+}$-substructure of T, and φ be a partial $\mathcal{L}_1^{V+}$-isomorphism from T to T' with domain A. Let $x \in T \setminus A$. By Fact 1 there exists a node n_x such that $x \wedge n_x$ is the maximal element of the set $\{x \wedge y; y \in A\}$.

1. Assume first that $x \in V^T \setminus A$; thus x is not a leaf; since $n_x \leq x$, n_x belongs to V^T. Hence, as in Fact 2, since A is an $\mathcal{L}_1^{V+}$-substructure, the $\mathcal{L}_1^{V+}$-substructure generated by A and x, $\langle A \cup \{x\}\rangle_V$, is the minimal subset containing A, x and n_x.

Assume furthermore that $x = n_x$, so $\langle A \cup \{x\}\rangle_V = A \cup \{x\}$. As in Fact 4, there exist $a \in A \cap V^T$ and $b \in A \cup \{-\infty\}$ such that $]b, a[\cap A = \emptyset$ and $x \in]b, a[$. Set $\varphi(-\infty) = -\infty$. Then, $\varphi(b) < \varphi(a)$ and $]\varphi(b), \varphi(a)[$ is included in $V^{T'}$. For any x' in this interval, $A \cup \{x\}$ and $\varphi(A) \cup \{x'\}$ are isomorphic $\mathcal{L}_1^{V+}$-structures. We extend φ on x by sending it to x'.

Now, we can assume that $n_x \neq x$ and $n_x \in A$. Since V has no leaf it is possible to find $x' \in V^{T'}$, $x' > \varphi(n_x)$. So $\langle A \cup \{x\}\rangle_V$ is $\mathcal{L}_1^{V+}$-isomorphic to $\langle \varphi(A) \cup \{x'\}\rangle_V$.

2. Assume now that $x \in T \setminus (V^T \cup A)$. By case 1 we may assume $x \wedge V \in A$, thus $x \wedge V \leq n_x$. So, the $\mathcal{L}_1^{V+}$-substructure $\langle A \cup \{x\}\rangle_V$ is the minimal subset containing A, x, n_x and $p(x)$. If $x \wedge V < n_x$, none of $x, p(x), n_x$ touch V. We use quantifier elimination of Σ in $\mathcal{L}_1^+$ to find $x' \in T'$ such that (A, x, n_x) and $(A', x', n_{x'})$ have same quantifier free $\mathcal{L}_1^+$-type. They must have same quantifier free $\mathcal{L}_1^{V+}$-type. If $x \wedge V = n_x$ then $n_x \in A$ and we have an analogue of Fact 3 (with its corresponding proof):

Let Γ be a cone at $a \in A$, such that $\Gamma \cap (A \cup V^T) = \emptyset$. Then there exists a cone Γ' of T' at $\varphi(a)$ such that $\Gamma' \cap (\varphi(A) \cup V^{T'}) = \emptyset$. Moreover, if Γ is infinite, resp. consists of a single leaf, then there is such a Γ' infinite, resp. consisting of a single leaf.

This allows us to find $x' \in T'$ such that (A, x) and (A', x') have same quantifier free $\mathcal{L}_1^{V+}$-type and achieves the forth proof. The back construction is the same.
So, we have proven elimination of quantifiers in the language $\mathcal{L}_1^{V+}$, completeness and $\aleph_0$-categoricity. This achieves the case $n = 1$.

General case. Let (T, V) be a countable model of Σ^V. Then, T is an n-solvable good tree, so by Proposition 6.10, $T = T_1 \rtimes T_{>1}$ where T_1 is a model of Σ_1 and $T_{>1}$ is an $(n-1)$-colored good tree model of $\Sigma_{>1}$. By $\aleph_0$-categoricity of Σ, T_1 is the unique countable model of Σ_1 and $T_{>1}$ is the unique countable or finite model of $\Sigma_{>1}$. Since V is included in the first level T_1, (T_1, V) is a model of $\Sigma_1^V := \Sigma_1 \cup \{\mathcal{V}\}$, the unique model in fact by the case $n = 1$. Thus (T, V) is the unique countable model of Σ^V. This proves $\aleph_0$-categoricity of Σ^V and its completeness.
To prove that Σ^V admits quantifier elimination in the language $\mathcal{L}_n^{V+}$, we will proceed as in the proof of Proposition 5.17.
Take any finite tuple from T and close it under e_1. Write it in the form $(x, y_1, \ldots, y_m)$ where x is a tuple from $(E_1)_\leq$, $y_1, \ldots, y_m$ tuples from $(E_1)_>$ such that all components of each y_i have same image under e_1, call it $e_1(y_i)$ (thus, $e_1(y_1), \ldots, e_1(y_m)$ are components of x), and $e_1(y_i) \neq e_1(y_j)$ for $i \neq j$. Take $(x', y'_1, \ldots, y'_m) \in T$ having same quantifier free $\mathcal{L}_n^{V+}$-type than $(x, y_1, \ldots, y_m)$. Since the complete theory Σ_1^V eliminates quantifiers, x and x' have same complete type in $(E_1)_\leq$ which embeds canonically in T_1, and there exists an $\mathcal{L}_1^{V+}$-automorphism σ of T_1 sending x to x'. Since $\Sigma_{>1}$ eliminates quantifiers in $\mathcal{L}_{n-1} \cup \{p, D_p, F_p\}$, the rest of the proof runs similarly with e_1, E_1 instead of e and E.
Indiscernibility and C-minimality of the set of leaves follow from quantifier elimination. $\square$

Lemma 8.13. *Let $a, b \in T(M)$, with $b < a$ and such that the interval $]b, a[$ is not empty, not a singleton and is dense. Assume that the canonical tree $\Gamma(]b, a[)$ of the pruned cone $\mathcal{C}(]b, a[)$ is an n-colored good tree with colors (m_i, μ_i) for $1 \leq i \leq n$ and that $]b, a[$ is contained in its first level. Let $\Sigma(]b, a[)^V$ be the complete theory of the tree $\Gamma(]b, a[)$ enriched with $]b, a[$. Assume furthermore that there is $c \in T(M)$, $c > a$, such that $]a, c[$ is not empty and $(\Gamma(]a, c[),]a, c[)$ is a model of $\Sigma(]b, a[)^V$. Then $(\Gamma(]b, c[),]b, c[)$ is a model of $\Sigma(]b, a[)^V$ iff there are at a exactly $m_1 + \mu_1$ cones and among those that do not contain c, m_1 are models of $\Sigma(]b, a[)_{>1}$ if $n > 1$ (respectively m_1 are leaves if $n = 1$) and $\mu_1 - 1$ models of $\Sigma(]b, a[)$.*

Proof: According to Lemma 8.12, $(\Gamma(]b, c[),]b, c[) \models \Sigma(]b, a[)^V$ iff $[\Gamma(]b, c[) \models \Sigma(]b, a[)$ and $]b, c[$ lies in the first level of $\Gamma(]b, c[)]$.
Assume first that $(\Gamma(]b, c[),]b, c[)$ is a model of $\Sigma(]b, a[)^V$. Since $]b, c[$ is included in the first level of the tree $\Gamma(]b, c[)$ and $a < c$, the color of a is (m_1, μ_1) and the cone of c at a is one of the μ_1 inner cones at a. Now all inner cones at a are models of $\Sigma(]b, a[)$. And all border cones at a are models of $\Sigma(]b, a[)_{>1}$.
For the converse we argue as in the proof of Lemma 8.11, case (c). Take any model

$(\mathcal{G}, V)$ of $\Sigma(]b,a[)^V$ and $v \in V$. So $\mathcal{G}$ is the disjoint union of the pruned cone $\Gamma(]-\infty, v[)$ (considered in $\mathcal{G}$), $\{v\}$, m_1 border cones at v and μ_1 inner cones at v; call $\mathcal{C}$ the inner cone intersecting V non trivially. Now, both $(\Gamma(]-\infty, v[),]-\infty, v[)$ and $(\mathcal{C}, \mathcal{C} \cap V)$ are models of $\Sigma(]b,a[)^V$, inner cones at v are models of $\Sigma(]b,a[)$ and border cones at v models of $\Sigma(]b,a[)_{>1}$. Following the hypotheses, there exists a similar decomposition of $\Gamma(]b,c[)$ with a in place of v. All involved theories are complete, which makes possible to carry on an infinite back and forth between $(\Gamma(]b,c[),]b,c[)$ and $(\mathcal{G}, V)$. $\square$

8.2. The labeled tree $\bar{\Theta}$

The automorphism group of $\mathcal{M}$ acts on Θ. Let $\overline{\Theta} := \{A_1, \ldots, A_s\}$ be the set of orbits of elements from Θ. Each A_i is a finite $\emptyset$-definable antichain of T^*.

Definition 8.14. *For A and B antichains in T^*, let us define:*
- the relation $a < b \ :\iff \forall a \in A,\ \exists b \in B,\ a < b$ and $\forall b \in B,\ \exists a \in A,\ a < b$ (given b this a is unique);
- let A and B be (finite) antichains in T^* such that $A < B$ and, for any $a \in A$, $b, c \in B$ with $a < b, c$, then either $b = c$ or $a = b \wedge c$; we define $]A, B[$ as the (definable) subset of M consisting of the union of cones of elements from B at nodes from A, with the thick cones at nodes from B removed. We extend this notation to $]\{-\infty\}, A[$, or still $]-\infty, A[$, which will denote the complement of the union of thick cones at all $a \in A$.

Lemma 8.15. *Let A and B be in $\overline{\Theta}$. Then*
- if there are $a \in A$ and $b \in B$ with $a < b$ (or $a = b$) then $A < B$ (or $A = B$).
- $(\overline{\Theta}, <)$ is a finite meet-semi-lattice tree; its root, say A_0, is a singleton (either $\{r\}$ if r is a root of T, or $\{-\infty\}$). It allows to define the predecessor A^- of an element $A \neq A_0$ of $\overline{\Theta}$.
- If $A < B$ there is $k \in \mathbb{N}^{\geq 1}$ such that each $a \in A$ is smaller than exactly k elements from B.
- If $A = B^-$, $a \in A, b, c \in B, a < b, a < c, b \neq c$ then $a = b \wedge c$.

Proof: By construction all elements of A have same type in $\mathcal{M}$. Now B is $\emptyset$-definable, thus if for some $a \in A$, there is $b \in B$ such that $a < b$, the same is true for any $a \in A$. For the same raison, if for some $b \in B$, there is $a \in A$ such that $a < b$, it is true for any $b \in B$. Same thing with $a = b$ instead of $a < b$. This show the first assertion.
The two next assertions are clear.
About the last one: by construction, $b \wedge c \in \Theta$, thus $b \wedge c$ belongs to some element of $\overline{\Theta}$, which must be A, since $A = B^-$ and $a \leq b \wedge c$. $\square$

We now aim to collect on $\overline{\Theta}$ and the indiscernible blocks M_i enough information to be able to reconstruct $\mathcal{M}$ from them. To each $A \in \overline{\Theta}$, associate
- its cardinality n_A;
- an integer s_A, complete theories $\Sigma_{A,1}, \ldots, \Sigma_{A,s_A}$ in $\mathcal{L}_1$ all different and coefficients

$k_{A,1}, \ldots, k_{A,s_A} \in \mathbb{N}^{\geq 1} \cup \{\infty\}$ such that, at each $a \in A$, there are exactly $k_{A,1} + \cdots + k_{A,s_A}$ cones containing no branch from Θ, $k_{A,1}$ of which are models of $\Sigma_{A,1}$,..., and k_{A,s_A} ones models of Σ_{A,s_A} (we are here applying Ryll-Nardzewski again);
- if $A \neq A_0$, $]A^-, A[\neq \emptyset$, $b \in A^-$, $a \in A$ and $b < a$, the complete $\mathcal{L}_1$-theory $\Sigma_{(A^-,A)}$ of $\Gamma(]b, a[)$.

We consider the s_A, $\Sigma_{A,i}$ and $k_{A,i}$ (respectively the $\Sigma_{(A^-,A)}$) as labels on the vertices (respectively the edges) of $\overline{\Theta}$ and Θ, and the n_A as labels on the vertices of $\overline{\Theta}$. The $\Sigma_{A,i}$ (respectively $\Sigma_{(A^-,A)}$) may also be understood as indexing those cones at any/some $a \in A$ (respectively pruned cones $\Gamma(]b, a[)$ for $b \in A^-$, $a \in A$, $b < a$) which are models of it.

Lemma 8.16. *1. Assume $A \neq A_0$. There is no theory $\Sigma_{(A^-,A)}$ labeling (A^-, A) iff $]A^-, A[= \emptyset$.*

2. For $A \in \overline{\Theta}$ and any/some $a \in A$, Θ has a unique branch at a iff there is a unique $B \in \overline{\Theta}$ such that $B^- = A$, and furthermore $n_A = n_B$ holds.

3. $T^* \neq T$ iff $s_{A_0} = 0$, A_0 has a unique successor in $\overline{\Theta}$, say B, and $n_B = 1$.

Proof: (1) holds by definition of the labels of $\overline{\Theta}$.
(2) is clear.
(3) The direction only if is clear. Let us prove the if direction. The unique element, say a_0, of A_0 is either $-\infty$ or the root of T. If A_0 has a successor, a_0 is not a leaf, and if different from $-\infty$ it must be a branching point of T. Now the hypotheses force $\overline{\Theta}$ to have a unique branch at its root. Therefore $a_0 = -\infty$. □

The next lemma gives a list of constraints.

Lemma 8.17. *Let A_0 and $A \in \overline{\Theta}$, A_0 the root of $\overline{\Theta}$.*

(1) If $A \neq A_0$, n_{A^-} divides n_A; $n_{A_0} = 1$.

(2) If A is maximal in $\overline{\Theta}$, then either $s_A = 0$, or $\Sigma_{1 \leq i \leq s_A} k_{A,i} \geq 2$.

(3) If $-\infty$ exists and $B \in \overline{\Theta}$ is such that $B^- = A_0$, then $]A_0, B[\neq \emptyset$.

(4) If Θ has a unique branch in any/some $a \in A$, and $A \neq \{-\infty\}$ if $-\infty$ exists, then $s_A \geq 1$.

(5) Assume $A \neq A_0$, $a \in A$, $b \in A^-$, $b < a$. If $]A^-, A[$ is not empty, then the theory of $\Gamma(]b, a[)$ considered as an $\mathcal{L}_1^V$-structure with $V =]b, a[$ is a theory of colored good tree enriched with a branch without leaf, as described in Lemma 8.12.

(6) Assume $s_A \neq 0$. Then at most one $k_{A,i}$ is infinite and the $\Sigma_{A,i}$ are complete theories of colored good trees.

(7) Assume A maximal in $\overline{\Theta}$, A not the root of $\overline{\Theta}$. If $]A^-, A[$ is empty then $s_A \geq 2$.

(8) Theories $\Sigma_{A,1},\dots,\Sigma_{A,s_A}$ are all different.

(9) Assume A maximal in $\overline{\Theta}$, A not the root of $\overline{\Theta}$ and such that $]A^-,A[$ is not empty. Assume that models of $\Sigma_{(A^-,A)}$ are n-colored trees with colors (m_i,μ_i) for $1\le i\le n$. Then, none of the following situations can appear:

(a) $m_1=0$, $n\ge 2$, $s_A=1$, $\Sigma_{A,1}=(\Sigma_{(A^-,A)})_{>1}$ and $k_{A,1}=m_2$.

(b) $m_1=0$, $s_A=1$, $\Sigma_{A,1}=\Sigma_{(A^-,A)}$ and $k_{A,1}=\mu_1$.

(c) $m_1\neq 0$, $\mu_1\neq 0$, $n=1$, $s_A=2$, $\Sigma_{A,1}=\Sigma_{(A^-,A)}$, $k_{A,1}=\mu_1$, $\Sigma_{A,2}=\Sigma_{(0,0)}$ (i.e. the theory of a tree consisting only of a leaf) and $k_{A,2}=m_1$.
$m_1\neq 0$, $\mu_1\neq 0$, $n\ge 2$, $s_A=2$, $\Sigma_{A,1}=\Sigma_{(A^-,A)}$, $k_{A,1}=\mu_1$, $\Sigma_{A,2}=(\Sigma_{(A^-,A)})_{>1}$, and $k_{A,2}=m_1$.

(10) Assume A not maximal, not the root of $\overline{\Theta}$ and such that $]A^-,A[$ is not empty. Assume furthermore that models of $\Sigma_{(A^-,A)}$ are n-colored trees with colors (m_i,μ_i) for $1\le i\le n$. Then the conjonction of the following conditions cannot appear:
- at least one wedge of $\overline{\Theta}$ starting at A has a label
- if B is the successor of A on such a wedge, the label of (A,B) is $\Sigma_{(A^-,A)}$
- either [$m_1\ge 1$, $\mu_1\ge 2$, $s_A=2$, $\Sigma_{A,1}=\Sigma_{(A^-,A)}$, $k_{A,1}=\mu_1-1$ and $\Sigma_{A,2}=(\Sigma_{(A^-,A)})_{>1}$] or [same condition after exchanging 1 and 2 in $\Sigma_{A,1}$, $\Sigma_{A,2}$ and $k_{A,1}$] or [$m_1=0$, $s_A=1$, $\Sigma_{A,1}=\Sigma_{(A^-,A)}$ and $k_{A,1}=\mu_1$] or [$\mu_1=1$, $s_A=1$, $\Sigma_{A,1}=(\Sigma_{(A^-,A)})_{>1}$ and $k_{A,1}=m_1$].

Proof. (1) n_{A^-} divides n_A by indiscernibility of elements from A. It has already been noticed in Fact 8.15 that A_0 is a singleton.
(2) If A is maximal in $\overline{\Theta}$, either any $a\in A$ is a leaf of $T(M)$ and then $s_A=0$, or any such a is a node in $T(M)$ where no branch of Θ goes through and then $\Sigma_{1\le i\le s_A}k_{A,i}\ge 2$.
(3) If $-\infty$ exists, no branch of T has a first element.
(4) Indeed a must be a node in $T(M)^*$.
(5) It is lemma 8.7.
(6) At most one $k_{A,i}$ is infinite by strong minimality of the node a, for any $a\in A$. Cones of M are C-minimal by Proposition 2.8 and $\aleph_0$-categorical since they are definable in $\mathcal{M}$. The cones considered here are furthermore indiscernible by construction, so their canonical trees are colored good trees by Theorem 7.1.
(7) It is a reformulation of Lemma 8.10.
(8) By construction.
(9) The situation has already been set out in Lemma 8.11, that we apply here with $b\in A^-$, $a\in A$ and $\mathcal{C}$ the thick cone at a. In this way $T(\mathcal{C}(]b,a[)\cup\mathcal{C})$ becomes the cone $\Gamma(b,a)$ of a at b. Condition (8) prevents $\mathcal{C}(b\wedge a,a)$ from being a model of $\Sigma_{(A^-,A)}$ hence indiscernible. Would it be the case, $\mathcal{C}(b\wedge a,a)$ would be as well indiscernible in M contradicting maximal indiscernibility of (the orbit of) $\mathcal{C}(]b,a[)$.

(10) Follows from Lemma 8.13. □

A last constraint is given by the next proposition.

Proposition 8.18. *(11) The tree $\overline{\Theta}$ labeled with coefficients n_A, s_A, $k_{A,i}$ and theories $\Sigma_{A,i}$ (and $\Sigma_{(A^-,A)}$) on its vertices (and edges) has no non trivial automorphism.*

By construction two elements from Θ having same type in $\mathcal{M}$ are identified in $\overline{\Theta}$. Thus, to prove the above proposition it is enough to show that, if $\mathcal{M}$ is the countable model, then any automorphism of $\overline{\Theta}$ lifts up to an automorphism of $\mathcal{M}$. This proof requires some new tools that we introduce now.

8.3. Connection and sticking

8.3.1. Connection $\sqcup$ of C-structures.

Let k_i, $i \in I$, be cardinals such that $\Sigma k_i > 1$ and $\mathcal{H}_i$, $i \in I$, C-structures. The underlying set of the *connection* $\mathcal{H} := \bigsqcup_{i\in I} \mathcal{H}_i.k_i$ is the disjoint union of k_i copies of H_i, $i \in I$. Its canonical tree is the disjoint incomparable union of k_i copies of $T(H_i)$, $i \in I$, plus an additional root, say r, id est: for $a, b \in T(H)$, $a \leq b$ in $T(H)$ iff $a = r$ or a and b are in a same copy of $T(H_i)$ for some i, and $a \leq b$ in this $T(H_i)$. For $i \in I$, we call $H_{i,j}$, $j \in k_i$, the different copies of H_i canonically embedded in H.

Language: Assumptions are as follows. Each $\mathcal{H}_i$ is a C-structure in the language $\mathcal{L}(\mathcal{H}_i)$. The structure on $\mathcal{H}_i$ is in fact given via its canonical tree: each $T(\mathcal{H}_i)$ is a structure in a language $\mathcal{L}(T(\mathcal{H}_i))$ such that $\mathcal{L}(T(\mathcal{H}_i)) \setminus \mathcal{L}_1$ consists only of predicate or unary function symbols. Among predicates are D_f and F_f for each unary function $f \in \mathcal{L}(T(\mathcal{H}_i))$ and the interpretation of the triple (f, D_f, F_f) in $T(\mathcal{H}_i)$ is required to satisfy Conditions (4$*$) of section 5.3.
The different languages $\mathcal{L}(T(\mathcal{H}_i)) \setminus \mathcal{L}_1$, $i \in I$, are disjoint.

We consider $T(H)$ in the language

$$\mathcal{L}(T(\mathcal{H})) := \mathcal{L}_1 \dot\cup \{T_i; i \in I\} \dot\cup \dot{\bigcup}_{i\in I} (\mathcal{L}(T(\mathcal{H}_i)) \setminus \mathcal{L}_1) \dot\cup \{E_r\}$$

where each T_i is a unary predicate for the union of the k_i copies $T(H_{i,j})$ of $T(H_i)$, E_r is a unary predicate interpreted as $\{r\}$ if r is the root of $T(H)$ and $\mathcal{L}(T(\mathcal{H}_i)) \setminus \mathcal{L}_1$ is interpreted in $T(\mathcal{H})$ as described now. On each $T(H_{i,j})$, $j \in k_i$, $\mathcal{L}(T(\mathcal{H}_i))$ has its natural interpretation. We interpret it "trivially" outside of the $T(H_{i,j})$: a unary function of $\mathcal{L}(T(\mathcal{H}_i))$ is defined as the identity outside of T_i and an n-ary predicate is taken to be empty outside of $\bigcup_{j\in k_i} T_{i,j}^n$. Note that each $T(\mathcal{H}_i)$ is an $\mathcal{L}(T(\mathcal{H}))$-substructure of $T(\mathcal{H})$.
We set $L_i := T_i \cap L$ id est L_i is a predicate for the subset $\bigcup_{j\in k_i} H_{ij}$ of H.

Lemma 8.19. *If I finite then $\bigsqcup_{i\in I} \mathcal{H}_i.k_i$ is completely axiomatized by the axioms and axiom schemes expressing for each $i \in I$:*

1. *C-structure with a root, say r, in its canonical tree;* $E_r = \{r\}$*;*

2. $\forall x\ (\bigvee_{k\in I} L_k(x))$ *and* $\forall x\ (L_i(x) \to \bigwedge_{j\neq i} \neg L_j(x))$*;*

3_i. *L_i is a union of cones at r;*

4_i. *L_i has exactly $\overline{k}_i$ cones at r, where $\overline{k}_i \in \mathbb{N}\cup\{\infty\}$ and $\overline{k}_i = k_i$ iff $k_i \in \mathbb{N}$;*

5_i. *$(x \notin D_f \to f(x) = x)$ and $(x \in D_f \to r < f(x) \leq x)$, for any unary function $f \in \mathcal{L}(T(\mathcal{H}_i))$;*

6_i. *$R \subseteq T_i^n$ and $\neg R(x)$ for any tuple x having among its coordinates x and y such that $E_r(x \wedge y)$, for any n-ary predicate $R \in \mathcal{L}(T(\mathcal{H}_i)) \setminus \mathcal{L}_1$;*

7_i. *by axioms 5_i, for any cone $\mathcal{C}$ at r, $T(\mathcal{C})$ is an $\mathcal{L}(T(\mathcal{H}_i))$-substructure for any $i \in I$; if $\mathcal{C}$ is contained in L_i, then $T(\mathcal{C})$ is required to be elementary equivalent to $T(\mathcal{H}_i)$ as an $\mathcal{L}(T(\mathcal{H}_i))$-structure.*

If for any $i \in I$, $T(\mathcal{H}_i)$ eliminates quantifiers (respectively is $\aleph_0$-categorical), then $T(\bigsqcup_{i\in I} \mathcal{H}_i.k_i)$ has the same property.

Proof: The three results, completeness, transfer of quantifier elimination, and transfer of $\aleph_0$-categoricity, are proved using a back-and-forth argument. Axioms 1 to 6 imply that:
- $\{r\}$ is an $\mathcal{L}(T(\mathcal{H}))$-substructure with a uniquely determined isomorphism type
- any cone at r in $T(H)$ is an $\mathcal{L}(T(\mathcal{H}))$-substructure (due to axioms 5)
- there is no interaction between these cones or $\{r\}$ via predicates or functions from $\mathcal{L}(T(\mathcal{H})) \setminus \mathcal{L}_1$ (due to axioms 6, indeed x and y are in different cones at r exactly when $x \wedge y = r$).

Consequently the $\mathcal{L}(T(\mathcal{H}))$-structure of the canonical tree of a model is completely determined by its restrictions to cones at r. To prove the lemma, we consider first the case where I is a singleton:

Claim 8.20. *Assume $k_i > 1$. Then the theory given by axioms 1, 3_i, 4_i, 5_i, 6_i and 7_i completely axiomatizes $\mathcal{H}_i.k_i$. If $\mathcal{H}_i$ is $\aleph_0$-categorical, so is $\mathcal{H}_i.k_i$. If $T(\mathcal{H}_i)$ eliminates quantifiers in $\mathcal{L}(T(\mathcal{H}_i))$, so does $T(\mathcal{H}_i.k_i)$ in $\mathcal{L}(T(\mathcal{H}_i)) \cup \{E_r\}$.*

Proof: For any model M of this theory, $T(M)$ is the disjoint union of $\{r\}$ and $\overline{k}_i$ cones at r, all elementary equivalent to $T(H_i)$ as $\mathcal{L}(T(H_i))$-structures. Take two $\aleph_0$-saturated models M and N of this theory and a finite tuple x from $T(M)$. By the considerations above we may assume x contains r and thus decomposes $x = (r, x_1, ..., x_n)$ with x_i a tuple consisting of elements all in the same cone at r and x_i and x_j in different cones for $i \neq j$. Two elements y and z are in the same cone at r iff $\neg E_r(y \wedge z)$. Consequently a tuple $y \in T(N)$ with same quantifier free 0-type as x decomposes in the same way $y = (r, y_1, ..., y_n)$. Let $a \in T(M)$ be a single element. Assume first a is in the same cone Γ

at r as, say x_1. Since Γ has the same theory as $T(H_i)$ it eliminates quantifiers and there is $b \in T(N)$ in the cone of y_1 at r such that (x_1, a) and (y_1, b) have the same quantifier free type in this cone (type in the theory of $T(H_i)$). If a is in the cone at r of none of the x_i, then as the number of such cones is $\overline{k}_i$ in both M and N, there exists $b \in T(N)$ in none of the cones of the y_i with same quantifier free type as a. In both cases (x, a) and (y, b) have same quantifier free type in $T(M)$. $\dashv$

An arbitrary model M of axioms of Lemma 8.19 is of the form $\bigsqcup_{i\in I} L_i(M)$ with $L_i(M) \equiv \mathcal{H}_i.k_i$ by the case where I is a singleton or trivially if $k_i = 1$. A finite tuple x from $T(\mathbb{M})$ containing r may be uniquely written $x = (r, (x_i)_{i\in I})$ with x_i a finite tuple in $T_i(M) \setminus \{r\}$. Another tuple y in a model with same quantifier free type as x is of the form $(r, (y_i)_{i\in I})$ with y_i in T_i with same type as x_i. As in the proof of previous claim we can carry on an infinite back-and-forth between two $\aleph_0$-saturated models. We argue with complete types for each component in some T_i to prove completeness and with complete qf types to transfer qe, using the claim above, or direct quantifier elimination in H_i if $k_i = 1$. The transfer of $\aleph_0$-categoricity is clear. $\square$

Lemma 8.21. *Assume I is finite, k_i is infinite for at most one $i \in I$, say i_0.*
If all $T(\mathcal{H}_i)$ are pure trees and $T(\mathcal{H}_i) \not\equiv T(\mathcal{H}_k)$ for $i \neq k$ in I, then $\bigsqcup_{i\in I} \mathcal{H}_i.k_i$ is a pure tree too.
If $T(\mathcal{H}_{i_0})$ is a pure colored good tree and for any $i \in I$, $\mathcal{H}_i$ is C-minimal then $\bigsqcup_{i\in I} \mathcal{H}_i.k_i$ is C-minimal too.

Proof: We extend each language $\mathcal{L}(T(\mathcal{H}_i))$ with new relations to get quantifier elimination in $T(\mathcal{H}_i)$. By Lemma 8.19, $T(\mathcal{H})$ eliminates quantifiers in $\mathcal{L}(T(\mathcal{H}))$. This shows that definable subsets of a model are Boolean combinations of definable subsets of the L_i. Since I and all k_i except at most one are finite, each L_i is a finite union of cones at r or complement of such an union.
This shows L_i is quantifier free definable with the pure C-relation and parameters. The condition "$T(\mathcal{H}_i) \not\equiv T(\mathcal{H}_k)$" provides a definition without parameters.
Since the L_i are quantifier free definable with the pure C-relation $\mathcal{H}$ is C-minimal if all L_i are. Let us prove L_i is C-minimal. For $i \neq i_0$ the argument is the same as just used: since k_i is finite definable subsets of L_i are Boolean combinations of definable subsets of its cones at r. As these cones are C-minimal (by Proposition 2.8), L_i is C-minimal too. For $i = i_0$ with k_{i_0} infinite, consider on the canonical tree T_{i_0} of L_{i_0} the singleton $E := \{r\}$, e the constant function sending T_{i_0} to r and $\sim$ the equivalence relation defined as in Lemma 5.11, case (2). Now Proposition 5.15 applies and shows that, if $T(H_{i_0})$ is a an n-colored good tree and X is a 1-colored good tree of color $(\infty, 0)$, then $T_{i_0} \equiv X \rtimes T(H_{i_0})$ as pure trees. Thus T_{i_0} is a pure $(n+1)$-colored good tree hence its set of leaves is C-minimal. $\square$

8.3.2. Sticking $\triangleleft$ in a pruned cone M of a C-structure $\mathcal{C}$ whose canonical tree has a root

Let be given two C-structures, first $\mathcal{C}$, which has a root in its canonical tree, and then $\mathcal{M} := (M, V)$, where V is a branch without leaf from $T(M)$. We define the C-structure $\mathcal{M} \triangleleft\ \mathcal{C}$, *sticking of* $\mathcal{C}$ *into* (M, V). The underlying set of $\mathcal{M} \triangleleft\ \mathcal{C}$ is the disjoint union $M\dot{\cup}\mathcal{C}$, its canonical tree the disjoint union $T(M)\dot{\cup}T(\mathcal{C})$ equipped with the unique order extending those of $T(M)$ and $T(\mathcal{C})$ such that $V = \{t \in T(M); t < T(\mathcal{C})\}$.

Canonicity: $\mathcal{M} \triangleleft\ \mathcal{C}$ is the unique C-set which is the union of M and $\mathcal{C}$ and where $\mathcal{C}$ becomes a thick cone with basis the supremum of V.

Language: As in previous subsection, we assume some additional structures given on the canonical trees by languages $\mathcal{L}(T(\mathcal{M}))$ and $\mathcal{L}(T(\mathcal{C}))$, which are such that $\mathcal{L}(T(\mathcal{M})) \setminus \mathcal{L}_1$ and $\mathcal{L}(T(\mathcal{C})) \setminus \mathcal{L}_1$ consist only of predicate or unary function symbols. Among predicates of $\mathcal{L}(T(\mathcal{M})) \setminus \mathcal{L}_1$ there is V. Among predicates are D_f and F_f for each unary function $f \in \mathcal{L}(T(\mathcal{M}))$ or $\mathcal{L}(T(\mathcal{C}))$ and the interpretation of the triple (f, D_f, F_f) in $T(M)$ is required to satisfy Conditions (4∗) of section 5.3

We consider $\mathcal{M} \triangleleft \mathcal{C}$ in the language

$$\mathcal{L}(T(\mathcal{M} \triangleleft\ \mathcal{C})) := \mathcal{L}_1\dot{\cup}\{E_a, E_{\geq a}, G_a\}\dot{\cup}(\mathcal{L}(T(\mathcal{M})) \setminus \mathcal{L}_1)\dot{\cup}(\mathcal{L}(T(\mathcal{C})) \setminus \mathcal{L}_1)\dot{\cup}\{\wedge_V\}$$

where $E_a, E_{\geq a}$ and G_a are unary predicates for the elements of, respectively, the singleton consisting of the basis, call it a, of the thick cone $\mathcal{C}$, $\mathcal{C}$ and M; $\mathcal{L}(T(\mathcal{M}))$ and $\mathcal{L}(T(\mathcal{C}))$ are naturally interpreted in $T(M)$ and $T(\mathcal{C})$ respectively, and then trivially (see below) outside of $T(M)$ and $T(\mathcal{C})$ respectively; $\wedge_V$ is the unary function sending a point $x \in T(M)$ to $x \wedge V$ and the identity on $T(\mathcal{C})$.

Lemma 8.22. *$\mathcal{M} \triangleleft\ \mathcal{C}$ is completely axiomatized by the axioms and axiom schemes expressing*

1. *C-set*
2. *$E_{\geq a}$ is a thick cone in the canonical tree, call a its basis*
3. *E_a is the singleton $\{a\}$*
4. *G_a is the complement of $E_{\geq a}$*
5. *$V = \{x \in G_a; x < a\}$*
6. *$G_a(x) \to \wedge_V(x) = x \wedge V$; $E_{\geq a}(x) \to \wedge_V(x) = x$*
7. *$x \notin D_f \to f(x) = x$ for any unary function $f \in \mathcal{L}(T(\mathcal{M} \triangleleft\ \mathcal{C}))$*
8. *$x \in D_f \to a \leq f(x) \leq x$, for any unary function $f \in \mathcal{L}(T(\mathcal{C}))$; $\neg R(x)$ for any tuple x having some coordinate in G_a and any predicate $R \in \mathcal{L}(T(\mathcal{C})) \setminus \mathcal{L}_1$*

9. *by axioms 7 and 8, $E_{\geq a}$ is an $\mathcal{L}(T(\mathcal{C}))$-substructure; it is required to be elementary equivalent to $T(\mathcal{C})$*

10. *$x \in D_f \to f(x) \leq x$, for any unary function $f \in \mathcal{L}(T(\mathcal{M}))$; $\neg R(x)$ for any tuple x having some coordinate in $E_{\geq a}$ and any predicate $R \in \mathcal{L}(T(\mathcal{M})) \setminus \mathcal{L}_1$*

11. *by axioms 7 and 10, G_a is an $\mathcal{L}(T(\mathcal{M}))$-substructure; it is required to be elementary equivalent to $T(\mathcal{M})$.*

If $T(\mathcal{M})$ and $T(\mathcal{C})$ eliminate quantifiers, or are $\aleph_0$-categorical, then $T(\mathcal{M} \triangleleft \mathcal{C})$ has the same property. If $\mathcal{M}$ and $\mathcal{C}$ are C-minimal then $\mathcal{M} \triangleleft \mathcal{C}$ has the same property.

Proof: The axioms imply that any model has a canonical tree of the form $G_a \triangleleft E_{\geq a}$, with the interpretation of the language we have considered. Consequently it is easy to carry on an infinite back and forth between two $\aleph_0$-saturated models. This shows all assertions except C-minimality. By transfer of quantifier elimination G_a and $E_{\geq a}$ are stably embedded in $G_a \triangleleft E_{\geq a}$. Since the set of leaves of $G_a \triangleleft E_{\geq a}$ is the union of those of G_a and $E_{\geq a}$, $\mathcal{M} \triangleleft \mathcal{C}$ is C-minimal if $\mathcal{M}$ and $\mathcal{C}$ are. □

8.4. Proof of proposition 8.18 and reconstruction of $\mathcal{M}$ from $\overline{\Theta}(\mathcal{M})$

Consider a finite meet-semi-lattice Ξ_0, A_0 its root and $A \in \Xi_0 \setminus \{A_0\}$. Vertices and edges are labeled as follows.
- All vertices are labeled. Labels of a vertex $A \in \Xi_0$ are of several types: two integers $n_A \geq 1$ and s_A, cardinals $k_{A,1}, \ldots, k_{A,s_A} \in \mathbb{N}^{\geq 1} \cup \{\infty\}$ and complete $\mathcal{L}_1$-theories $\Sigma_{A,1}, \ldots, \Sigma_{A,s_A}$ which are not, at this point, supposed all different.
- Some edges are labeled by a complete $\mathcal{L}_1$-theory. For $A \neq A_0$, the complete $\mathcal{L}_1$-theory possibly labeling (A^-, A) is denoted $\Sigma_{(A^-,A)}$.
We must now reformulate conditions (1) to (10) of Lemma 8.17, and (11) of Proposition 8.18 in terms of meet-semi-lattice and labels only. For example, due to Lemma 8.16, the condition "$-\infty$ exists in $T(M)$" will be replaced by "$s_{A_0} = 0$, A_0 has a unique successor (in Ξ_0), say B and $n_B = 1$". So conditions $(1'), (2'), (3'), (6'), (7'), (8'), (9')$ and $(10')$ are the same as $(1), (2), (3), (6), (7), (8), (9)$ and (10) in Lemma 8.17 and $(11')$ is the same as (11) in Proposition 8.18 with $\overline{\Theta}$ replaced with Ξ_0, "$-\infty$ exists in $T(M)$" replaced as indicated and "$]A^-, A[$ not empty" replaced with "there is an $\mathcal{L}_1$-theory labeling (A^-, A)". The other conditions are:

(4') Assume $A \neq A_0$. If A has a unique successor, say B and $n_B = n_A$, then $s_A \geq 1$. (This reformulation of (4) into (4') uses Lemma 8.23.)

(5') An $\mathcal{L}_1$-theory possibly labeling an edge of Ξ_0 is a complete theory of colored good tree.

Lemma 8.23. *Given a finite meet-semi-lattice tree Ξ_0, A_0 its root, Ξ_0 labeled with a coefficient n_A to each $A \in \Xi_0$ and satisfying (1'), there is a unique ordered set Ξ which is the disjoint union of antichains U_A, $A \in \Xi_0$, and satisfying that, for all $A, B \in \Xi_0$:*
(a) $|U_A| = n_A$,
(b)(for all $a \in U_A$, exists $b \in U_B$, $a < b$ in Ξ) iff $A < B$ in Ξ_0,
(c) if $B^- = A$ and $a \in U_A$, then there are exactly $|n_B/n_A|$ elements $b \in U_B$ such that $b > a$.
Furthermore:
(d) Ξ is a meet-semi-lattice tree,
(e) the set of the U_A ordered by the order induced by the order of Ξ is isomorphic to Ξ_0,
(f) any automorphism of the labeled tree Ξ_0 lifts to an automorphism of the tree Ξ,
(g) given two points in Ξ belonging to the same antichain of Ξ_0, there is an automorphism of Ξ sending one to the other one,
(h) for $A \in \Xi_0$, Ξ has a unique branch starting from some (or any) $a \in A$ iff (Ξ_0 has a unique branch starting from A and, if $B^- = A$ then $n_B = n_A$).

Proof: We define inductively an order on $\Xi := \dot{\bigcup}_{A \in \Xi_0} U_A$. We take U_{A_0} a singleton, as it should be. Let $\Xi_1 \subseteq \Xi_0$ satisfying $[(A, B \in \Xi_0 \;\&\; A < B \;\;\&\; B \in \Xi_1) \Rightarrow A \in \Xi_1]$ and assume $\dot{\bigcup}_{A \in \Xi_1} U_A$ already ordered in such a way that the U_A are antichains and satisfy (a), (b) and (c) for $A, B \in \Xi_1$. Let $X \in \Xi_0 \setminus \Xi_1$ such that $X^- =: B \in \Xi_1$. Since $X^- = B$, n_B divides n_X which allows us to take for each $y \in U_B$ an antichain W_y with $n_X(n_B)^{-1}$ elements and $U_X := \dot{\bigcup}_{y \in U_B} W_y$; for $x \in U_X$ and $y \in U_B$ we set $x > y$ iff $x \in W_y$, with no other order relation between elements from $U_B \cup U_X$. So we have extended the order from $\dot{\bigcup}_{A \in \Xi_1} U_A$ to $\dot{\bigcup}_{A \in \Xi_1} U_A \dot{\cup} U_X$. Due to (a), (b) and (c) we made the only possible choice. By construction (a), (b) and (c) are true on $\Xi_1 \cup \{X\}$.
(d) The order Ξ we constructed is a meet-semi-lattice tree because Ξ_0 is one and $n_{A_0} = 1$.
(e) and (h) are clear.
(f) is proven by induction. Let σ be an automorphism of the labeled tree Ξ_0, $\Xi_1 \subseteq \Xi_0$, X and B as at the beginning of the proof of (a), (b) and (c) but we assume now furthermore Ξ_1 closed under σ. We assume also there is τ a partial automorphism of the tree Ξ defined on $\dot{\bigcup}_{A \in \Xi_1} U_A$ and lifting $\sigma \restriction \Xi_1$. Let $\mathcal{X} = \{X, \sigma(X), \sigma^2(X), \ldots, \sigma^{r-1}(X)\}$ be the orbit of X under σ. Since σ preserves the order, $\mathcal{X}$ is an antichain and $\sigma^i(X)^- = \sigma^i(B)$ which belongs to Ξ_1 since Ξ_1 is closed under σ. So we can extend τ on $\dot{\bigcup}_{A \in \Xi_1 \cup \mathcal{X}} U_A$ by taking any bijective map $U_{\sigma^i(X)} \to U_{\sigma^{i+1}(X)}$ for any i, $0 \leq i < r$.
(g) Let $A \in \Xi_0$ and $x, y \in U_A$. We carry on the induction of the proof of (f) starting with σ the identity of Ξ_0, τ the identity on $\bigcup_{\{X \in \Xi_0; \neg(X \geq A)\}} U_X$ and choosing a function $U_A \to U_A$ sending x to y. $\square$

Theorem 8.24. *Given a finite meet-semi-lattice tree Ξ_0 labeled with coefficients and theories satisfying (1') to (7'), consider the language $\mathcal{L} := \{C\} \cup \{P_{A,i}; A \in \Xi_0, 1 \leq i \leq s_A\} \cup \{P_{A^-,A}; A \in \Xi_0, A \neq A_0\}$ where all new symbols represent unary predicates. Then*

there exists a unique finite-or-countable $\mathcal{L}$-structure $\mathcal{M}$ such that the tree Ξ built from Ξ_0 and $\{n_A; A \in \Xi_0\}$ according to Lemma 8.23 embeds in $T(M)^*$ in such a way that for any $A \in \Xi_0$, $A \neq A_0$:
(a) Let $A, B \in \Xi_0$, $B = A^-$, and $a, b \in \Xi$, $a \in A$, $b \in B$, $b < a$; then, either there is no theory labeling the edge (B, A) and b is the predecessor of a in $T(M)$, or $(\Gamma(]b, a[),]b, a[)$ is model of $\Sigma^V_{(B,A)}$ (as defined in Lemma 8.12 from the theory labeling (B, A)); $P_{B,A}$ is the union of all pruned cones $\mathcal{C}(]b, a[)$ for a and b as above. Any cone at b which does not contain a is contained in one of the $P_{B,i}$ and, for each $i \leq s_B$, $P_{B,i} \cap \mathcal{C}(b)$ consists of exactly $k_{B,i}$ cones at b, all with a canonical tree model of $\Sigma_{B,i}$.
(b) "Pieces" $P_{A,i}$ and $P_{A^-,A}$, $A \in \Xi_0, 1 \leq i \leq s_A$, are stably and purely embedded in $\mathcal{M}$ and the structure $\mathcal{M}$ is induced by them, in the sense that the definable sets of $\mathcal{M}$ are exactly the Boolean combinations of definable sets of these pieces.
Then $\mathcal{M}$ is C-minimal and $\aleph_0$-categorical and any automorphism of the labeled tree Ξ that preserves the class in Ξ_0 extends to an automorphism of $T(\mathcal{M})$.

Unlike the proof of Lemma 8.23, here we use a downward induction, more precisely an induction of the depth of vertices, that we now define.

Definition 8.25. *Let Ξ be a finite semi-lattice tree. The* depth *of a vertex in Ξ is the minimal function from Ξ to ω such that:*
- if a is a maximal element of Ξ, $depth(a) = 0$,
- if $x < y$, $depth(x) \geq depth(y) + 1$.

Proof: We will define simultaneously C-structures $\mathcal{M}_a$ and $\mathcal{N}_a$ for $a \in \Xi$, by induction on $depth(a)$, $\mathcal{M}_a$ for each of these a and $\mathcal{N}_a$ if furthermore a is not the root of Ξ. The M_a are intended to become thick cones in $\mathcal{M}$ and the N_a cones, and they will be the only possible choice thanks to the canonicity of both constructions of connection and sticking. Their languages are, if $a \in A \in \Xi_0$, $\mathcal{L}(\mathcal{M}_a) := \{C\} \cup \{P_{B,i}; B \in \Xi_0, B > A, 1 \leq i \leq s_B\} \cup \{P_{B^-,B}; B \in \Xi_0, B > A\}$ and, if $N_a \neq M_a$, $\mathcal{L}(\mathcal{N}_a) := \mathcal{L}(\mathcal{M}_a) \cup \{P_{A^-,A}\}$. As previously we work with canonical trees: $T(\mathcal{M}_a)$ and $T(\mathcal{N}_a)$ will be shown by induction to eliminate quantifiers in languages $\mathcal{L}(T(\mathcal{M}_a))$ and $\mathcal{L}(T(\mathcal{N}_a))$ respectively, and to be $\aleph_0$-categorical trees. By induction too the $\mathcal{M}_a$ and the $\mathcal{N}_a$ are C-minimal.
Let us start.
Theories such as $\Sigma_{A,i}$ or $\Sigma_{(A^-,A)}$, $A \in \Xi_0$, appear among the labels. By (6') each theory $\Sigma_{A,i}$ is the theory of some n-colored good tree for some integer n and we consider $\Sigma_{A,i}$ in its elimination language $\mathcal{L}(\Sigma_{A,i}) := \mathcal{L}^+_n$. Let $\Gamma_{A,i}$ be the unique finite-or-countable model of $\Sigma_{A,i}$ and $\mathcal{C}_{A,i}$ the C-set with canonical tree $\Gamma_{A,i}$.
By (5') if the label $\Sigma_{(A^-,A)}$ exists, consider $\Sigma^V_{(A^-,A)}$, its enrichment as in Lemma 8.12. It eliminates quantifiers in the language $\mathcal{L}(\Sigma^V_{(A^-,A)}) := \mathcal{L}^{V+}_n$. Let $(\Gamma_{(A^-,A)}, V_A)$ be the unique finite-or-countable model of $\Sigma^V_{(A^-,A)}$ and $\mathcal{C}_{(A^-,A)}$ the C-set with canonical tree $\Gamma_{(A^-,A)}$.

- Let A be maximal in Ξ_0 and $a \in A$. Due to axiom $(2')$ either $s_A = 0$ or $\Sigma_{1\leq i\leq s_A} k_{A,i} \geq 2$. If $s_A = 0$ we take for $\mathcal{M}_a$ a singleton and $\mathcal{L}(T(\mathcal{M}_a)) := \mathcal{L}_1$. If $\Sigma_{1\leq i\leq s_A} k_{A,i} \geq 2$ we define $\mathcal{M}_a := \bigsqcup_{1\leq i\leq s_A} \mathcal{C}_{A,i} \cdot k_{A,i}$. Each $\Gamma_{A,i}$ is considered in its elimination language $\mathcal{L}(\Sigma_{A,i})$ and $\mathcal{L}(T(\mathcal{M}_a))$ is given by Lemma 8.19. It eliminates quantifiers. It is to be noticed that in both cases $T(\mathcal{M}_a)$ has a as a root.

- If A is not maximal in Ξ_0 and $a \in A$, we take for $\mathcal{M}_a$ the connection of $k_{A,i}$ copies of $\mathcal{C}_{A,i}$ and $(n_B : n_A)$ copies of $\mathcal{N}_b$, for $1 \leq i \leq s_A$ and $B^- = A, b \in B, b > a$. Due to condition (4') this connection is well defined since the number of connected C-structures is at least 2. Here again the $\Gamma_{A,i}$, the $\mathcal{N}_b$ and $T(\mathcal{M}_a)$ are considered in their elimination languages (some $\mathcal{L}^+_{n_{A,i}}$ for the $\Gamma_{A,i}$, given by induction hypothesis for the $\mathcal{N}_b$, and by Lemma 8.19 for $T(\mathcal{M}_a)$) and $T(\mathcal{M}_a)$ has a as a root.

- For A different from the root A_0 of Ξ_0, if there is a theory $\Sigma_{(A^-,A)}$ we set $\mathcal{N}_a = \mathcal{M}_a \triangleright (\mathcal{C}_{(A^-,A)}, V_A)$. If there is no theory labeling (A^-, A) we set $\mathcal{N}_a = \mathcal{M}_a$.

- In the case where $s_{A_0} = 0$, A_0 has a unique successor B in Ξ_0 with $n_B = 1$, call b the unique element of B; we define $\mathcal{M} = \mathcal{N}_b$; then $T(M)$ has no root and A_0 embeds in $T(M)^*$ as $\{-\infty\}$. Else, we define $\mathcal{M} = \mathcal{M}_{a_0}$, where $A_0 = \{a_0\}$.

We look now a bit more carefully at languages in the above construction. An easy downwards induction shows that, for $a, c \in A \in \Xi_0$, the two structures $(T(M_a), \mathcal{L}(T(\mathcal{M}_a)))$ and $(T(M_c), \mathcal{L}(T(\mathcal{M}_c)))$ are isomorphic, as are $(T(N_a), \mathcal{L}(T(\mathcal{N}_a)))$ and $(T(N_c), \mathcal{L}(T(\mathcal{N}_c)))$ when $A \neq A_0$. And indeed we choose to identify the languages $\mathcal{L}(T(\mathcal{M}_a))$ and $\mathcal{L}(T(\mathcal{M}_c))$ on one hand and $\mathcal{L}(T(\mathcal{N}_a))$ and $\mathcal{L}(T(\mathcal{N}_c))$ on the other hand. This means that in the situation where $a, c > b$, $b \in A^-$ when constructing $\mathcal{M}_b$ by a connection, $T(\mathcal{N}_a)$ and $(T(\mathcal{N}_c)$ are considered as two copies of the same structure, like $\mathcal{H}_{i,j}$ and $\mathcal{H}_{i,k}$ in Subsection 8.3.1. We do not do any other identification: if for example the same language $\mathcal{L}_n$ appears as elimination language in $\mathcal{N}_a$ and $\mathcal{M}_b$ or in $\mathcal{N}_a$ and $\mathcal{N}_c$ for two nodes a and c which do not belong to the same antichain, then it will be duplicated, one avatar for each node.

Note that the $\mathcal{L}(T(\mathcal{M}_a))$-structure of $TM_a)$ is definable in $\mathcal{L}(\mathcal{M}_a)$ and the $\mathcal{L}(T(\mathcal{N}_a))$-structure of $T(N_a)$ definable in $\mathcal{L}(\mathcal{N}_a)$. Hence the $\mathcal{L}(T(\mathcal{M}))$-structure of $T(M)$ is definable in $\mathcal{L}(\mathcal{M})$. By construction Ξ embeds into $T(M)^*$ and $\mathcal{M}$ satisfies properties (a) and (b) ((b) follows from quantifier elimination). By induction this $\mathcal{M}$ is unique (above Ξ) due to $\aleph_0$-categoricity of labels theories and canonicity of connection and sticking. It is $\aleph_0$-categorical and C-minimal due to Lemmas 8.19, 8.22 and 8.21.

Let τ be an automorphism of Ξ preserving the projection $\Xi \to \Xi_0$. We define by induction an automorphism ρ of $T(\mathcal{M})$ extending τ. Again there are two induction steps. Either ρ is defined on $\Xi \cup T(M_a)$ (or on $\Xi \cup T(N_b)$ for each $b, b^- = a$) and we want to extend it to $\Xi \cup T(N_a)$ (or to $\Xi \cup T(M_a)$). Since τ preserves classes in Ξ_0 it preserves labels, and the conclusion follows by $\aleph_0$-categoricity of involved theories and canonicity of the sticking (or connection) construction. $\square$

Proof of Proposition 8.18: As already noticed just after the statement of Proposition 8.18, it is enough to prove that any automorphism of the labeled tree $\overline{\Theta}(\mathcal{M})$ lifts up to an automorphism of $T(\mathcal{M})^*$ ($\mathcal{M}$ is here the countable model). Thus Proposition 8.18 follows immediately from Lemma 8.23 and Theorem 8.24. $\square$

Theorem 8.26. *If the labeled tree Ξ_0 satisfies furthermore (8'), (9'), (10') and (11') then $\mathcal{M}$ is a pure C-set, $\Theta(\mathcal{M}) = \Xi$ and $\overline{\Theta}(\mathcal{M}) = \Xi_0$.*

Proof: Let $\Xi_{\geq a} := \{x \in \Xi; x \geq a\}$. We show, by induction on vertices depth, that $\Xi_{\geq a} = \Theta(\mathcal{M}_a)$ and, if $N_a \neq M_a$ and $b \in B := A^-$, $b < a$, $\Theta(\mathcal{N}_a) = \Xi_{\geq a} \cup \{b\}$ where b plays here the role of $-\infty$ for the tree $\Theta(\mathcal{N}_a)$.
1. $a \in \Theta(\mathcal{M}_a)$: this means that $\mathcal{M}_a$ is not indiscernible, which follows from (9') for A maximal in Ξ_0 and from (10') if A is not maximal.
2. a remains in $\Theta(\mathcal{N}_a)$ either trivially if a has a predecessor in $T(M)$ or because of (10'). Since a is in $\Theta(\mathcal{N}_a)$ it is $\emptyset$-definable (in $\mathcal{N}_a$) and the tree $\Xi_{\geq a}$ remains in $\Theta(\mathcal{N}_a)$.
3. So Ξ embeds in $\Theta(\mathcal{M})$. Elements of Ξ are thus $\emptyset$-algebraic. Elements of Ξ_0 are $\emptyset$-definable due to (11'). An induction (using Lemma 8.21 and (8')) shows that $\mathcal{M}$ is a pure C-set.
4. Any point in $T(M) \setminus \Xi$ is in some canonical copy of either some pruned cone $\Gamma_{(A^-,A)}$ or some cone $\Gamma_{A,i}$. Since C-sets associated to these trees are indiscernible, an element of $\Gamma_{(A^-,A)}$ or $\Gamma_{A,i}$ can belong to $\Theta(\mathcal{M})$ only if it belongs to U (see Definition 8.3), which is impossible in both situations.
This proves that $\Theta(\mathcal{M})$ is exactly Ξ and consequently $\bar{\Theta}(\mathcal{M})$ is Ξ_0. $\square$

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DEFINABLE COMPLETENESS OF P-MINIMAL FIELDS AND APPLICATIONS

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Abstract. We show that every definable nested family of closed and bounded subsets of a P-minimal field K has non-empty intersection. As an application we answer a question of Darnière and Halupczok showing that P-minimal fields satisfy the "extreme value property": for every closed and bounded subset $U \subseteq K$ and every interpretable continuous function $f\colon U \to \Gamma_K$ (where Γ_K denotes the value group), $f(U)$ admits a maximal value. Two further corollaries are obtained as a consequence of their work. The first one shows that every interpretable subset of $K \times \Gamma_K^n$ is already interpretable in the language of rings, answering a question of Cluckers and Halupczok. This implies in particular that every P-minimal field is polynomially bounded. The second one characterizes those P-minimal fields satisfying a classical cell preparation theorem as those having definable Skolem functions, generalizing a result of Mourgues.

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Symmetric Real Semigroups. A summary of results.

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1 Introduction

The subject of this paper originates in the preprint [D]. The motivation of the latter was the search for examples of real semigroups which are fans (RS-fans) beyond those given in [DP2]. In [D] a detailed study was carried out of the real spectrum of and the real semigroups associated to the rings $F[\![G]\!]$ of formal power series with coefficients in a formally real (i.e., orderable) field F and exponents in the positive cone G^+ of an arbitrary totally ordered abelian group G.

For the reader's benefit we begin with a reminder of the main results proved in [D] that bear a relation with those summarized in the later sections §§ 4 − 8. To abridge we will frequently write A for rings of type $F[\![G]\!]$.

2 A summary of results from [D].

2.1 Basics. (a) [D, Fact 2.7]. The set of all prime ideals of $A = F[\![G]\!]$ is in an inclusion reversing bijective correspondence with the set of convex subgroups of G; hence it is totally ordered under inclusion.

(b) [D, Proposition 2.8]. Every prime ideal I of A is real (i.e., $\sum_{i=1}^n a_i^2 \in I$ with $a_1, \ldots, a_n \in A$ implies $a_1, \ldots, a_n \in I$).

(c) **Notation.** • The rings $A = F[\![G]\!]$ carry a natural (Krull) valuation defined by:

$$v\left(\textstyle\sum_{g\in G^+} a_g X^g\right) = \min \operatorname{supp}\left(\textstyle\sum_{g\in G^+} a_g X^g\right),$$

where $\operatorname{supp}\left(\sum_{g\in G^+} a_g X^g\right) = \{g \in G^+ \mid a_g \neq 0\}$ for a non-null series, and $v(0) = \infty$. For a non-zero series $a \in A$ we denote by $a_{v(a)}$ the smallest non-zero coefficient of a.

• (Real semigroups.) For the definition, notation and basic results concerning *real semigroups* (RS) the reader is referred to [DP1]. The most important examples are the RSs, G_R, associated to (commutative, unitary) rings R, see [DST, 13.6.5, pp. 536-539]. The element in G_R corresponding to an element $a \in R$ is denoted by $\overline{a}$. We denote by X_G the set of *characters* of the RS G, i.e., the RS-homomorphisms $G \longrightarrow \mathbf{3}$, where $\mathbf{3} = \{1, -1, 0\}$ is endowed with its (uniquely determined) structure of RS, cf. [DP1, Corollary 2.4, p. 109] or [DP2, Example 2.8, p. 13]. The *zero-set* of an element $a \in G$ is $Z(a) := \{h \in X_G \mid h(a) = 0\}$ and the *zero-set* of a character $h \in X_G$ is $Z(h) := h^{-1}[0] \subseteq G$.

(d) [D, Theorem 3.1]. (*i*) *The family* $\{Z(\overline{a}) \mid a \in A\}$ *of zero-sets of elements of* G_A *is totally ordered under inclusion.*

(*ii*) *The zero-sets of characters of* G_A *are in one-one, inclusion preserving, correspondence with the prime ideals of* G_A. *Hence, they are totally ordered under inclusion. It follows that* G_A *has a unique maximal ideal.* □

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2.2 Characterizing the elements of Sper $(F[\![G]\!])$**.**

The elements of the real spectrum of the rings $A := F[\![G]\!])$ are determined by objects of three types:

• A total order T of the coefficient field F;

• A convex subgroup C of G (possibly improper);

• A subgroup H of C of index 1 or 2,

as follows:

Definition 2.2.1 Given a triple of sets T, C, H as above, we define:

$$\begin{aligned}\alpha_{T,C,H} = \{\, a \in A \mid v(a) > C, \ \ &\text{or } \ v(a) \in H \ \text{ and } \ a_{v(a)} >_T 0, \\ &\text{or } \ v(a) \in C \setminus H \ \text{ and } \ a_{v(a)} <_T 0 \,\} \cup \{0\},\end{aligned}$$

and prove:

Theorem 2.2.2 (i) ([D, Proposition 4.3]) *For all parameters* T, C, H *as above,* $\alpha_{T,C,H} \in$ Sper (A).

(ii) ([D, Proposition 4.4]) *Let* $\alpha \in$ Sper (A). *Then, there are parameters* T, C, H *such that* $\alpha = \alpha_{T,C,H}$. □

2.3 The order structure of Sper $(F[\![G]\!])$.

For ready reference we recall:

(a) The **connected components** of a root system $(X, \preceq)$ are the equivalence classes of X under the relation of having a common $\preceq$-upper bound, see [DP2, Def. 11.1 (a), p. 62]. For the definition of a root systrem, see [DST, Appendix A5 (i), p. 584].

(b) Every spectral space comes endowed with a partial order, called **specialization**, see [DST, 1.1.3, p. 3]. Further, every element of the space lies under a maximal element for this order, see [DST, Proposition 4.1.2, p. 103]. The specialization partial order of the real spectrum of any ring is set-theoretic inclusion (and is a root system). For more details, cf. [DST, Chapter 13]. □

Proposition 2.3.1. ([D, Proposition 5.1]) *For* $i = 1, 2$, *let* T_i *be total orders of* F, C_i *be convex subgroups of* G, *and* H_i *be a subgroup of* C_i *such that* $[C_i : H_i] \le 2$. *Then,*

$$\alpha_{T_2,C_2,H_2} \subseteq \alpha_{T_1,C_1,H_1} \Leftrightarrow \ T_1 = T_2, \ C_1 \subseteq C_2 \ \text{ and } \ H_1 = H_2 \cap C_1. \qquad \square$$

The following Proposition describes the exact structure of the maximal inclusion chains and the connected components of the real spectrum of $A =$ Sper $(F[\![G]\!])$.

Proposition 2.3.2. ([D, Proposition 5.2]) *Let* T *be a total order of* F. *Then,*

(i) *For every subgroup* H *of* G *of index* ≤ 2, *the set*

$$\mathcal{C}_{T,H} = \{\alpha_{T,C,H \cap C} \mid C \text{ convex subgroup of } G\}$$

is a maximal inclusion chain of Sper (A) *with minimal element* $\alpha_{T,G,H}$ *and maximal element* $\alpha_{T,\{0\}}$ $(= \alpha_{T,\{0\},\{0\}})$.

(ii) *The set*

$$\mathcal{C}_T = \{\alpha_{T,C,H \cap C} \mid C \text{ convex subgroup of } G;\ H \text{ subgroup of } G \text{ of index } \le 2\},$$

is a connected component of the spectral root system Sper (A), *having* $\alpha_{T,\{0\}}$ *as top element. Every connected component of* Sper (A) *is of this form for a unique* T. □

2.4 The real semigroup associated to the ring $F[\![G]\!]$.

Fact 2.4.1. ([D, Fact 6.1]) *For every $a \in A \setminus \{0\}$, its image $\overline{a} \in G_A$ is identical to $\overline{a_{v(a)} X^{v(a)}}$, the image in G_A of the smallest non-zero term of the series a.* $\square$

The value of terms of the form $\overline{fX^g}$ at elements of the real spectrum of A can be explicitely computed:

Lemma 2.4.2. ([D, Lemma 6.4]) *For $g \in G^+, f \in F \setminus \{0\}$ and parameters T, C, H as in Definition 2.2.1, we have:*

$$\overline{(fX^g)}(\alpha_{T,C,H}) = \begin{cases} 0 & \text{if } g > C \\ \mathrm{sgn}_T(f) & \text{if } g \in H \\ -\mathrm{sgn}_T(f) & \text{if } g \in C \setminus H. \end{cases} \qquad \square$$

Theorem 2.4.3. (Characterization of "bar equality".) *Let $a, b \in A \setminus \{0\}$. The following are equivalent:*

(1) $\overline{a} = \overline{b}$.

(2) *(i)* $a_{v(a)} \cdot b_{v(b)} \in \sum F^2 \setminus \{0\}$ *(i.e., $a_{v(a)}$ and $b_{v(b)}$ have the same sign in all orders of F.)*

(ii) $v(a) \sim v(b)$ *(i.e., $v(a)$ and $v(b)$ belong to the same convex subgroups of G.)*

(iii) $v(a) \equiv v(b) \pmod{2G}$. $\square$

2.5 Characterization of the real semigroups $G_{F[\![G]\!]}$ that are fans.

The initial motivation to study the real semigroups associated to the rings $A = F[\![G]\!]$ of formal power series was to obtain examples of rings R whose associated RS G_R is a RS-fan in the sense of [DP2]. The analysis summarized above achieved a complete answer to this query, and reads as follows:

Theorem 2.5.1. *Let F be a formally real field and let G be a totally ordered abelian group; $A := F[\![G]\!]$ denotes the ring of formal power series with coefficients in F and exponents in $G^+ = \{g \in G \,|\, G \geq 0\}$. The following are equivalent:*

(1) *The real semigroup G_A associated to A is a RS-fan.*

(2) *The preorder $\sum F^2$ of F is a (field) fan.*[1] $\square$

3 From real spectra to character spaces.

The set-theoretic framework set up for the real spectra of rings is equivalent to a function-theoretic one for their associated real semigroups. This later approach is akin to our presentation of symmetric real semigroups in § 4 below. We begin by explicitly stating this equivalence for rings in general, and then develop some of its consequences in the case of rings of formal power series.

(a) Given a ring R, to each $\alpha \in \mathrm{Sper}\,(R)$ there corresponds a unique character $h_\alpha \in X_R$ defined by: for $a \in R$, $h_\alpha(\alpha) := \mathrm{sgn}_\alpha(\pi_\alpha(a))$, i.e., the sign ($1$, -1 or 0) of $\pi_\alpha(a)$ at the total order $\leq_\alpha$ of $R/(\alpha \cap -\alpha)$ determined by α (with $\pi_\alpha : R \longrightarrow R/(\alpha \cap -\alpha)$ canonical). Explicitly,

$$h_\alpha(\overline{\alpha}) = \begin{cases} 0 & \text{if } a \in \alpha \cap -\alpha \\ 1 & \text{if } a \in \alpha \setminus (-\alpha) \\ -1 & \text{if } a \in (-\alpha) \setminus \alpha; \end{cases}$$

i.e., $h_\alpha(\overline{\alpha})(\overline{a}) = \overline{a}(\alpha)$ ($\overline{a}$ = image of $a \in R$ in G_R). In other words, h_α is the map dual to the

[1] I.e., for $f_1, f_2 \in F$ so that $-f_1 f_2 \notin \sum F^2$, we have $f_1 \cdot \sum F^2 + f_2 \cdot \sum F^2 = (f_1 \cdot \sum F^2) \cup (f_2 \cdot \sum F^2)$.; cf. [La, Chapter 5].

element $\overline{a} \in G_R$ (seen itself as a map). Obviously, $Z(h_\alpha) = \alpha \cap -\alpha$.
The compositional inverse of the assignement $\alpha \longmapsto h_\alpha$ is the map

$$X_{G_R} \ni h \longmapsto \alpha_h \in \mathrm{Sper}\,(R),$$

where $\alpha_h = \{a \in R \,|\, h(\overline{a}) \in \{0,1\}\}$.
(b) Recall (2.3 (b)) that the specialization order of Sper (R) is set-theoretic inclusion; for $\alpha, \beta \in$ Sper (R) we have $\alpha \subseteq \beta \Leftrightarrow h_\alpha \rightsquigarrow h_\beta$.
(c) Back to the case $A := F[\![G]\!]$, from 2.3.2 we know that every connected component of A is the set of $\subseteq$-predecessors of the precone $\alpha_{T,\{0\}}$, for a unique total order T of F, and its minimal elements are all of the form $\alpha_{T,G,H}$ with H a subgroup of G of index ≤ 2. We have:

Fact 3.1 *(i) By maximality of $\alpha_{T,\{0\}}$ $Z(\alpha_{T,\{0\}})$ is the unique (cf. 2.1 (d.ii)) maximal ideal of G_A.*
(ii) Minimality of $\alpha_{T,G,H}$ entails $Z(h_{T,G,H}) = \{\overline{0}\}$.
Proof. (i) is clear.
(ii) Assume there is $\overline{a} \in Z(h_{T,G,H})$ such that $\overline{a} \neq 0$. From 2.4.1 it follows that $\overline{a} = \overline{fX^g}$ for some $f \in F^\times$ and $g \in G^+$. Since also $-\overline{a} \in Z(h_{T,G,H})$, by Lemma 2.4.3 we have

$$\overline{a}(\alpha_{T,C,H}) = \begin{cases} \mathrm{sgn}_T(f) & \text{if } g \in H \\ -\mathrm{sgn}_T(f) & \text{if } g \in C \setminus H, \end{cases}$$

and

$$(-\overline{a})(\alpha_{T,C,H}) = \begin{cases} \mathrm{sgn}_T(-f) = -\mathrm{sgn}_T(f) & \text{if } g \in H \\ -\mathrm{sgn}_T(-f) = \mathrm{sgn}_T(f) & \text{if } g \in C \setminus H, \end{cases}$$

a contradiction. □

In the terminology of characters we have proved:

Proposition 3.2 *Let A be a ring of formal power series. For every $h \in X_{G_A}$ there are $h', h'' \in X_{G_A}$ such that $h'' \rightsquigarrow h \rightsquigarrow h'$, $Z(h') = \mathfrak{m}_{G_A}$, and $Z(h'') = \{\overline{0}\}$.* □

Our next result about the character space of X_{G_A} is:

Theorem 3.3 (Coherence principle) *If $h_1, h_2 \in X_{G_A}$ are in the same connected component and h' is any character of G_A, then the product $h_1 h_2 h'$ is also a character of G_A.*

The proof of this result requires a more sophisticated argument than that of the previous Proposition. Auxiliary results will be stated within the proof.

Proof. Since $h_1, h_2 \in X_{G_A}$ are in the same connected component, by 2.3.2 they are of the form $h_i = h_{T,C_i,H_i}$ for some (and the same) total order T of F and parameters C_i and H_i as in 2.2. Let $h' = h_{T',C',H'}$ (possibly $T' \neq T$). We must prove:

(†) There are a convex subgroup C of G and a subgroup H of C of index ≤ 2 such that $h_1 h_2 h' = h_{T',C,H}$.

(Note, by 2.3.1, that if $h_1 h_2 h' = h_{T'',C,H}$, then $T'' = T'$.)

Let $\overline{a} \neq 0$ be an arbitrary element of $G_A \setminus \{\overline{0}\}$. By 2.4.1 we may assume that $\overline{a} = \overline{fX^g}$, with $f \in F^\times$ and $g \in G^+$. If $g > C_1 \cap C_2 \cap C'$ (i.e., g larger that one of C_1, C_2 or C' as these are comparable under inclusion) then, by 2.4.2, $k(\overline{a}) = k(\overline{fX^g}) = 0$ for k equal to one of h_1, h_2 or h', and hence $(h_1 h_2 h')(\overline{a}) = 0$. Then, a likely candidate for C in (†) is $C_1 \cap C_2 \cap C'$. For this guess to be confirmed we need to find a subgroup H of C so that $[C : H] \leq 2$ and (†) holds. The answer is provided by:

Proposition A. *Let G be an abelian group and let $\mathcal{S}_2(G)$ be the set of subgroups of G of index at most two. Then $\mathcal{S}_2(G)$ is an abelian group under the group operation $*$ defined, for $H_1, H_2 \in \mathcal{S}_2(G)$, by:*

$$H_1 * H_2 = (H_1 \cap H_2) \cup (H_1^c \cap H_2^c) = (H_1 \triangle H_2)^c.$$

where $\triangle$ denotes symmetric difference. G is the unit for the operation $*$. $\mathcal{S}_2(G)$ is of exponent two. □

Lemma B. *Let G be an abelian group and let $H_1, \ldots, H_n \in \mathcal{S}_2(G)$ with n odd. Then*

$$H_1 * \ldots * H_n = H_1 \triangle \ldots \triangle H_n. \qquad \square$$

Applying these results to the previous situation (where $n = 3, H_3 = H'$), the operation $(H_1 \triangle H_2 \triangle H') \cap C$ is well-defined, i.e., gives a subgroup H of index ≤ 2 of the convex subroup C of G. Direct verification using the identity

$$H_1 \triangle H_2 \triangle H' = (H_1 \cap H_2 \cap H') \cup (H_1^c \cap H_2^c \cap H') \cup (H_1^c \cap H_2 \cap H'^c) \cup (H_1 \cap H_2^c \cap H'^c),$$

where $K^c := C \setminus K$ for $K \in \{H_1, H_2, H'\}$, and item (b) in § 3 above, prove (†) and therefore $h_1 h_2 h' \in X_{G_A}$.

We illustrate this verification by checking a couple of cases (with $\overline{a} = \overline{fX^g}$):

- If $g \in H_1 \cap H_2 \cap H'$, by 2.4.2 we have $h_i(\overline{a}) = \mathrm{sgn}_T(f)$ for $i = 1, 2$ and $h'(\overline{a}) = \mathrm{sgn}_{T'}(f)$, whence $(h_1 h_2 h')(\overline{a}) = \mathrm{sgn}_T(f)^2 \cdot \mathrm{sgn}_{T'}(f) = \mathrm{sgn}_{T'}(f)$.
- If, say, $g \in H_1^c \cap H_2 \cap H'^c$, then $h_1(\overline{a}) = -\mathrm{sgn}_T(f)$, $h_2(\overline{a}) = \mathrm{sgn}_T(f), h'(\overline{a}) = -\mathrm{sgn}_{T'}(f)$, then, again by 2.4.2:

$$(h_1 h_2 h')(\overline{a}) = -\mathrm{sgn}_T(f) \cdot \mathrm{sgn}_T(f) \cdot (-\mathrm{sgn}_{T'}(f)) = -1 \cdot (-\mathrm{sgn}_{T'}(f)) = \mathrm{sgn}_{T'}(f). \qquad \square$$

Theorem 3.3 can be generalized by a natural extension of the arguments used in the preceding proof. The following notion is useful in these generalizations:

Definition 3.4 Let G be a RS. We say that a finite sequence $\langle h_1, \ldots, h_n \rangle$ of characters in X_G is **coherent** if the product $\prod_{i=1}^n h_i$ is a character. [2]

In terms of this notion we get:

Theorem 3.3′. *Let A be a ring of formal power series. Given characters $h_1, h_2, g_1, \ldots, g_{n-2} \in X_{G_A}$ such that h_1 and h_2 are in the same connected component of X_{G_A}, the sequence $\langle h_1, h_2, g_1, \ldots, g_{n-2} \rangle$ is coherent if and only if $\langle g_1, \ldots, g_{n-2} \rangle$ is coherent.* □

In fact, this result holds for any RS satisfying the conditions of Proposition 3.2.

Remark 3.5 A rather remarkable fact is that the coherence principle stated in Theorem 3.3′ is equivalent *for any RS satisfying the conditions of Proposition 3.2* to the following (apparently) far more general property:

(Generalized Coherence Principle) *Given characters $f_1, \ldots, f_n, g_1, \ldots, g_n \in X_G$ such that for all $i \in \{1, \ldots n\}$, g_i and f_i are in the same connected component of X_G, if one of these sequences is coherent, so is the other.*

In fact, this is just one of a series of (generalized) coherence principles, all equivalent under the hypotheses on the RS G stated above. A particular case that will be used below (see axiom [SYM.3] in Definition 4.1) is when the zero-set of all f_i is the maximal ideal of G and $g_i \rightsquigarrow f_i$ for all $i = 1, \ldots, n$ □

4 Symmetric real semigroups.

The properties of the real semigroups associated to the rings of type $F[[G]]$ proved in § 3 (see 3.2, 3.3, 3.3′, 3.5), suggest the following notion:

[2] The given sequence may have repetitions.

Definition 4.1 A real semigroup, G, is called **symmetric** if it satisfies the following requirements:

[Z] The family $\{Z(a) \mid a \in G\}$ of zero-sets of elements of G is totally ordered under inclusion. Equivalently, for all $a, b \in G$, either $a^2b^2 = a^2$ or $a^2b^2 = b^2$.

[SYM.1] For all $h \in X_G$ there exists $h_0 \in X_G$ such that $h_0 \rightsquigarrow h$ and $Z(h_0) = \{0\}$.

[SYM.2] For all $h \in X_G$ there exists $h_{\mathfrak{m}_G} \in X_G$ such that $h \rightsquigarrow h_{\mathfrak{m}_G}$ and $Z(h_{\mathfrak{m}_G}) = \mathfrak{m}_G$.[3]

[SYM.3] (Coherence Axiom) Given characters $h_1, \ldots, h_n, g_1, \ldots, g_n \in X_G$ $(n \geq 1)$ such that $Z(h_i) = \mathfrak{m}_G$ and $g_i \rightsquigarrow h_i$ for $i = 1, \ldots, n$, if the sequence $\langle h_1, \ldots, h_n \rangle$ is coherent[4], so is the sequence $\langle g_1, \ldots, g_n \rangle$. □

Remarks 4.2 (1) The coherence axiom [SYM.3] is a weakening —and a relative form— of the condition defining the notion of a fan in the category **ARS** of abstract real spectra (ARS-fan): an ARS-fan is an ARS closed under the product of any three (and hence any odd number) of its members; see [DP2, Definition 1.3 (2), p. 8]. [Note that the requirement "$\langle h_1, \ldots, h_n \rangle$ is coherent" implies $(\prod_{i=1}^n h_i)(-1) = -1$, and hence that n is odd.]

(2) Fans in the category of real semigroups (RS-fans) and symmetric RS are related but not identical notions: each of these classes contains members that are not in the other. In fact, from [DP2, Theorem 7.1, p. 39] it follows that

> *A symmetric real semigroup G is a RS-fan if and only if every quotient $(G/I) \setminus \{\pi_I(0)\}$ by a proper ideal I of G is a fan in the category of reduced special groups.*

(3) Theorem 4.5 below gives the most significant relationship between the spaces of characters of RS-fans and of symmetric RS, a relationship that justifies the noun "symmetric" given to the class defined above. □

Proposition 4.3 ([DP3, Corollary 7.11]) *Let G be a real semigroup with the following property: For all $h, g \in X_G$, we have $h^2g \in X_G$.*[5] *Let G^* be the group of invertible elements of G. With the representation relation induced from G, G^* is a reduced special group.* □

The following property characterizes the connected components (cf. 2.3 (a)) of the space of characters of symmetric real semigroups.

Proposition 4.4 ([DP3, Proposition 7.15]) *Let G be a real semigroup satisfying condition* [Z] *in* 4.1 *and*

[C] *Given $g, h \in X_G$, if $Z(g) \subseteq Z(h)$, there is $h' \in X_G$ so that $Z(h) = Z(h')$ and $g \rightsquigarrow h'$.*

Let $h_1, h_2 \in X_G$. Then the following are equivalent:

(i) *h_1 and h_2 are in the same connected component of X_G.*

(ii) *There exists a character $h \in X_G$ such that $h_1 \rightsquigarrow h$, $h_2 \rightsquigarrow h$ and $Z(h) = \mathfrak{m}_G$.*

(iii) *For all $g \in G^*$, $h_1(g) = h_2(g)$.* □

The next Theorem and its Corollary furnish essential tools to study the class of symmetric RS. They also justify the name "symmetric" given to it.

Theorem 4.5 ([DP3, Theorem 7.16]) *Any symmetric real semigroup, G, satisfies the following conditions:*

(i) *For every $h \in X_G$, the set $P_h = \{g \in X_G : g \rightsquigarrow h\}$ is an ARS-fan.*

(ii) *For all characters $h_1, h_2 \in X_G$ such that $Z(h_1) = Z(h_2)$, the sets P_{h_1} and P_{h_2} are isomorphic as ARS-fans.* □

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[3] If G is a RS satisfying condition [Z], $\mathfrak{m}_G$ denotes the maximal ideal of G.

[4] See 3.4.

[5] Equivalently, for all $a, b \in G$, $Z(a) \subset Z(b)$ implies $D^t_G(a, b) = \{a\}$.

Corollary 4.6 ([DP3, Corollary 7.17]) *Let G be a symmetric real semigroup. Then every connected component of X_G is an ARS-fan. Moreover, any two connected components of X_G are isomorphic as ARS-fans and, in particular, as abstract real spectra.* $\square$

The following results gives a useful criterion for the isomorphism of *finite* symmetric real semigroups.

Theorem 4.7 ([DP3, Theorem 7.21]) *Let G, H be finite symmetric real semigroups and let G^*, H^* be the reduced special groups of their invertible elements (see 4.3). The following are equivalent:*

(1) *G and H are isomorphic (as RS).*

(2) (*i*) *G^* and H^* are isomorphic as reduced special groups.*

(*ii*) *$(X_G, \underset{G}{\rightsquigarrow})$ and $(X_H, \underset{H}{\rightsquigarrow})$ are order-isomorphic posets, where $\underset{G}{\rightsquigarrow}$ and $\rightsquigarrow_H$ denote the specialization partial orders of X_G and X_H, respectively.* $\square$

5 Representation of symmetric real semigroups by rings of formal power series.

In this section we address the natural question whether every symmetric real semigroup is isomorphic to the RS associated to some ring of formal power series (in the sense of § 2). The answer turns out to be negative in general —a counterexample is (briefly) presented in Proposition 5.2 and Thorem 5.3 below. However, it is positive for *finite* symmetric RSs, as shown by:

Theorem 5.1 ([DP3, Theorem 8.3]) *Let T be a finite symmetric real semigroup and let F be a field whose space of orders is isomorphic, as an abstract space of orders, to the space of characters X_{T^*} of the reduced special group T^* of invertible elements of T* [6]. *Then there exists a totally ordered abelian group G satisfying the following conditions:*

(*i*) *G has finite rank*[7] *and a finite number of subgroups of index 2.*

(*ii*) *X_T is isomorphic (as an abstract real spectrum) to $\mathrm{Sper}(A)$, where $A = F[[G]]$, and hence G_A is isomorphic to T.* $\square$

Remark. We note that Theorem 4.7 is an essential ingredient in the proof of this representation theorem. $\square$

Proposition 5.2 ([DP3, Proposition 8.5 and Corollary 8.6]) *Let G be a group of exponent two with a distinguished element $-1 \neq 1$. Then, there exists an (explicitly constructed) RS-fan, T_G, whose space of characters is a two-level root-system, where:*

(*i*) *The set of elements of level 0 (i.e., those whose zero-set is $\{0\}$) are in a one-one correspondence with $\chi(G)$* [8]*, the space of characters of the* RSG-fan *associated to G.*

(*ii*) *The level 1 consists of a unique character h_1 with $Z(h_1) = G \cup \{0\}$.*

(*iii*) *h_1 specializes (the character of T_G corresponding to) each $h \in \chi(G)$.*

These properties guarantee that T_G is a symmetric RS. $\square$

[6] See Proposition 4.3. The existence of such a field —in fact, a considerably more general result— is proved in [M, Theorem 4.2.2, p. 65, Remark (2), p. 66, and Corollary 8.7.5, pp. 174-176].

[7] The *rank* of a totally ordered abelian group is the cardinality of the set of its convex subgroups.

[8] $\chi(G)$ denotes the set of *group characters* of G to ± 1, identical with the space of characters of the (unique) RSG-fan structure on G.

Finally, the announced counterexample is given by:

Theorem 5.3 ([DP3, Theorem 8.7]) *Let H be a group of exponent two and dimension strictly larger than $2^{\aleph_0}$ (as a vector space over the two-element field $\mathbb{F}_2$). Then the symmetric RS-fan T_H is not realizable by any ring of the form $F[[G]]$, where F is a formally real field and G is a totally ordered abelian group.*

To see why this construction yields the desired counterexample we include the (brief) argument proving the contention.

Proof. Suppose, towards a contradiction, that there exists a formally real field F and a totally ordered abelian group G such that G_A is isomorphic to T_H, where $A = F[[G]]$. Since T_H has only two proper ideals, G has only two convex subgroups, G and $\{0\}$. Therefore, G is Archimedean and hence isomorphic to a subgroup of the additive group $<\mathbb{R}, 0, +>$. On the other hand, since X_{T_H} has only one connected component, F has only one order, P. As the character space of H may be identified with the elements of Sper$\,(A)$ of the form $\alpha_{P,G,K}$ where K is a subgroup of G of index ≤ 2, and the cardinality of G is $\leq 2^{\aleph_0}$, it follows that the cardinality of $\chi(H)$ is at most $2^{2^{\aleph_0}}$. On the other hand, the dimension of H over $\mathbb{F}_2$ being larger than $2^{\aleph_0}$, it follows that its space of characters, or, equivalently, its dual space in the category of vector spaces over $\mathbb{F}_2$, has cardinality strictly larger than $2^{2^{\aleph_0}}$, a contradiction. $\square$

6 Extensions of reduced special groups by 3-semigroups.

6.1 Preliminaries. An important construction in the theory of special groups (SG) is that of the **extension** of a SG by a group of exponent 2. As a motivation for an analogous construction in the realm of real semigroups presented in this section, we begin by briefly recalling the very basics of the notion of extension for SGs. For a more detailed treatment of this construction and its basic properties see [ABR, Chapter IV, § 2, pp. 91-95] and [DM, pp. 11-14 and Example 5.18, p. 88].

Definition. Given a special group, G (not necessarily reduced), and a group Γ of exponent 2, the **extension** $G[\Gamma]$ is the group $G \times \Gamma$ with unit $\mathbf{1} := (1, 1)$, distinguished element $-\mathbf{1} := (-1, 1)$, and representation relation defined by [9]: for $g \in G, \gamma \in \Gamma$,

$$D_{G[\Gamma]}(\mathbf{1}, (g, \gamma)) = \begin{cases} D_G(1, g) \times \{1\} & \text{if } g \neq -1, \gamma = 1 \\ G \times \Gamma & \text{if } g = -1, \gamma = 1 \\ \{\mathbf{1}, (g, \gamma)\} & \text{if } \gamma \neq 1. \end{cases}$$

A direct proof that $G[\Gamma]$ is a special group appears in [Li]. If the SG, G, is reduced, a proof using the dual framework of abstract order spaces (also called spaces of orderings) can be found in [ABR, Proposition IV.2.13 (a), p. 93].

A typical example of a extension is the reduced special group of the *field* of formal power series $F((X))$ with a formally real field F of coefficients: the RSG $G_{F((X))}$ is the extension of the RSG $G(F) = F^\times/(\sum F^2)^\times$ of F by the group $\mathbb{Z}_2 = \{\pm 1\}$. See also [ABR, Corollary VI.1.4 (b) and Remark VI.1.5 (a), p. 147].

Our concern in this section is whether (and to what extent) there exists a notion of extension in the realm of real semigroups. We do not know at present of any analog of this notion applying to RSs in full generality. However, below we present a construction of the extension of *reduced special groups* by 3-semigroups (see 6.2 (ii)) *that produces real semigroups.* This construction

[9] The binary representation relation in a SG, G, is determined by defining the set $D_G(1, \cdot)$, as for arbitrary $x, y, z \in G$ we have $z \in D_G(x, y) \Leftrightarrow xz \in D_G(1, xy)$.

fits well the framework of symmetric RSs and, indeed, has an important role in it, as we will show in § 7. $\square$

Definition 6.2 (i) A **ternary semigroup** (abbreviated TS) is a structure $\langle S, \cdot, 1, 0, -1\rangle$ with individual constants $1, 0, -1$, and a binary operation " $\cdot$ " such that:

[TS1] $\langle S, \cdot, 1\rangle$ is a commutative semigroup with unit.

[TS2] $x^3 = x$ for all $x \in S$.

[TS3] $-1 \neq 1$ and $(-1)(-1) = 1$.

[TS4] $x \cdot 0 = 0$ for all $x \in S$.

[TS5] For all $x \in S$, $x = -1 \cdot x \Rightarrow x = 0$.

We shall write $-x$ for $-1 \cdot x$.

(ii) The semigroups satisfying conditions [TS1] and [TS2] (no constants other than 1) will be called **3-semigroups**. $\square$

Remarks and Notation 6.3 (i) The notion of a ternary semigroup is already a part of that of a real semigroup; see [DP2, Definitions 2.1, pp. 10-11, and 2.7, p. 13]. However, we only need it explicitly in these last sections.

(ii) Given a group of exponent 2 with a distinguished element $-1 \neq 1$ we denote by X_G the set of group homomorphisms $G \longrightarrow \{\pm 1\}$ sending -1 to -1. Likewise, X_T will denote the set of (TS-)homomorphisms of a ternary semigroup T into $\mathbf{3}$ (they preserve all three constants of T), and the set of (RS-)homomorphisms of a real semigroup T into $\mathbf{3}$ (they preserve the constants and the representation relations of T).

(iii) Given a 3-semigroup Δ, we denote by $\chi(\Delta)$ the set of all semigroup homomorphisms of Δ into $\mathbf{3}$. Morphisms in $\chi(\Delta)$ will be denoted by Greek characters.

(iv) Note that a 3-semigroup may or may not have an absorbent element 0 and it *does not* have a distinguished element -1. Thus, the morphisms of $\chi(\Delta)$ are only required to preserve product and send 1 to 1. In particular, if Δ does not contain an absorbent element, the constant map sending all of Δ to 1 is in $\chi(\Delta)$. If Δ has an absorbent element 0, it is understood that the morphisms of $\chi(\Delta)$ map $0 \in \Delta$ to $0 \in \mathbf{3}$. Clearly, an absorbent element (if any) is unique.

(v) If Δ is a 3-semigroup (with or without a 0), then $\Delta \setminus \{0\}$ is also a 3-semigroup. $\square$

Definition 6.4 Let G be a group of exponent 2 and let Δ be a 3-semigroup. We set $G[\Delta] := (G \times \Delta) \cup \{0\}$, where $0 \notin G \times \Delta$, and define a binary operation $\cdot$ in $G[\Delta]$ as follows: if $x, y \in G[\Delta]$,

$$x \cdot y = \begin{cases} 0 & \text{if } x = 0 \vee y = 0 \\ (gg', dd') & \text{if } x = (g, d) \wedge y = (g', d'), \end{cases}$$

where $g, g' \in G$ and $d, d' \in \Delta$. $\square$

Proposition 6.5 ([DP3, Propositions 0.3 and 0.5, Appendix]) *Let G be a group of exponent 2 with a distinguished element -1 $(\neq 1)$ and let Δ be a 3-semigroup. Let $\widehat{1}$ denote the unit of Δ. Then $(G[\Delta], \cdot, (1, \widehat{1}), (-1, \widehat{1}), 0)$ is a ternary semigroup. If Δ satisfies condition [Z] (i.e., for all $a, b \in \Delta$, $a^2b^2 = a^2$ or $a^2b^2 = b^2$), then $G[\Delta]$ also satisfies condition [Z],.* $\square$

Our next task will be to analyze the structure of the set $X_{G[\Delta]}$ of TS-characters of the ternary semigroup $G[\Delta]$.

Proposition 6.6 *Let G be a group of exponent 2 with a distinguished element -1 ($\neq 1$) and let Δ be a 3-semigroup with unit $\widehat{1}$. Then the following conditions hold:*

(i) If $h \in X_G$ and $\tau \in \chi(\Delta)$, the map $h \cdot \tau : G[\Delta] \longrightarrow \mathbf{3}$ defined by $(h \cdot \tau)(g, d) = h(g) \cdot \tau(d)$ if $(g, d) \in G \times \Delta$, and $(h \cdot \tau)(0) = 0$, is a character of ternary semigroups, i.e., $h \cdot \tau \in X_{G[\Delta]}$.

(ii) Conversely, given $p \in X_{G[\Delta]}$ there are unique characters $h \in X_G$ and $\tau \in \chi(\Delta)$ such that $p = h \cdot \tau$. □

Let Δ be a 3-semigroup. An **ideal** of Δ is a non-empty subset I of Δ such that $I \cdot \Delta \subseteq I$. Note that I is a proper ideal of Δ if and only if $\widehat{1} \notin I$.

The next result characterizes the ideals of the ternary semigroup $G[\Delta]$ in terms of the ideals of Δ.

Proposition 6.7 ([DP3, Proposition 0.6, Appendix]) *Let G be a group of exponent 2 with a distinguished element -1 and let Δ be a 3-semigroup. Then the following conditions hold:*

(i) If I is an ideal of Δ, then $\widehat{I} = (G \times I) \cup \{0\}$ is an ideal of $G[\Delta]$.

(ii) If J is an ideal of $G[\Delta]$ and $J \neq \{0\}$, there exists a unique ideal I of Δ such that $J = \widehat{I}$. □

Definition 6.8 Let G be a reduced special group and let Δ be a 3-semigroup satisfying condition $[Z]$. We define ternary relations $D_{G[\Delta]}$ and $D^t_{G[\Delta]}$ in $G[\Delta]$ by the following prescription, where $x, y, z \in G[\Delta]$:

$$x \in D_{G[\Delta]}(y, z) \ \Leftrightarrow\ \forall h \in X_G\, \forall \tau \in \chi(\Delta)[(h \cdot \tau)(x) \in D_{\mathbf{3}}((h \cdot \tau)(y), (h \cdot \tau)(z)))],$$

and, similarly, replacing $D_{\mathbf{3}}$ by $D^t_{\mathbf{3}}$, where $\cdot$ is the product of characters defined in 6.6. □

The following Theorem is of central importance insofar it gives a tractable characterization of transversal representation in $G[\Delta]$. Its proof is long and delicate.

Theorem 6.9 ([DP3, Theorem 0.8, Appendix]) *Let G be a reduced special group and let Δ be a 3-semigroup satisfying condition $[Z]$. Then the transversal representation relation $D^t_{G[\Delta]}$ (6.8) satisfies the following formula:*

$$[F] \qquad D^t_{G[\Delta]}(x, y) = \begin{cases} \{x\} & \text{if } Z(x) \subset Z(y) \text{ or } y = 0 \\ \{y\} & \text{if } Z(y) \subset Z(x) \text{ or } x = 0 \\ \{x, y\} & \text{if } Z(x) = Z(y),\ x = (g, d),\ y = (g', d') \text{ and } d \neq d' \\ D_G(g, g') \times \{d\} & \text{if } Z(x) = Z(y),\ x = (g, d),\ y = (g', d) \text{ and } g \neq -g' \\ x^2 \cdot G[\Delta] & \text{if } x = -y, \end{cases}$$

where $Z(x) = \{h \cdot \tau \mid (h \cdot \tau)(x) = 0,\ h \in X_G$ and $\tau \in \chi(\Delta)\}$. □

Corollary 6.10 ([DP3, Corollary 1, Appendix]) *Let G be a reduced special group and let Δ be a 3-semigroup satisfying condition $[Z]$. Then the transversal representation relation $D^t_{G[\Delta]}$ defined in 6.8 satisfies the following condition:*

$[C]$ If $x \neq -y$ and $z \in D^t_{G[\Delta]}(x, y)$ then either $z \in \{x, y\}$ or there are $g_1, g_2, g_3 \in G[\Delta]^*$ such that $g_3 \in D_{G[\Delta]}(g_1, g_2)$ and $zg_3 = xg_1 = yg_2$.

Moreover, if $\Delta^* = \{\widehat{1}\}$, transversal representation in $G[\Delta]$ may be restated in terms of condition $[C]$, as follows:

Let $x, y, z \in G[\Delta]$ be such that $x \neq -y$ and $Z(x) = Z(y)$. Then $z \in D^t_{G[\Delta]}(x, y)$ if and only if either $z \in \{x, y\}$ or there are $g_1, g_2, g_3 \in G[\Delta]^*$ such that $g_3 \in D_{G[\Delta]}(g_1, g_2)$ and $zg_3 = xg_1 = yg_2$,

where Δ^* is the set of invertible elements of the 3-semigroup Δ. □

The following Theorem confirms that, in the present context, extensions of RSGs by 3-semigroups do yield real semigroups. The proof is delicate.

Theorem 6.11 ([DP3, Theorem 2, Appendix]) *Let G be a reduced special group and let Δ be a 3-semigroup satisfying condition $[Z]$. Then $(G[\Delta], D_{G[\Delta]}, \cdot, (1,\widehat{1}), (-1,\widehat{1}), 0)$ is a real semigroup.* $\square$

The next Proposition shows how the RSs of rings of formal power series can be naturally presented as extensions.

Proposition 6.12 ([DP3, Proposition 0.9, Appendix]) *Let F be a formally real field and let G be a totally ordered abelian group. Let $A = F[[G]]$ be the ring of formal power series with coefficients in F and exponents in G^+. Then the real semigroup G_A is isomorphic to the extension of the reduced special group $G(F) = F^\times/(\sum F^2)^\times$ of F by the 3-semigroup $\Delta = \{\overline{X^g} : g \in G^+\}$.* $\square$

Our final result in this section characterizes those 3-semigroups Δ for which all extensions $G[\Delta]$ with G a reduced special group, are symmetric.

Theorem 6.13 *Let Δ be 3-semigroup satsfying condition $[Z]$. Then the following conditions are equivalent.*

(i) If $h \in \chi(\Delta)$, there exists $h' \in \chi(\Delta)$ such that $h' \rightsquigarrow h$ and $Z(h') = \{0\}$.

(ii) If $p, q, x \in \Delta$ and $px = py$, then $xy \in \mathrm{Id}(\Delta)$.

Moreover, for every reduced special group G the extension $G[\Delta]$ is a symmetric RS if and only if Δ satisfies condition (ii). $\square$

7 Symmetric real semigroups as extensions.

We begin this section by showing that if G is a symmetric real semigroup, the inclusion $i : G^* \cup \{0\} \hookrightarrow G$ has a retract (Theorem 7.5). This function turns out to be an essential tool in proving the main result, Theorem 7.6, that every symmetric real semigroup is the extension of the RSG of its invertible elements (see 4.3) by a suitably chosen 3-semigroup.

Notation 7.1 Given a RS, G, we denote by $X_G^{\max}$ (resp., $X_G^{\min}$) the set of maximal (resp., minimal) points for the specialization partial order $\rightsquigarrow$ of the character space X_G of G. These sets are not empty; indeed, for every x in <u>any</u> spectral space, X, there are $y \in X^{\min}$ and $z \in X^{\max}$ such that $y \rightsquigarrow x \rightsquigarrow z$; cf. [DST, Proposition 4.1.2, p. 103, and Corollary 4.1.4, pp. 104-105]. $\square$

Remarks 7.2 (i) Let G be a symmetric real semigroup and fix $h_0 \in X_G^{\max}$. By Theorem 4.5 (i) the set $P_{h_0} = \{g \in X_G : g \rightsquigarrow h_0\}$ is a ARS-fan (a fan as an abstrat real spectrum), and [DP3, Theorem 7.16] shows that the set $F_{h_0} = \{g \in X_G^{\min} \mid g \rightsquigarrow h_0\}$ is a subfan of it.

(ii) Moreover, $\Delta_{h_0} = \bigcap_{g \in F_{h_0}} \ker(\widehat{g})$ is a saturated subgroup of the reduced special group $G_0 = (G/\{0\}) \setminus \{\pi_0(0)\}$ and the quotient G_0/Δ_{h_0} is a RSG-fan whose space of characters is isomorphic to F_{h_0} ([DP3, Proposition 3, Appendix]). [10]

(iii) $\pi_0 : G \longrightarrow G/\{0\}$ and $\pi_{h_0} : G_0 \longrightarrow G_0/\Delta_{h_0}$ denote the respective canonical quotient maps. $\square$

Theorem 7.3 *Let G be a symmetric real semigroup. Let $(x_i)_{i \in I}$ be a family of elements of G such that $(\pi_{h_0}(\pi_0(x_i)))_{i \in I}$ is a basis of the fan G_0/Δ_{h_0} as a $\mathbb{Z}_2$-vector space (cf.7.2 (ii)). Then for every $f \in \{\pm 1\}^I$ and every $h \in X_G^{\max}$ there exists a <u>unique</u> $g_h \in U_f$ such that $g_h \rightsquigarrow h$ (i.e., $g_h \in F_h \cap U_f$), where $U_f := \bigcap_{i \in I} U(f(i)x_i) \cap X_G^{\min}$.* $\square$

[10] Recall that $\widehat{g} : X_G \longrightarrow \mathbf{3}$ is the map "evaluation at g": for $h \in X_G$, $\widehat{g}(h) := h(g) \in \mathbf{3}$.

Theorem 7.4 *Let G be a symmetric real semigroup and let $U := \bigcap_{i\in I} U(x_i) \cap X_G^{\min}$ be the set U_f defined in* 7.3 *for the function $f \in \{\pm 1\}^I$ with constant value* 1. *Let $\mu : U \to X_G^{\max}$ be the map $\mu(g) = h_g$ where h_g is the unique character in $X_G^{\max}$ such that $g \rightsquigarrow h_g$. Then μ is an isomorphism of spaces of orderings.* □

Theorem 7.5 *Let G be a symmetric real semigroup. Then there exists a retract $r : G \longrightarrow G^* \cup \{0\}$ of the inclusion map $i : G^* \cup \{0\} \hookrightarrow G$.*

Sketch of proof. For each $g \in G$, let $\widehat{\pi_0(g)} : X_G^{\min} \longrightarrow \mathbf{3}$ be the map defined by $\widehat{\pi_0(g)}(h) = h(g)$ for $h \in X_G^{\min}$. $\widehat{\pi_0(g)}$ is a continuous function in the topology induced by X_G, that preserves the product of any coherent triple in $X_G^{\min}$. In particular, if $g \neq 0$ we have $\widehat{\pi_0(g)} : X_G^{\min} \to \mathbf{2} = \{-1, 1\}$. Let $\mu : U \to X_G^{\max}$ be the isomorphism of spaces of orderings defined in Theorem 7.4. Then, the composition $\widehat{\pi_0(g)} \circ \mu^{-1} : X_G^{\max} \to \mathbf{3}$ is a continuous function preserving the product of any three elements in $X_G^{\max}$. It follows from [M, Corollary 3.2.4] that there exists a unique $g^* \in G^*$ such that $\widehat{g^*} = \widehat{\pi_0(g)} \circ \mu^{-1}$. We define the map $r : G \longrightarrow G^* \cup \{0\}$ as follows:

$$r(g) = \begin{cases} g^* & \text{if } g \neq 0 \\ 0 & \text{if } g = 0. \end{cases}$$

This map r is the desired retract of the inclusion map $i : G^* \cup \{0\} \hookrightarrow G$. We only check that r is a retract. Let $g \in G^* \cup \{0\}$. Since $r(0) = 0$, it is enough to see that $\widehat{\pi_0(g)} \circ \mu^{-1} = \widehat{g}$, for $g \in G^*$. Let $h \in X_G^{\max}$. Since $\mu^{-1}(h) \rightsquigarrow h$, then $\mu^{-1}(h)(g) = g$ or, equivalently, $\widehat{\mu^{-1}(h)}(\pi_0(g)) = h(g)$. Since $\widehat{g}(h) = h(g)$, we conclude that r is a retract. □

With the map r in hand, below we indicate the main steps leading to:

Theorem 7.6 *Let G be a symetric real semigroup and let $r : G \longrightarrow G^* \cup \{0\}$ be the retract of the inclusion map $i : G^* \cup \{0\} \hookrightarrow G$ constructed in the preceding Theorem* 7.5. *Let $\Delta = \{x \in G : r(x) = 1\}$. Then G is isomorphic to $G^*[\Delta]$, the extension of the reduced special group of invertible elements of G by the* 3*-semigroup Δ.* □

Proposition 7.7 ([DP3, Proposition 2, Appendix]) *With notation as in Theorem* 7.6, *the following conditions hold:*

(i) $\Delta = \bigcap_{h\in U} \ker(h)$, *where $U = \bigcap_{i\in I} U(x_i) \cap X_G^{\min}$ and $(x_i)_{i\in I}$ is a family of elements of G satisfying the hypothesis of Theorem* 7.3.

(ii) If $x \in G\setminus\{0\}$, then $x \in \Delta \Leftrightarrow \pi_0(x) \in D_{G_0}(\langle\langle \pi_0(x_{i_1}), \ldots, \pi_0(x_{i_r}) \rangle\rangle)$ for some finite subset $\{i_1, \ldots, i_r\}$ of I, where $\langle\langle \pi_0(x_{i_1}), \ldots, \pi_0(x_{i_r}) \rangle\rangle$ denotes the Pfister form generated by the coefficients $\{\pi_0(x_{i_1}), \ldots, \pi_0(x_{i_r})\}$. □

The following result is the key step in the proof of Theorem 7.6:

Theorem 7.8 ([DP3, Theorem 3, Appendix]) *With notation as in Theorem* 7.6, *if the family $(x_i)_{i\in I}$ of elements of G satisfies the hypothesis of Theorem* 7.3, *we have*

$$x \in \Delta \text{ if and only if either } \pi_0(x) = \pi_0(1) \text{ or } \pi_0(x) = \pi_0\left(\textstyle\prod_{j=1}^r x_{i_j}\right)$$

for some finite subset $\{i_1, \ldots, i_r\}$ of I. □

Theorem 7.9 ([DP3, Theorems 5 and 6, Appendix]) *Let G be a symmetric real semigroup and let $x, y, z \in G\setminus\{0\}$ be such that $x \neq -y$, $Z(x) = Z(y)$ and $z \in D_G^t(x, y)$. If $r : G \longrightarrow G^* \cup \{0\}$ is the retract constructed in Theorem* 7.5, *we have:*

- $xr(x) \neq yr(y) \Rightarrow z \in \{x, y\}$.
- $xr(x) = yr(y) \Rightarrow zr(z) = xr(x) = yr(y)$. □

The proof of this last result is long and delicate.

8 First-order axiomatizability.

Our definition of the symmetric real semigroups is couched in terms of characters (cf. 4.1). A natural question, then, is whether this class admits a first-order axiomatization in the language $\mathcal{L}_{\mathrm{RS}} = \{\cdot, 1, 0, -1, D^t\}$ for real semigroups. The affirmative answer is given by:

Theorem 8.1 *Let G be a real semigroup. Then G is a symmetric real semigroup if and only if it satisfies the following first-order axioms in the language $\mathcal{L}_{\mathrm{RS}}$ of real semigroups:*

[*Z*] $\forall a, b \in G\ (a^2b^2 = a^2 \vee a^2b^2 = b^2)$. (*Condition* 4.1.[*Z*])

[SRS.1] *Given $a, b_1, \ldots, b_n, x \in G\setminus\{0\}$, if $ax^2 \in D_G(\langle\langle b_1, \ldots, b_n \rangle\rangle)$ and $Z(ab_1) \subseteq \ldots Z(ab_n)$, then $ab_n^2 \in D_G(\langle\langle b_1, \ldots, b_n \rangle\rangle)$, where $\langle\langle b_1, \ldots, b_n \rangle\rangle$ is the Pfister form generated by the coefficients $b_1, \ldots, b_n$.*

[SRS.2] $\forall a, b \in G\ (Z(a) \subset Z(b) \Rightarrow D^t(a, b) = \{a\})$.

[SRS.3] *For all $a, b \in G$, if $a \neq -b$ and $c \in D^t_G(a, b)$, then either $c \in \{a, b\}$ or there are $g_1, g_2, g_3 \in G^*$ such that $g_3 \in D_G(g_1, g_2)$ and $cg_3 = ag_1 = bg_2$.* □

Remark. Each of the axioms [SRS.i] ($i = 1, 2, 3$) above is equivalent to the corresponding axiom [SYM.i] in Definition 4.1. By far the trickiest of these equivalences is for $i = 3$. Proving this equivalence required, as a preliminry step, to show that any symmetric RS, G, is isomorphic to the extension $G^*[\Delta]$ (Theorem 7.6) which, in turn, needed the construction of the retract r (Theorem 7.5) and, previously, to develop the theory of extensions presented in § 6. □

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DEFINABLY COMPLETE DENSE C-MINIMAL STRUCTURES

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Abstract. The C-minimal structures M we consider here are dense, definably complete and in their canonical tree $T(M)$ there is no definable bijection from an interval $]a, b[$ to an interval $]c, d[$, for any $b \in M$ and $d \notin M$. We show that in such an M every definable nested family of closed and bounded subsets has non-empty intersection, as long as the family is indexed by an interval Y of the canonical tree of the form $]a, e[$ for some e in M. As a consequence, for every closed and bounded subset $U \subseteq M$ and every interpretable continuous function f from U to Y, $f(U)$ is bounded above in Y.

1. Introduction

P-minimal fields are definably complete in a strong sense and the structure induced on the valuation group is the pure ordered group. The situation is very different for C-minimal fields or general C-minimal structures. In particular, a C-minimal field may have any o-minimal expansion of its valuation group and need not be definably complete. But as soon it is definably complete and satisfies a weak purity assumption on its valuation group, then it satisfies the strong form of definable completeness. Let us be more precise now.

Definition. Let M be a C-structure.

(1) Let $X \subseteq Y \times M$ be a definable set, where $(Y, <)$ is an ordered set interpretable in M. We say that X is a *definable nested family*, in short *dnf*, if
  - (a) for every $\gamma \in Y$, the fiber $X_\gamma := \{x \in M\,; (x, \gamma) \in X\}$ is non-empty and
  - (b) $X_{\gamma'} \subseteq X_\gamma$ for every $\gamma, \gamma' \in Y$ such that $\gamma < \gamma'$.

  We say X is *final* if furthermore Y is of the form $]a, e[$ for some $a \in T(M)$ and e in M ordered by the order of the canonical tree.
  We say that X has *non-empty intersection* if $\bigcap_{\gamma \in Y} X_\gamma \neq \emptyset$.

(2) We say M has the *bound property* when, for any $U \subseteq M$ a closed and bounded subset, any $e \in M$, $a \in T(M)$, $a < e$, $Y := \,]a, e[\, \subseteq Br(e)$, and every interpretable continuous function $f\colon U \to Y$, then $f(U)$ is bounded above in Y.

(3) M is called *definably complete*[1] if every final dnf of cones has non empty intersection (cones are the generalization of open balls in the context of C-relations).

(4) M is called *good* if it is definably complete and satisfies the condition (NoBij): in the canonical tree $T(M)$ there is no definable bijection from an interval $]a, b[$ to an interval $]c, d[$, for any $b \in M$ and $d \notin M$.

Theorem (A). *In every good C-minimal structure every definable final nested family of closed and bounded subsets has non-empty intersection.*

Theorem (B). *Every good C-minimal structure has the bound property.*

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[1]About this definition, see Section 5.

Our main motivation for proving this result is the construction of a group law in non trivial geometric C-minimal structures. Fares Maalouf achieved the construction when the C-structure is locally modular (see [M1] and [M2]) and Theorem (B) allows us to carry it in any geometric good C-structure, see [DMS].
The form of the present paper ought much to [CuD] where similar questions are treated in P-minimal fields. In such fields any definable interval of the canonical tree which has no maximal element has the form $]a, e[$, $e \in M$. In fact any definable branch of the canonical tree has the structure of a pure $\mathbb{Z}$-group as shown by Raf Cluckers (see [C] Lemma 2 and Theorem 6). The situation is very different in C-minimal fields and other C-minimal structures. Assuming Condition (NoBij) makes things similar again. Our paper is organized as follows. Section 2 is devoted to some preliminaries. Section 3 gives the proofs of Theorems (A) and (B). In Section 4 we present a technical variation of Theorems (A) and (B), Proposition 4.2, with the aim of removing assumption (NoBij) from the group construction. In Section 5 we discuss the terms of our statements.

2. Preliminaries

C-relations can be understood as a slight weakening of ultrametric spaces. In such a space the C-relation is given as: $C(x, y, z)$ ssi $d(x, y) = d(x, z) > d(y, z)$. The presentation below comes essentially from [AN].

Definition 2.1. A *C-relation* is a ternary relation, usually called C, satisfying the four axioms:
1. $C(x, y, z) \to C(x, z, y)$
2. $C(x, y, z) \to \neg C(y, x, z)$
3. $C(x, y, z) \to [C(x, y, w) \vee C(w, y, z)]$
4. $x \neq y \to C(x, y, y)$.
The relation is called *dense* if it satisfies furthermore
5. $\exists x, y,\ x \neq y$ and $\forall x, y\ (x \neq y \to \exists z \neq y,\ C(y, x, z))$.
A *C-set* is a set equipped with a C-relation.

Adeleke and Neumann's representation theorem of C-sets shows a close connection between C-sets and trees, see [AN] slightly modified in [D]).

Definition 2.2. A *tree* is an order in which for any element x the set $\{y; y \leq x\}$ is linearly ordered.
Call a tree *good* if :
- it is a meet semi-lattice (i.e. any two elements x and y have an infimum, $x \wedge y$, which means: $x \wedge y \leq x, y$ and $(z \leq x, y) \to z \leq x \wedge y$),
- it has maximal elements, or *leaves*, everywhere (i.e. $\forall x, \exists y\ (y \geq x \wedge \neg \exists z > y)$)
- and any of its elements is a leaf or a *node* (i.e. of form $x \wedge y$ for some distinct x and y).
A *branch* of a tree is a maximal subchain.

A branch may have or not have a leaf. Leaves of a tree T may be identified to branches via the map $x \mapsto Br(x) := \{\alpha \in T; \alpha \leq x\}$. The set of branches of T carries the *canonical C-relation*: $C(\alpha, \beta, \gamma)$ iff $\alpha \cap \beta = \alpha \cap \gamma \subsetneq \beta \cap \gamma$.

Proposition 2.3. *C-sets and good trees are bi-interpretable classes. More precisely, any C-set M interprets a good tree, called the* canonical tree *of M and denoted $T(M)$, such that (M, C) is definably isomorphic to the set of leaves of $T(M)$ equipped with the canonical C-relation.*

We can define in C-sets the notions generalizing open and closed balls of ultrametric spaces:

Definition 2.4. (1) For x and y two distinct elements of M, $\Lambda(x \wedge y, y) := \{z \in M; C(x,y,z)\}$ is called the *cone* of y at $x \wedge y$. We also use the notation, for elements $y > x$ from $T(M)$, $\Lambda(x,y) := \Lambda(x,\alpha)$ for any (or some) $\alpha \in M$ such that $Br(\alpha)$ contains y, and we say that $\Lambda(x,y)$ is the cone of y at x.

(2) For x and y in M, $\Lambda(x \wedge y) := \{z \in M; \neg C(z,x,y)\} = \{z; x \wedge y \leq z\}$ is called the 0-*levelled set* at $x \wedge y$.

(3) A subset U of M is *bounded* if it is contained in some cone.

The canonical tree appears to be the set of 0-level sets, ordered by inclusion.

As it happens to balls in ultrametric spaces, the intersection of two cones or 0-leveled sets is either empty or one of the two intersected sets. Cones form a base of a totally disconnected topology.

Let us now define C-minimality. It has been introduced by Deirdre Haskell, Dugald Macpherson and Charlie Steinhorn as the minimality notion suitable to C-relations (see [HM], [MS]).

Definition 2.5. A *C-structure* is a C-set possibly equipped with additional structure. A C-structure M is called *C-minimal* iff for any structure $N \equiv M$ any definable subset of N is definable by a quantifier free formula in the pure language $\{C\}$ or equivalently, is a Boolean combination of cones and 0-levelled sets.

Proposition 2.6. *Let M be a C-minimal C-structure and T its canonical tree. Then*

(1) any node c of T is strongly minimal in the sense that, any definable set of cones at c is finite or cofinite;

(2) for each $x \in M$ the branch $Br(x)$ of T is o-minimal in the sense that, any subset of $Br(x)$ definable in T is a finite union of intervals with bounds in $Br(x) \cup \{-\infty\}$.

Proof. Haskell and Macpherson [HM] Lemma 2.7 (ii) and (i). □

Corollary 2.7. *Let M be a C-minimal C-structure. Then:*

(1) No infinite set of cones at a same node can be linearly ordered by a definable relation.

(2) For any x and y in M and any definable partial function $f\colon Br(x) \to Br(y)$, the domain of f can be partitioned into finitely many intervals such that, on each of them, the restriction of f is either increasing, or decreasing or constant.

(3) For some $x \in M$ let t be a complete 1-type over $Br(x)$. Then t remains complete as the type on M of an imaginary element.

Proof. (1) and (3) follow directly from Proposition 2.6, respectively (1) and (2). For (2) we go back to classical results and proofs in o-minimality. In [PS] Pillay and Steinhorn prove that in an o-minimal structure O, given any definable function defined on an interval $]a,b[\subseteq O$, $]a,b[$ can be partitioned into finitely many intervals, on each of which the function is either constant or order preserving or reversing. The key tool is Lemma 4.3. Now, the proof of this lemma only uses that $]a,b[$ is o-minimal in the sense of Proposition 2.6 (2). □

In fact item (3) of Corollary 2.7 is true for any complete type over $Br(x)$ thanks to Theorem 1.4 in [P].

Proposition 2.8. *A bounded definable subset of M is a finite union of cones, 0-levelled sets and sets of the form $\Lambda(\delta,\alpha) \setminus \Lambda(\alpha)$ for some $\delta, \alpha \in T(M)$, $\delta < \alpha$.*

Note that the above statement is a very weak form of cellular decomposition as it does not take into account the way the decomposition depends on parameters.

3. Proofs

Lemma 3.1. *Let M be a C-minimal dense C-structure satisfying* (NoBij)*, $r \in T(M)$ and $\ell \geqslant 1$ an integer. Let $e \in M$, $a \in T(M)$, $a < e$, $Y :=]a, e[\subseteq Br(e)$ and $(A_\gamma)_{\gamma \in Y}$ be a definable family of finite subsets of $T(M)$, each of cardinality ℓ. Assume that elements of each A_γ are above r. Then one of the following holds:*

1. *there is some $\alpha \in T(M)$ such that $\alpha \in A_\gamma$ for each big enough $\gamma \in Y$;*
2. *There is I a definable initial segment of (some branch of) $T(M)$, without maximal element, containing r and such that, for all $y \in I$ and γ big enough in Y, there is $z \in A_\gamma(M) \cap I$ such that $y < z$.*
   If M is definably complete, then I is of the form $Br(x) \setminus \{x\}$ for some $x \in M$.

Proof. Fix any $x \in T(M) \setminus M$, $x > r$. Define $x_{\leq} := \{y \in T(M) \ ; \ y \leq x\}$. For $\gamma \in Y$, since A_γ has exactly ℓ elements, there is an integer $n_\gamma \leqslant \ell$ such that $|\{x \wedge y : y \in A_\gamma\}| = n_\gamma$. Since $x_{\leq}$ is linearly ordered, there are ℓ definable functions $r_{x,i} \colon Y \to x_{\leq}$, $1 \leqslant i \leqslant \ell$, such that $\{r_{x,i}(\gamma); 1 \leqslant i \leqslant n_\gamma\} = \{x \wedge y : y \in A_\gamma\}$ and

$$r_{x,1}(\gamma) < r_{x,2}(\gamma) < \cdots < r_{x,n_\gamma}(\gamma) = r_{x,n_\gamma+1}(\gamma) = \cdots = r_{x,\ell}(\gamma).$$

As the function $r_{x,i}$ is bounded above by x and below by r, by Proposition 2.7 (2) and Condition (NoBij), it becomes constant near infinity: for all $x \in T(M) \setminus M$ there are $\delta_x \in Y$ and $(z_{x,i})_{1 \leqslant i \leqslant \ell} \in T(M)^\ell$ such that

$$(\forall \gamma \in Y_{>\delta_x})(\forall y \in A_\gamma)(\bigvee_{i=1}^{\ell} x \wedge y = z_{x,i}),$$

where by definition, $Y_{>\delta_x} := \{\gamma \in Y \ ; \ \gamma > \delta_x\}$. Define more precisely $z_{x,i} := r_{x,i}(\gamma)$ for any $\gamma > \delta_x$ and set $z_x := z_{x,\ell} = max\{z_{x,i} \ ; 1 \leqslant i \leqslant \ell\}$.
Consider on $Br_M(e)$ the type e^-. By Corollary 2.7 (3) it remains complete as the type on M of an imaginary element. This implies that, for all γ realizing this type in some elementary extension of M, sets A_γ have same type on M. Call this type A_{e^-}, $t_1, \ldots, t_m$, $m \leqslant \ell$, the different types on M of its elements and define $I_j := \{y \in T(M) \ ; \ y \leq t_j\}$ for $1 \leq j \leq m$. By construction each z_x lies in some I_j and for any given j, the set of the $z_x \leq t_j$ is cofinal in I_j. The type e^- is definable hence A_{e^-} too. Since there are only finitely many t_j each one is definable too. Case (1) of the statement of the lemma occurs when one of the t_j is realized in $T(M)$.

Claim 3.2. *If (1) does not hold, the set $H := \{z_x \ ; \ x \in T(M) \setminus M, x > r\}$ has no maximal element.*

Proof of the claim. We show by induction on i that, if (1) does not hold then, for every $\gamma \in Y$, every $x \in T(M) \setminus M$, $x > r$, and every $1 \leqslant i \leqslant \ell$, the element $z_{x,i}$ is not a maximal element of H. For $i = 1$, by definition of $z_{x,1}$,

$$(\forall \gamma \in Y_{>\delta_x})(\forall y \in A_\gamma)(x \wedge y \geqslant z_{x,1}).$$

If there are $\gamma \in Y_{>\delta_x}$ and $y \in A_\gamma$ such that $x \wedge y > z_{x,1}$ we are done. Otherwise,

$$(\forall \gamma \in Y_{>\delta_x})(\forall y \in A_\gamma)(x \wedge y = z_{x,1}).$$

Since (1) does not hold, we can (definably) choose δ'_x such that, for all $\gamma \in Y_{>\delta'_x}$ and $y \in A_\gamma$, $z_{x,1} < y$. Replace δ_x by $max\{\delta_x, \delta'_x\}$. Thus the map

$$\{ \text{ cones at } z_{x,1}\} \to Y_{>\delta_x}, \Gamma \mapsto sup\{\gamma \in Y_{>\delta_x} \ ; A_\gamma \cap \Gamma \neq \emptyset\},$$

is well defined on a non empty subset of $\{$ cones at $z_{x,1}\}$. It ranges is finite by Corollary 2.7 (1). Now this range must be cofinal in Y. Thus one of the cone lies in the image of arbitrary large γ hence of any large enough γ. Since (1) does not hold, for all sufficiently large $\gamma, \delta \in Y$ there are $y \in A_\gamma, z \in A_\delta$, such that $y \wedge z > x \wedge y = x \wedge z$. Choose more precisely $\gamma \in Y_{>\delta_x}$, $y \in A_\gamma$, $\delta > max\{\delta_x, \delta_y\}$ and $z \in A_\delta$. Then $y \wedge z = z_{y,i}$ for some $1 \leq i \leq \ell$ and the result also follows.

We handle now the case $i > 1$. Define $B_\gamma := A_\gamma \cap \{\alpha \in T(M) \; ; \; \alpha \geq z_{x,i}\}$. By o-minimality of Y the cardinality of B_γ becomes constant for big enough γ, say $\gamma > \gamma_0$. By definition of the $z_{x,j}$, $1 \leq |B_\gamma| \leq \ell$ holds for $\gamma > \gamma_0$. Now, the dnf $(B_\gamma)_{\gamma \in Y_{>\gamma_0}}$ fits in case $i = 1$, which shows the result. $\dashv$

We are left to prove the assertion for a definably complete M. Such an M contains a point $x > I$. The map $Y \to Br(x)$, $\gamma \mapsto max\{y \in I \; ; y \in A_\gamma\}$ induces a bijection from a final interval of Y to a final interval of I. By (NoBij) I is cofinal in $Br(x) \setminus \{x\}$. $\square$

Theorem (A). *Let M be a good C-minimal dense C-structure. Let $X \subseteq Y \times M$ be a final dnf of closed and bounded sets. Then $\bigcap_{\gamma \in Y} X_\gamma \neq \emptyset$.*

Proof. We begin with some remarks about definable subsets of M. For $Z \subseteq M$ a definable subset, we consider

$$\Theta(Z) := \{\nu \in T(M); \exists x, y \in M, x, y > \nu, x \in Z \text{ and } y \notin Z\}.$$

By C-minimality Z is a finite Boolean combination of cones and 0-levelled sets, say of Γ_i where each Γ_i is of the form $\Lambda(\alpha_i)$ or $\Lambda(\alpha_i; a_i)$ with $\alpha_i \in T(M)$ and $a_i \in M$, $a_i > \alpha_i$. Clearly if ν is in $\Theta(Z)$ then:
- any $\mu \in T(M)$, $\mu < \nu$, is in $\Theta(Z)$ too,
- $\nu \leq \alpha_i$ for some i.

As a consequence $\Theta(Z)$ is a meet sub-semi-lattice of $T(M)$ and a tree with finitely many branches. Define $U(Z)$ the set of suprema of branches of $\Theta(Z)$, suprema which exist in $T(M)$ by o-minimality of branches and definability of $\Theta(Z)$. So $U(Z)$ is a finite set. Take $\alpha \in U(Z)$. Then either
(a) $\alpha \in \Theta(Z)$; in this case any cone at α is entirely contained either in Z or in its complement and both situations do exist; or
(b) $\alpha \notin \Theta(Z)$ (which implies that α is upper limit in $T(M)$); in this case $\Lambda(\alpha)$ is entirely contained either in Z (case b.1) or in its complement (case b.2). In case (b.2), by cellular decomposition, there is some $\delta \in T(M)$, $\delta < \alpha$, such that $\Lambda(\delta, \alpha) \setminus \Lambda(\alpha) \subseteq Z$.
Let us apply these considerations to $Z = X_\gamma$ and set $U_\gamma := U(X_\gamma)$. Take $\alpha \in U_\gamma$. If X is definable with parameters c, consider A_γ the set of elements of $T(M)$ with same type as α over (c, γ). Clearly $A_\gamma \subseteq U_\gamma$ and all elements of A_γ are in the same case, (a), (b.1) or (b.2). Replace Y by a final subset on which the cardinality of A_γ is (finite and) independent of γ and the case of its elements is independent of γ as well. Call ℓ the cardinality of A_γ and apply Lemma 3.1.
(1) If (1) holds let $\alpha \in T(M)$ belonging to all A_γ.
- Assume first α is in case (a). Take any cone Γ at α. If Γ is not contained in every X_γ (recall: they are decreasing) there is a minimal γ, say γ_Γ, such that $\Gamma \subseteq \neg X_\gamma$ for all $\gamma > \gamma_\Gamma$ (remember: the family X_γ is decreasing). The definable partial function $f : \{$ cones at α $\} \to Y$, $\Gamma \mapsto \gamma_\Gamma$ must have finite range. Let γ_1 be the maximal element of its range and take $\gamma > \gamma_1$. If f is defined on any cone at α, X_γ does not intersect $\Lambda(\alpha)$, contradiction. Hence there is some Γ contained in every X_γ.
- In case (b.1) all X_γ contain $\Lambda(\alpha)$.

- Assume now we are in case (b.2). If $\alpha \in M$, then α is in every X_γ since this set is closed, this contradicts case (b.2). So α is a node. Let δ_γ be minimal such that $\Lambda(\delta_\gamma, \alpha) \setminus \Lambda(\alpha) \subseteq X_\gamma$. So δ_γ is definable on (c, γ, α). On a final subset of Y, δ_γ is either increasing or decreasing or constant. By (NoBij) it can not be decreasing as X is bounded below. Neither can it be increasing since $\alpha \notin M$. Hence it is constant, say equal to β hence all X_γ contain $\Lambda(\beta, \alpha) \setminus \Lambda(\alpha)$.
(2) If (2) holds let $x \in M$, $x > r$, be such that, for all $y \in T(M)$, $y < x$ and γ big enough in Y, there is $z \in A_\gamma(M)$ such that $z > y$. This implies that there is $t \in X_\gamma$, $t > y$. Since X is decreasing it follows that x is an accumulation point of each X_γ, thus $x \in X_\gamma$ as this set is closed. □

In the next theorem the topology on Y is the order topology.

Theorem (B). *Let M be a good C-minimal dense C-structure and $U \subseteq M$ a closed and bounded subset. Then, for every interpretable continuous function $f : U \to Y$, $f(U)$ is bounded above in Y.*

Proof. Assume for a contradiction that $f(U)$ contains a final subset Y' of Y. Take Y' closed in Y. Replace Y with Y' and U with $U' := f^{-1}(Y')$ which is still closed by continuity of f. Let cl denote the topological closure. For each $\gamma \in Y'$ let

$$X_\gamma = cl\left(\bigcup\{(f^{-1}(\gamma')\,;\gamma' \in Y' \text{ and } \gamma \leq \gamma'\}\right), \text{ and}$$

$$X = \bigcup_{\gamma \in Y'} (\{\gamma\} \times X_\gamma).$$

Let us show that X is a dnf of closed and bounded sets. Each fiber X_γ is closed by definition. Since U' is closed and f is continuous, $X_\gamma \subseteq U'$ for each $\gamma \in Y'$. Therefore, since U' is bounded, so is X_γ. Finally, for $\gamma, \gamma' \in U'$ such that $\gamma < \gamma'$ the inclusion $X_{\gamma'} \subseteq X_\gamma$ holds by definition of X. So, by Theorem (A), there exists $x \in U'$ such that, $x \in X_\gamma$ for all $\gamma \in Y'$. In particular, $x \in U'$ so take $\gamma_0 := f(x)$ and $\gamma \in Y'$ such that $\gamma > \gamma_0$. Since $x \in X_\gamma$, there is $\gamma' \geq \gamma$ such that $x \in cl(f^{-1}(\gamma'))$ which contradicts that $f(x) = \gamma_0$. □

4. Une variation sans hypothèse (NoBij)

Nous présentons maintenant un cas où une fonction donnée satisfaisant les hypothèses de la fonction f de la propriété de la borne, est bornée, sans que (NoBij) soit supposée. Pour prouver cet énoncé, on commence par établir une version adaptée du lemme 3.1.

Lemma 4.1. *Let M be a definably complete C-minimal dense C-structure, $r \in T(M)$ and ℓ an integer. Let $e \in M$, $a \in T(M)$, $a < e$, $Y :=]a, e[\subseteq Br(e)$ and $(A_\gamma)_{\gamma \in Y}$ be a definable family of finite subsets of $T(M)$, each of cardinality ℓ and with all its elements above r. Let V be a cone definable on M such that all elements of V have same type over the (imaginary) canonical parameter of A. Then, one of the following holds:*

- *(3) $A_\gamma \cap T(V) = \emptyset$ for all γ big enough in Y; or*
- *(4) if α is the basis of the cone V, for all $\beta \in T(V)$ there is $\gamma \in Y$ and $x \in A_\gamma \cap T(V)$ such that $\beta > x > \alpha$.*

Proof. By o-minimality of Y, $|A_\gamma \cap T(V)|$ is constant for γ big enough in Y, say $\gamma > \gamma_0$. If $A_\gamma \cap T(V)$ is not cofinally empty, we begin applying the proof of Lemma 3.1 to the family $(A_\gamma \cap T(V))_{\gamma \in Y_{>\gamma_0}}$. When removing assumption (NoBij), it appears the new possibility that, for some $x \in T(M)$, $r_{x,i}$ tends monotonously to a limit: there is some $\alpha \in x_{\leq}$ such that,
- either: $\forall \beta \in T(M)$, $\beta < \alpha$, $\forall \gamma$ big enough in Y, $\exists \delta \in A_\gamma, \beta < \delta < \alpha$;
- or: $\forall \beta \in T(M)$, $x > \beta > \alpha$, $\forall \gamma$ big enough in Y, $\exists \delta \in A_\gamma, \beta > \delta > \alpha$.

In the first case one of the t_j (recall: the t_j are the different types of elements of A_{e^-}) must be the type α^- of $x_\leq$ (still complete as a type over $T(M)$); in the second case, one of the t_j is the type α^+ of $x_\leq$. Such an α is algebraic over the canonical parameter of A. Call a this canonical parameter.
Since V is a cone and all its elements have same type over a, $\alpha \notin T(V)$.
In situation $t_j = \alpha^-$ or t_j realized by α as in (1) in the statement of lemma 3.1, t_j does not intersect $T(V)$ for γ big enough in Y. In situation $t_j = \alpha^+$, t_j intersects (and is included in) $T(V)$ only when α is the basis of the cone V, $T(V)$ has no root and t_j is the unbounded type of $T(V)$.
Lorsqu'aucune des situations (3) ou (4) n'est réalisée, la preuve se déroule comme celle de 3.1 via le Claim 3.2 et montre la première assertion de (2). Puisque M est définissablement complet, I a un supremum x dans $T(M)$. Puisque le type e^- est complet et que les A_γ sont de cardinalité fixée finie, cette définition de x l'algébrise sur a. Comme x est limite supérieure dans $T(M)$ et que tous les éléments de V ont même type sur a, x ne peut appartenir à $T(V)$, ni aucun de ses conjugués sur a. En conséquence, cofinalement, A_γ n'intersecte pas $T(V)$. $\square$

Proposition 4.2. *Let M be a dense and definably complete C-minimal structure.*
$Y =]a, e[$ as previously, $B \subseteq M$ a set of parameters.
$U \subset V$ cones in M, all elements of V having same type over B.
Then,

(a) for any dnf $X \subseteq Y \times M$ of closed subsets of M, X definable with parameters from B, $X \cap (Y \times U)$ has either ultimately empty sections or ultimately section U;
(b) for any function $f \colon V \to Y$ definable with parameters from B and continuous, $f(U)$ is bounded in Y.

Proof. We begin to prove (a) from 4.1 as we have proven Theorem (A) from 3.1. We have $A_\gamma \cap T(U) = \emptyset$ for all γ big enough in Y. Thus, either $X_\gamma \cap T(U) = \emptyset$ for all γ big enough in Y, or $X_\gamma \cap T(U) = T(U)$ for all γ big enough. So (a) is true.
The proof of (b) from (a) is the same as for proving Theorem (B) from Theorem (A). $\square$

5. Two remarks

5.1. If there is some $\beta \in T(M)$ which is not a leaf and is the supremum of $\{\gamma \in T(M); \gamma < \beta\}$ Theorem B fails when replacing the condition " $f(U)$ bounded above in Y " by " $f(U)$ has a maximal element ". As a counter-example, take such a β, $\alpha \in T(M)$, $\alpha < \beta$, $U := \Gamma(\alpha; \beta) \setminus \Lambda(\beta)$ and $f \colon U \to]\alpha, \beta[$, $x \mapsto x \wedge \beta$.

5.2. Paulo Ribenboim a dégagé dans les espaces ultramétriques les notions de complétude sphérique (toute intersection décroissante de boules a une intersection non vide) et de complétude (la même condition lorsque, de plus, le rayon des boules tend vers 0). Dans le cadre d'une C-relation il n'est possible de comparer des distances qu'à un même point, c'est-à-dire sur une même branche de l'arbre canonique. Dire que le rayon tend vers 0 signifie que le rayon devient plus petit que toute distance apparaissant dans l'espace, or ceci ne peut être estimé avec la seule C-relation. Ainsi, seule la définition de la complétude sphérique a un sens : toute intersection décroissante de cônes a une intersection non vide. De façon inattendue, la distinction entre les deux complétudes réapparaît lorsqu'on considère des objets définissables. Nous avons en effet choisi d'indexer nos boules par un intervalle d'une branche. À partir de ce moment-là, il est aisé de d'isoler le cas où ce segment est cofinal sous la feuille. Ainsi la distinction entre complétude sphérique (ou plutôt son équivalent dans le cadre d'une C-relation, que nous appellerons "complétude conique" (toute intersection décroissante de cônes indexée

par un intervalle d'une branche a une intersection non vide) et la notion plus faible, que nous avons simplement appelée complétude (la même condition lorsque, de plus, l'intervalle des indices est cofinal sous la feuille) réapparaît naturellement dans le cadre définissable.
Renforçons ainsi l'hypothèse de complétude définissable en "complétude conique définissable" (toute intersection décroissante de cônes indéxés par un intervalle d'une branche a une intersection non vide). Cela ne permet pas de renforcer la conclusion du Théorème A en autorisant des dnf indexées par n'importe quel intervalle d'une branche. D'une part le cas (b.2) de la preuve coince et d'autre part il est facile de recycler le contre-exemple de la remarque 5.1 : $\Gamma(a_\gamma, b) \setminus \Lambda(b)$ avec $b \notin M$ et $a_\gamma < b$ est l'exemple d'une dnf de fermés bornés d'intersection vide.

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Distinguished subfields of Hahn fields

Salma Kuhlmann

Abstract

Let k be a field, G a totally ordered abelian group. The maximal field of generalised power series $k((G))$, endowed with its canonical valuation, plays a fundamental role in the classification of valued fields (Kaplansky, 1942 and 1945). In this talk, we describe the group of valuation preserving automorphisms of any Hahn field K, i.e. a subfield of the maximal Hahn field, which contains the minimal Hahn field $k(G)$ (the fraction field of the group ring $k[G]$). Under the assumption that K satisfies two lifting properties we prove a structure theorem decomposing into a 4-factor semi-direct product of notable subgroups. We identify a large class of fields satisfying the two aforementioned lifting properties. We then focus on the group of strongly additive automorphisms of K. We give an explicit description of the group of strongly additive internal automorphisms in terms of the groups of homomorphisms of G into $k^{\times}$ and of G into the group of 1-units of the valuation ring of K. To illustrate the power of our methods, we apply our results to some special cases, such as the field of Laurent series (Schilling, 1944) and that of Puiseux series (Deschamps, 2005).

This is joint work with Michele Serra and has been published as part of the paper *The automorphism group of a valued field of generalised formal power series; Journal of Algebra, 605, 339-376 (2022)* .

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THE EXISTENTIAL THEORY OF DISCRETE EQUICHARACTERISTIC HENSELIAN VALUED FIELDS

ARNO FEHM

Abstract. From a model theoretic point of view, local fields of positive characteristic, i.e. fields of Laurent series over finite fields, are much less well understood than their characteristic zero counterparts - the fields of real, complex and p-adic numbers. I will discuss different approaches to axiomatize and decide at least their existential theory in various languages and under various forms of resolution of singularities.

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STABLE AND HYPERBOLIC POLYNOMIALS AND THEIR DETERMINANTAL REPRESENTATIONS

VICTOR VINNIKOV

A homogeneous polynomial of degree d with real coefficients is called hyperbolic with respect to a point if any real line through this point intersects the corresponding hypersurface in d real points (counting multiplicities). Hyperbolic polynomials are in a sense the opposite of strictly positive polynomials: they have as many real zeroes as possible. Hyperbolic polynomials were first introduced by Gärding in the 1950s [7, 8] (who often refers to Petrovsky [22] as a source) in the study of linear partial differential equation with constant coefficients, see also Lax [20] and the later paper [1] and [11] a good survey, as well as [26, 17, 18, 19] for an interesting generalization of hyperbolicity to higher codimensional subvarieties of the projective space. Gärding showed that a hyperbolic polynomial determines a convex cone, called a hyperbolicity cone. In recent years hyperbolic polynomials and hyperbolicity cones came to play an important in convex programming [10, 2, 24, 25] as well as combinatorics and other areas [21].

Much like a representation as a sum of squares certifies the positivity of a polynomial, its hyperbolicity is certified by a representation as a determinant of a matrix of linear forms, with the coefficient matrices of the linear forms satisfying some positivity conditions. I will describe some of what is known about the existence of these determinantal representations, usually "with denominators", see [27] for a survey and [17, 16] for some recent progress. One fruitful approach uses a Hermitian Positivstellensatz [23, 12, 9] that gives a representation of a polynomial satisfying matrix positivity conditions as a weighted sum of hermitian squares. This approach uses also a relation between hyperbolic polynomials and polynomials with complex coefficients that are $\mathbb{H}^d$-stable, i.e., have no zeroes in $\mathbb{H}^d$, where $\mathbb{H} = \{z \in \mathbb{C} \colon \Im z > 0\}$ is the upper half plane [3, 4, 5], or more generally Ω_C-stable, i.e., have no zeroes in Ω_C, where $\Omega_C \subseteq \mathbb{C}^d$ is a Siegel domain of the first kind (a tube domain over a convex cone as a base): $\Omega_C = \mathbb{R}^d + iC$ with $C \subseteq \mathbb{R}^d$ an (open) convex cone [13, 14, 15, 6].

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Elimination of quantifiers for a theory of real closed rings.

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Abstract

Let T^* be the theory of lattice-ordered subrings, without minimal (non zero) idempontents, convex in von Neumann regular real closed rings that are divisible-proyectable and sc-regular ([12]). In this paper, a local divisibility binary relation is introduced in order to prove quantifier elimination for the theory T^* in the language of lattice-ordered rings adding the (usual) divisibility relation, the radical relation associated to the minimal prime spectrum ([20]) and this new local divisibility relation.

1 Introduction.

Real closed rings, in one of its general presentations, were introduced by Niels Schwartz in [23]. Leaving aside the case of real closed fields, the first model theoretic results concerning real closed rings was the model-completeness of the von Neumann regular real closed rings without (non zero) minimal idempotents in the language of lattice-ordered rings, proved by Macintyre in [17]. The question of quantifier elimination for this theory evolved thereafter in many different ways that strongly depended on the language under consideration. The first result was proved by Weispfenning in [28] for the language of lattice-ordered rings where an unary function symbol $*$ was added to the language, representing the quasi-inverse in von Neumann regular rings. This same result was later proved using simpler techniques by Boffa-Cherlin in [4]. In [20], this elimination result was substantially improved by Prestel-Schwartz by replacing the function symbol $*$ by a binary radical relation, namely the radical relation associated to the minimal prime spectrum.

Examples of integral real closed rings that are not fields are the real closed valuation rings; this theory was introduced and well studied from the model theoretic point of view by Cherlin and Dickmann in [7]; they showed quantifier elimination in the language of ordered rings with divisibility as an additional binary relation symbol.

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It is a well known fact that von Neumann regular rings are Boolean products of fields, cf. [17, section 5], or cf. [10]. Boolean products of real closed valuation rings have been caracterized in [12] as real closed projectable rings satisfying the first convexity property (i.e.: $\forall a \forall b (0 < a < b \rightarrow b \mid a)$) that are sc-regular (see pp. 5 and 7 below) and divisible-projectable (see p. 5). Following the quantifier elimination for von Neumann regular real closed rings proved in [28] and [4], the author gave in [12] a quantifier elimination result for this theory, that we presently call T^*, in the language of lattice-ordered rings with an extra binary function symbol $\mathrm{div}(a, b)$ representing (locally) the quotient of b by a if it exists, and 0 if not.

In this paper, quantifier elimination for the theory T^* is significantly improved, replacing the function symbol $\mathrm{div}(\cdot, \cdot)$ by the binary radical relation associated to the minimal prime spectrum (as Prestel and Schwartz did in [20]) and by adding to the language a binary relation that represents local divisibility (the usual "global" divisibility is also present in the language).

We think that an important feature of the present result is that the new primitive notions used to prove quantifier elimination for T^* have an interesting mathematical meaning. In the case of the radical relation this has been amply demonstrated in [20] and subsequent work. For the local divisibility relation this can be unravelled from the arguments of the present paper. We hope, in future work, to be able to examine the mathematical significance of this relation and pursue some applications.

In Section 2 we present the basic material needed in the rest of the paper, including the all important local divisibility relation and its basics properties. In the third section we prove model-completeness of the theory T^* in the language of lattice-ordered rings enlarged by the radical and the local divisibility relations. The model-completeness of T^* in our present context is caused by the following fact: the preservation of the local divisibility by "global" homomorphisms implies the preservation of (usual) divisibility "locally" in the fibers, cf. Theorem 3.1.

In the fourth section we study (and characterize) the universal part of the theory T^* in various languages; see Theorem 4.14, Proposition 4.20, and Theorem 4.21. This entails a noteworthy result on model-companionship, Theorem 4.22.

The hypothesis of projectability of the ring under consideration provides tools necessary to prove the preservation of divisibility and local divisibility. However, this notion has a shortcoming: it is not expressed by a universal (first order) formula in any of the languages under consideration. To overcome this obstacle we found a set of universal axioms in the language of rings —the *divisibility glueing axiom scheme* (cf. Definition 4.9)— and —the *local divisibility property*, (cf. Definition 4.19), in the language enriched with the radical and the local divisibility relations—, that advantageously replace the tools furnished by the projectability property in the proof of the (downward) preservation of divisibility and local divisibility.

The notion of local divisibility introduced in Section 2 is not sufficient for a direct proof of quantifier elimination of the theory T^*. Once again, the propery of divisible-projectability turns out to be of crucial importance for the introduction of the relation of *maximal local divisibility*, a stregthening of the notion of local divisibility. In analogy to the situation for model-completeness, this stronger notion makes it possible to show that the preservation of local divisibility in models of $T^*_\forall$, the universal part of T^*, yields local preservation of (usual) divisibility in the fibers (cf. Theorem 6.9). Use of this "preservation transfer" result leads, finally, to the proof of quantifier elimination for T^*, via the amalgamation property in models of $T^*_\forall$. This is the subject matter of the last Section 6 of the paper.

The author spent the months of June and July 2022 working at the Logic Group of the Université Paris-Cité [1]. During those months the author had fruitful and enlightening conversations with Max Dickmann, who helped him in the elaboration of the fifth and sixth sections of this paper. He also read many versions of this work, improving it by his suggestions and comments. In addition, the ideas of those sections benefited from valuable comments by Françoise Delon and Françoise Point during a sesion of the Delon-Dickmann-Gondard seminar at the begining of June 2022, where sections 2 to 4 of the present work were presented. The author wishes to express his gratitude to the colleagues and institutions mentioned in this paragraph.

2 Basic notions; the local divisibility relation.

The main aim of this section is to introduce the basic facts and notions of the theory T^* of real closed rings considered in this paper. Amongst them, a *local divisibility* relation, variants (in section 5) of which will be crucial for the proof of the main quantifier elimination result proved in this paper.

$\mathcal{L}_{\mathrm{or}} = \{0, 1, +, \cdot, <\}$ will be the language of ordered rings and $\mathcal{L}_{\mathrm{lor}} = \{0, 1, +, \cdot, \wedge\}$ will be the language of lattice-ordered rings. All rings considered in this paper are conmutative with unity.

An **f-ring** is a subdirect product of totally ordered rings. This notion can be expressed by a first-order formula in $\mathcal{L}_{\mathrm{lor}}$ (see [3, 9.1.2]). For an f-ring A, the **absolute value** of $a \in A$ is $|a| = a \vee -a$; two elements $a, b \in A$ are **orthogonal** if $|a| \wedge |b| = 0$ (we denote this by $a \perp b$); the **polar** of $a \in A$ is $a^\perp = \{b \in A : a \perp b\}$ and the **bipolar** of a is $a^{\perp\perp} = \{b \in A : b \perp c \text{ for all } c \in a^\perp\}$. An f-ring A is **projectable** if $A = a^\perp + a^{\perp\perp}$, for all $a \in A$. Note that this notion is expressed by a first order formula in $\mathcal{L}_{\mathrm{lor}}$. A ring is **reduced** if it doesn't have nilpotent elements other than zero. By [3, 9.3.1], if A is a reduced f-ring, then $\forall x \forall y (x \perp y \leftrightarrow xy = 0)$ is valid formula in A, and therefore:

$$b \in a^{\perp\perp} \Longleftrightarrow a^\perp \subseteq b^\perp \Longleftrightarrow \mathrm{Ann}(a) \subseteq \mathrm{Ann}(b),$$

for any $a, b \in A$.

Let L be a first-order language, $\{\mathfrak{A}_x : x \in X\}$ a family of L-structures and $\mathfrak{A}$ a L-structure. We say that $\mathfrak{A}$ is a **Boolean product** of $\{\mathfrak{A}_x : x \in X\}$ in L, denoted by $\mathfrak{A} \in \Gamma^a_L\big(X, (\mathfrak{A}_x)_{x\in X}\big)$, cf. [5], if the following conditions holds:

(i) X is a Boolean space.

(ii) $\mathfrak{A}$ is a subdirect product of $\{\mathfrak{A}_x : x \in X\}$.

(iii) For every atomic L-formula $\Phi(v_1, \ldots, v_n)$ and every $a_1, \ldots, a_n \in |\mathfrak{A}|$,

$$[\![\, \Phi(a_1, \ldots, a_n) \,]\!] =_{\mathrm{def}} \big\{x \in X : \mathfrak{A}_x \models \Phi\big(a_1(x), \ldots, a_n(x)\big)\big\}$$

[1] Formerly called Université Paris Diderot, ex Université Paris VII.

is a clopen subset of X.

(iv) Patchwork property: For every $a, b \in \mathfrak{A}$ and any clopen set N of X, the element $c = a_{\upharpoonright N} \cup b_{\upharpoonright X \setminus N}$ defined by

$$c(y) = \begin{cases} a(y) & \text{if } y \in N \\ b(y) & \text{if } y \in X \setminus N, \end{cases}$$

belongs to $|\mathfrak{A}|$. We say that $\mathfrak{A}$ is an **elementary Boolean product** of $\{\mathfrak{A}_x : x \in X\}$ in L, denoted by $\mathfrak{A} \in \Gamma_L^e\big(X, (\mathfrak{A}_x)_{x\in X}\big)$, if $\mathfrak{A}$ is a Boolean product of $\{\mathfrak{A}_x : x \in X\}$ in L and condition (iii) is verified for **all** L-formulas $\Phi(v_1, \ldots, v_n)$. These notions come from [5].

If A is a (unitary) reduced and projectable f-ring, then in [16, 6.12] it is proved that:

$$A \in \Gamma^a_{\mathcal{L}_{\mathrm{or}}}\big(\pi A, (A/p)_{p\in\pi A}\big),$$

where $\pi A = \{p \in \mathrm{Spec}(A) : p \text{ is a minimal prime ideal}\} = \mathrm{Specmin}(A)$. In that case:

$$\begin{aligned} b \in a^{\perp\perp} &\iff [\![\, b \neq 0 \,]\!] \subseteq [\![\, a \neq 0 \,]\!] \\ &\iff \mathrm{supp}(b) \subseteq \mathrm{supp}(a) \\ &\iff [\![\, a = 0 \,]\!] \subseteq [\![\, b = 0 \,]\!] \\ &\iff \forall p \in \pi A\, (a \in p \Rightarrow b \in p). \end{aligned}$$

Radical relations were introduced in [19] and used in [20] to study the model theory of von Neumann regular real closed rings (cf. [23] or [22]) without minimal idempotents different from zero. Radical relations are defined in [20] by:

(1) $a \preceq a$, for all $a \in A$;

(2) if $a \preceq b$ and $b \preceq c$ then $a \preceq c$, for all $a, b, c \in A$;

(3) if $a \preceq c$ and $b \preceq c$ then $a + b \preceq c$, for all $a, b, c \in A$;

(4) if $a \preceq b$ then $ac \preceq bc$, for all $a, b, c \in A$;

(5) $a \preceq 1$, for all $a \in A$ and $1 \not\preceq 0$;

(6) $b \preceq b^2$, for all $b \in A$.

The original definition in [19] was the previous one with the relation $\preceq$ reversed. In this context, it is proved in [19, Theorem 2.5] that for any radical relation $\preceq$, there exists a subset $X \subseteq \mathrm{Spec}(A)$ such that:

$$a \preceq b \iff \forall p \in X\, (a \notin p \Rightarrow b \notin p).$$

This radical relation is denoted by $\preceq_X$. The case where $X = \pi A$ is a relevant one studied in [20], and there it is proved that:

$$\begin{aligned} a \preceq_{\pi A} b &\iff \mathrm{Ann}(b) \subseteq \mathrm{Ann}(a) \\ &\iff \forall x (bx = 0 \to ax = 0) \\ &\iff \forall x (ax \neq 0 \to bx \neq 0). \end{aligned}$$

Therefore the radical relation $\preceq_{\pi A}$ has all these possible definitions:

$$\begin{array}{rcccl} a \preceq_{\pi A} b & \Longleftrightarrow & \mathrm{Ann}(b) \subseteq \mathrm{Ann}(a) & \Longleftrightarrow & \forall x (bx = 0 \to ax = 0) \\ & \Longleftrightarrow & \forall x (ax \neq 0 \to bx \neq 0) & \Longleftrightarrow & \forall p \in \pi A \, (a \notin p \Rightarrow b \notin p) \\ & \Longleftrightarrow & \forall p \in \pi A \, (b \in p \Rightarrow a \in p) & \Longleftrightarrow & [\![\, b = 0 \,]\!] \subseteq [\![\, a = 0 \,]\!] \\ & \Longleftrightarrow & [\![\, a \neq 0 \,]\!] \subseteq [\![\, b \neq 0 \,]\!] & \Longleftrightarrow & \mathrm{supp}(a) \subseteq \mathrm{supp}(b) \\ & \Longleftrightarrow & b^{\perp} \subseteq a^{\perp} & \Longleftrightarrow & a \in b^{\perp\perp}. \end{array} \quad (\dagger)$$

Henceforth, the radical relation $\preceq_{\pi A}$ will be denoted by $\preceq$. In [20], the elimination of quantifiers of the theory of von Neumann regular real closed rings without minimal non-zero idempotents is given in the language $\mathcal{L}_{\mathrm{lor}} \cup \{ \preceq \}$ of lattice-ordered rings with a symbol for this radical relation.

Notation 2.1 *For any ring A and $a, b \in A$, we write $a =_s b$ for $a \preceq b$ and $b \preceq a$.*

According to [7], a **real closed valuation ring** is an ordered domain that satisfies the intermediate value property for polynomials in one variable that is not a field. In [7], the autors showed that this theory is complete and has quantifier elimination in the language $\mathcal{L}_{\mathrm{or}} \cup \{|\}$ of ordered rings with the (usual) divisibility relation.

In [12, Definition 2.5], a lattice ordered ring A is called **divisible-projectable** if:
$\forall x \forall y \Big(y \neq 0 \to \exists z \exists w \big(x = z + w \wedge z \perp w \wedge y \mid z \wedge \forall w' (w' \neq 0 \wedge w' \perp (w - w') \to y \nmid w') \big) \Big)$
is valid in A.

In [12, Definition 2.8], a ring A is called **sc-regular** if there exists an element $u \in A$ such that $\mathrm{Ann}(u) = \{0\}$ (or $1 \preceq u$) and $u \nmid e$ for every non-zero idempotent $e \in A$. By [12, Proposition 3.4 (i), Corollary 2.11 and Proposition 2.6], a ring A is a projectable real closed ring with the first convexity property that satisfies the sc-regularity and divisible-projectability if and only if

$$A \in \Gamma^{e}_{\mathcal{L}_{\mathrm{or}} \cup \{|\}} \big(\pi A, (A/p)_{p \in \pi A} \big),$$

where A/p is a real closed valuation ring, for every $p \in \pi A$.

Let T^* be the theory of projectable real closed rings with the first convexity property that satisfies the sc-regularity and divisible-projectability properties, and without minimal non-zero idempotetnts. By [13, Theorem 10], a ring A is a model of T^* if and only if A is a convex lattice-ordered subring of a von Neumann regular real closed ring, has no minimal non-zero idempotents, and satifies the divisible-projectability and sc-regularity properties.

By [12, Proposition 4.6(iii)], the theory T^* admits quantifier elimination in $\mathcal{L}_{\mathrm{lor}} \cup \{\mathrm{div}(\cdot,\cdot)\}$, where $\mathrm{div}(\cdot,\cdot)$ is a binary function symbol defined by:

$$\begin{array}{rcl} T^* \vdash \mathrm{div}(x, y) = c & \longleftrightarrow & c \in y^{\perp\perp} \wedge \exists z \exists w \big(x = z + w \wedge z \perp w \wedge cy = z \wedge \\ & & \forall w' (w' \neq 0 \wedge w' \perp (w - w') \to y \nmid w') \big). \end{array}$$

Remark that the definition of this binary funcion symbol $\mathrm{div}(\cdot,\cdot)$ can be written using the radical relation $\preceq$ by:

$$\begin{array}{rcl} T^* \vdash \mathrm{div}(x, y) = c & \longleftrightarrow & c \preceq y \wedge \exists z \exists w \big(x = z + w \wedge z \perp w \wedge cy = z \wedge \\ & & \forall w' (w' \neq 0 \wedge w' \perp (w - w') \to y \nmid w') \big). \end{array}$$

In order to study the theory T^* from the point of view of existential formulas or model completeness, it will be usefull to introduce the following binary predicate:

$$\begin{aligned} R(y,w) &\longleftrightarrow \forall w'(w' \neq 0 \wedge w' \perp (w-w') \rightarrow y \nmid w') \\ &\longleftrightarrow \forall w'(w' \neq 0 \wedge w'(w-w') = 0 \rightarrow y \nmid w'), \end{aligned}$$

that expresses the fact that y does not locally divide w. It will be convenient to rewrite the relation $\neg R$ in the following form:

$$\begin{aligned} \neg R(y,w) &\longleftrightarrow \exists w'(w' \neq 0 \wedge w' \perp (w-w') \wedge y \mid w') \\ &\longleftrightarrow \exists w'(w' \neq 0 \wedge w'(w-w') = 0 \wedge y \mid w'). \end{aligned}$$

Note that the last term in the preceding equivalences is a formula in the language of rings.

Note that if $\neg R(y,0)$ is valid in a reduced ring A, there is $w' \in A$ with $w' \neq 0$, $w'(-w') = 0$ and $y \mid w'$. Therefore, $w' \neq 0$ and ${w'}^2 = 0$, contradicting that A is reduced. Therefore we redefine:

Definition 2.2 (Local divisibility) *For any ring A, and $y, w \in A$, we say that y* **divides locally** *w if:*

$$w = 0 \vee \exists w'(w' \neq 0 \wedge w'(w-w') = 0 \wedge y \mid w').$$

We denote this by $y \mid_{\mathrm{loc}} w$.

The following proposition gives some elementary properties of this new local divisibility relation.

Proposition 2.3 *Let A be any ring, let $y, w, c \in A$ and $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The following properties hold in A.*

- *(i) if $y \mid w$ then $y \mid_{\mathrm{loc}} w$,*
- *(ii) $y \mid_{\mathrm{loc}} 0$ and $1 \mid_{\mathrm{loc}} w$,*
- *(iii) if $0 \mid_{\mathrm{loc}} w$ then $w = 0$,*
- *(iv) if $cy \mid_{\mathrm{loc}} w$ then $y \mid_{\mathrm{loc}} w$,*
- *(v) if $y^n \mid_{\mathrm{loc}} w$ then $y \mid_{\mathrm{loc}} w$,*
- *(vi) $y \mid_{\mathrm{loc}} y^n$,*
- *(vii) $y \mid_{\mathrm{loc}} w$ if and only if $-y \mid_{\mathrm{loc}} w$, if and only if $y \mid_{\mathrm{loc}} -w$, if and only if $-y \mid_{\mathrm{loc}} -w$,*
- *(viii) if A is a domain, then $y \mid w$ if and only if $y \mid_{\mathrm{loc}} w$.*

Proof: Routine checking. ∎

The following lemma is needed in the proof of Proposition 2.5 below.

Lemma 2.4 *Let A be a lattice-ordered ring and let $w, w' \in A$ be such that $w' \perp w - w'$. Then $|w'| \leqslant |w|$.*

Proof: Since $|w'| \wedge |w| \leqslant |w'|, |w|$ the following inequality holds:

$$\begin{aligned} |w'| = |w'| \wedge |w'| = |w'| \wedge |w' - w + w| &\leqslant |w'| \wedge \big(|w'-w| + |w|\big) \\ &= \big(|w'| \wedge |w'-w|\big) + \big(|w'| \wedge |w|\big). \end{aligned}$$

From $w' \perp w - w'$, comes $|w'| \wedge |w - w'| = 0$, and one obtains:

$$|w'| \leqslant 0 + \big(|w'| \wedge |w|\big) = |w'| \wedge |w| \leqslant |w'|.$$

Then $|w'| \wedge |w| = |w'|$, and hence $|w'| \leqslant |w|$. ∎

Proposition 2.5 *Let A be a lattice-ordered ring and let $y, w_1, w_2 \in A$. If $y \mid_{\rm loc} w_1$ and $y \mid_{\rm loc} w_2$ with $w_1 \perp w_2$, then $y \mid_{\rm loc} w_1 + w_2$.*

Proof: Suppose that $y \mid_{\rm loc} w_1$ and $y \mid_{\rm loc} w_2$ with $w_1 \perp w_2$. We consider several cases:

- The result is clear if one of w_1 or w_2 is 0.
- Let $w_1 \neq 0$ and $w_2 \neq 0$. If $w_1 + w_2 = 0$, by definition we have $y \mid_{\rm loc} w_1 + w_2$. Suppose, then, that $w_1 + w_2 \neq 0$. Since $y \mid_{\rm loc} w_i$ and $w_i \neq 0$ $(i = 1, 2)$, there is $w'_i \in A$, $w'_i \neq 0$ such that $w'_i \perp w_i - w'_i$ and $y \mid w'_i$. If $w'_1 + w'_2 = 0$ then $w'_2 = -w'_1$ and therefore:

$$|w'_1| \wedge |w'_2| = |w'_1| \wedge |-w'_1| = |w'_1| \wedge |w'_1| = |w'_1|.$$

By Lemma 2.4 one has $|w'_i| \leqslant |w_i|$ $(i = 1, 2)$. Then:

$$|w'_1| \wedge |w'_2| \leqslant |w_1| \wedge |w_2|.$$

Since $w_1 \perp w_2$, the previous inequality entails $|w'_1| \wedge |w'_2| = 0$. By the asumption one gets $|w'_1| = 0$, whence $w'_1 = 0$, a contradiction; hence $w'_1 + w'_2 \neq 0$. Next, we want to see that:

$$w'_1 + w'_2 \perp (w_1 + w_2) - (w'_1 + w'_2).$$

We have the following inequalities:

$$\begin{aligned}
0 &\leqslant \big|w'_1 + w'_2\big| \wedge \big|(w_1 + w_2) - (w'_1 + w'_2)\big| = \big|w'_1 + w'_2\big| \wedge \big|(w_1 - w'_1) + (w_2 - w'_2)\big| \\
&\leqslant \big|w'_1 + w'_2\big| \wedge \big(|w_1 - w'_1| + |w_2 - w'_2|\big) \leqslant \big(|w'_1| + |w'_2|\big) \wedge \big(|w_1 - w'_1| + |w_2 - w'_2|\big) \\
&= \big(|w'_1| \wedge |w_1 - w'_1|\big) + \big(|w'_1| \wedge |w_2 - w'_2|\big) + \big(|w'_2| \wedge |w_1 - w'_1|\big) + \big(|w'_2| \wedge |w_2 - w'_2|\big) \\
&= 0 + \big(|w'_1| \wedge |w_2 - w'_2|\big) + \big(|w'_2| \wedge |w_1 - w'_1|\big) + 0 \\
&= \big(|w'_1| \wedge |w_2 - w'_2|\big) + \big(|w'_2| \wedge |w_1 - w'_1|\big) \\
&\leqslant \big(|w'_1| \wedge \big(|w_2| + |w'_2|\big)\big) + \big(|w'_2| \wedge \big(|w_1| + |w'_1|\big)\big) \\
&= \big(|w'_1| \wedge |w_2|\big) + \big(|w'_1| \wedge |w'_2|\big) + \big(|w'_2| \wedge |w_1|\big) + \big(|w'_2| \wedge |w'_1|\big) \\
&= \big(|w'_1| \wedge |w_2|\big) + 2\big(|w'_1| \wedge |w'_2|\big) + \big(|w'_2| \wedge |w_1|\big).
\end{aligned}$$

Using again Lemma 2.4, $w'_i \perp (w_i - w'_i)$ entails $|w'_i| \leqslant |w_i|$ $(i = 1, 2)$. Returning to the inequalities one gets:

$$\begin{aligned}
0 &\leqslant \big|w'_1 + w'_2\big| \wedge \big|(w_1 + w_2) - (w'_1 + w'_2)\big| \\
&\leqslant \big(|w'_1| \wedge |w_2|\big) + 2\big(|w'_1| \wedge |w'_2|\big) + \big(|w'_2| \wedge |w_1|\big) \\
&\leqslant \big(|w_1| \wedge |w_2|\big) + 2\big(|w_1| \wedge |w_2|\big) + \big(|w_2| \wedge |w_1|\big) = 4\big(|w_1| \wedge |w_2|\big) = 4 \cdot 0 = 0,
\end{aligned}$$

for $w_1 \perp w_2$. This shows $\big|w'_1 + w'_2\big| \wedge \big|(w_1 + w_2) - (w'_1 + w'_2)\big| = 0$, i.e., $(w'_1 + w'_2) \perp (w_1 + w_2) - (w'_1 + w'_2)$. Since $y \mid w'_i$ for $i = 1, 2$, then $y \mid w'_1 + w'_2$. Setting $w' = w'_1 + w'_2$, we have proved $w' \neq 0$, $w' \perp (w_1 + w_2) - w'$ and $y \mid w'$, i.e., $\exists w'\big(w' \neq 0 \wedge w'\big(w' - (w_1 + w_2)\big) \wedge y \mid w'\big)$ holds in A, showing that $y \mid_{\rm loc} w_1 + w_2$, as required.

■

Let A be a reduced f-ring. The sc-regularity of A states the existence of an element $u \in A$ such that $1 \preceq u$ and $\forall e(e \neq 0 \wedge e^2 = e \to u \nmid e)$. Observe that:

$$\begin{aligned}
u \mid_{\rm loc} 1 &\longleftrightarrow \exists w'(w' \neq 0 \wedge w'(w' - 1) = 0 \wedge u \mid w') \\
&\longleftrightarrow \exists w'(w' \neq 0 \wedge w'^2 - w' = 0 \wedge u \mid w') \\
&\longleftrightarrow \exists e(e \neq 0 \wedge e^2 = e \wedge u \mid e).
\end{aligned}$$

Therefore:

$$u \nmid_{\rm loc} 1 \longleftrightarrow \forall e(e \neq 0 \wedge e^2 = e \to u \nmid e).$$

So, the condition that A is sc-regular can be restated as $A \models \exists u(1 \preceq u \wedge u \nmid_{\rm loc} 1)$.

3 Model completeness.

In this section we will work with the language $\mathcal{L} = \mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |_{\mathrm{loc}}\}$. Let A and B be two reduced f-rings satisfying the first convexity property and let us suppose that A is a substructure of B in the language $\mathcal{L}$; in particular A is a lattice-ordered subring of B.

Let us denote by $i\colon A \hookrightarrow B$ the inclusion, and the (dual) functorial (continuous) map:

$$\mathrm{Spec}(i)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A), q \mapsto i^{-1}(q) = q \cap A.$$

Since $A \subseteq_{\mathcal{L}} B$ and the radical relation $\preceq$ belongs to the language, then:

$$a \preceq_A a' \Longleftrightarrow i(a) \preceq_B i(a'),$$

for all $a, a' \in A$. Let us denote $\pi B = \mathrm{Specmin}(B) = \{q \in \mathrm{Spec}(B) : q \text{ is a minimal prime ideal }\} \subseteq \mathrm{Spec}(B)$ and similarly for the ring A. Using [20, Theorem, p. 23, and Proposition (a) y (b), p. 22] one has:

$$i^* = \mathrm{Spec}(i)_{\restriction_{\overline{\pi B}^{\mathrm{con}}}} \colon \overline{\pi B}^{\mathrm{con}} \to \overline{\pi A}^{\mathrm{con}},$$

where $\overline{\pi B}^{\mathrm{con}}$ and $\overline{\pi A}^{\mathrm{con}}$ are the closures of πB and πA in the constructible topology of the spectral spaces $\mathrm{Spec}(B)$ and $\mathrm{Spec}(A)$; i^* is surjective. As the (unitary) f-rings A and B are projectable, by [16, 6.11] the spaces πB and πA are compact (and Hausdorff). Remark that the topology of πA inherited by the space of irreducible ℓ-ideals of A is the Zariski topology on πA, cf. p. 29. By [26, Corollary 2.7], the subspaces πB and πA are proconstructible and therefore $\overline{\pi B}^{\mathrm{con}} = \pi B$ and $\overline{\pi A}^{\mathrm{con}} = \pi A$.

Henceforth, we assume that A and B are reduced and projectable f-rings. Under this assumption we get:

$$i^* = \mathrm{Spec}(i)_{\restriction_{\pi B}} \colon \pi B \to \pi A,$$

and i^* is surjective.

For $q_1, q_2 \in \pi B$, we set $q_1 \sim q_2$ if and only if $q_1 \cap A = q_2 \cap A$, if and only if $i^*(q_1) = i^*(q_2)$. Clearly $\sim$ is an equivalence relation on πB. Since $i^*\colon \pi B \to \pi A$ is surjective, πA can be endowed with the quotient topology of πB induced by i^* or by the equivalence relation $\sim$. By [29, Theorem 9.2, p. 60] the original topology of πA coincides with the topology induced by i^* whenever the function i^* is either open or closed. Since the f-rings A and B are projectable, by [16, 6.11], the spaces πA and πB are compact (and Hausdorff). Since the map $i^*\colon \pi B \to \pi A$ is continuous, by [29, p. 120], it is a closed function. Hence, the original topology on πA and the quotient topology on πB induced by the equivalence relation $\sim$ are the same. Therefore

$$j\colon \pi B/{\sim} \to \pi A, \quad q/{\sim} \mapsto i^*(q),$$

is a homeomorphism of topological spaces and of Boolean spaces.

Now let $p \in \pi A$ and $q \in (i^*)^{-1}\big(\{p\}\big)$, i.e., $i^*(q) = q \cap A = p$. Consider the map

$$h_{pq}\colon A/p \to B/q, \quad a + p \mapsto a + q.$$

Since $p \subseteq q \cap A$, h_{pq} is well defined, and $q \cap A \subseteq p$ implies that h_{pq} is injective (straightforward checking), proving the injectivity of h_{pq}. It is clear that h_{pq} is a ring homomorphism.

Let us now see that h_{pq} preserves order. Let $a, a' \in A$ be such that $a+p \leqslant a'+p$ in A/p. Then there exists $c \in p$ such that $c > 0$ and $a + c \leqslant a'$ in A. Since A is an $\mathcal{L}$-substructure of B and the order is in the language $\mathcal{L}$ then $a+c \leqslant a'$ in B. Since $p \subseteq q \cap A$ we get $c \in q$, whence $a+q \leqslant a'+q$ in B/q. Thus, $h_{pq}(a) \leqslant h_{pq}(a')$ in B/q. Since the orders on A/p and B/q are total, the reverse implication follows at once from the one we just proved. Therefore $h_{pq}\colon A/p \to B/q$ is an injective homomorphism of ordered rings. In this setting, one has:

Theorem 3.1 *Let A and B be reduced projectable f-rings satisfying the first convexity property such that $A \subseteq_{\mathcal{L}} B$, where $\mathcal{L} = \mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |_{\mathrm{loc}}\}$. If in addition A and B are divisible-projectable, then the homomorphism of ordered rings h_{pq} ($p \in \pi A$ and $q \in (i^*)^{-1}(\{p\})$) defined above preserves divisibility.*

Proof: We must prove: for any $a, a' \in A$,

$$a+p \mid a'+p \text{ in } A/p \text{ if and only if } a+q \mid a'+q \text{ in } B/q.$$

($\Rightarrow$) Suppose $a+p \mid a'+p$ in A/p, i.e., $(a+p)(c+p) = a'+p$ for some $c+p \in A/p$, i.e., $ac-a' \in p$. Since $p \subseteq q \cap A$, we have $ac - a' \in q$, which means $(a+q)(c+q) = a'+q$, i.e., $a+q \mid a'+q$ en B/q.

($\Leftarrow$) Assuming $a+q \mid a'+q$ in B/q, we have to show $a+p \mid a'+p$ in A/p.

• If $a'+q = 0$ then $a' \in q$, whence $a' \in q \cap A = p$. So $a'+p = 0$, and therefore $a+p \mid a'+p$ in A/p.

• If $a'+q \neq 0$ then $a' \notin q$. Then $a' \notin p$ and so $a'+p \neq 0$. Let us suppose in this case that $a+p \nmid a'+p$ en A/p. Consider $N := [\![\, a \nmid a' \,]\!]_{\pi A} \cap [\![\, a' \neq 0 \,]\!]_{\pi A}$ a clopen subset of πA. (Here we use the hypothesis that A is divisible projectable, see [12, Proposition 2.6]). Since $p \in N$ we have $N \neq \emptyset$. Let us define $\alpha' = a'_{\restriction_N} \cup 0_{\restriction_{\pi A \smallsetminus N}} \in A$. Since $N \neq \emptyset$, then $\alpha' \neq 0$.

Now suppose that $A \models a \mid_{\mathrm{loc}} \alpha'$. Since $\alpha' \neq 0$, we have

$$A \models \exists w'(w' \neq 0 \wedge w'(w'-\alpha') = 0 \wedge a \mid w').$$

Since $w' \neq 0$, there exists $\bar{p} \in \pi A$ such that $w'(\bar{p}) \neq 0$. From $w'(w'-\alpha') = 0$ comes $w'(\bar{p}) = \alpha'(\bar{p})$. By the definition of α' and $w'(\bar{p}) \neq 0$, one gets $\bar{p} \in N$ and $\alpha'(\bar{p}) = a'(\bar{p})$. Since $a \mid w'$, there exists $c \in A$ such that $ac = w'$, whence $a(\bar{p})c(\bar{p}) = w'(\bar{p}) = \alpha'(\bar{p}) = a'(\bar{p})$; so $a(\bar{p}) \mid a'(\bar{p})$ in $A/\bar{p}$, contradicting $\bar{p} \in [\![\, a \nmid a' \,]\!]_{\pi A}$. Conclusion: $A \models a \nmid_{\mathrm{loc}} \alpha'$. Since A is an $\mathcal{L}$-substructure of B and $|_{\mathrm{loc}}$ belongs to the language, then $B \models a \nmid_{\mathrm{loc}} \alpha'$. Since $\alpha' \neq 0$ then:

$$B \models \forall w'(w' \neq 0 \wedge w'(w'-\alpha') = 0 \to a \nmid w').$$

From our initial assumption $a+q \mid a'+q$ in B/q, it follows that $q \in [\![\, a \mid a' \,]\!]_{\pi B}$. By case assumption we also have $a'+q \neq 0$, i.e., $q \in [\![\, a' \neq 0 \,]\!]_{\pi B}$. From $p \in N$ comes $\alpha'(p) = a'(p)$, i.e., $\alpha'+p = a'+p$. From $p = q \cap A$ we get $\alpha'+q = a'+q$ in B/q and therefore $q \in [\![\, \alpha' = a' \,]\!]_{\pi B}$. The set $M := [\![\, a \mid a' \,]\!]_{\pi B} \cap [\![\, a' \neq 0 \,]\!]_{\pi B} \cap [\![\, \alpha' = a' \,]\!]_{\pi B}$, is a clopen set of πB with $q \in M$ and $M \neq \emptyset$ (here we use again that B is divisible-projectable).

Let us now set $w'' := \alpha'_{\restriction_M} \cup 0_{\restriction_{\pi B \smallsetminus M}} \in B$. Since $M \neq \emptyset$, for $\bar{q} \in M$ one has $w''(\bar{q}) = \alpha'(\bar{q}) = a'(\bar{q}) \neq 0$. Then $w'' \neq 0$. Let us prove next that $w''(w''-\alpha') = 0$. Let $\bar{q} \in \pi B$. If $\bar{q} \in \pi B \smallsetminus M$ then $w''(\bar{q}) = 0$ and so $[w''(w''-\alpha')](\bar{q}) = w''(\bar{q})(w''-\alpha')(\bar{q}) = 0$. If $\bar{q} \in M$, by the definition of w'' we have $w''(\bar{q}) = \alpha'(\bar{q})$, so $(w''-\alpha')(\bar{q}) = 0$, and $[w''(w''-\alpha')](\bar{q}) = 0$. In either case, $[w''(w''-\alpha')](\bar{q}) = 0$ for all $\bar{q} \in \pi B$, whence $w''(w''-\alpha') = 0$. Since $w'' \in B$ is such that $w'' \neq 0$ and $w''(w''-\alpha') = 0$, then $a \nmid w''$ in B.

On the other hand, for $\bar{q} \in \pi B$ one has the following:

• If $\bar{q} \in \pi B \smallsetminus M$, then $w''(\bar{q}) = 0$ and therefore $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$.

• If $\bar{q} \in M$ then $\bar{q} \in [\![\, a \mid a' \,]\!]_{\pi B} \cap [\![\, \alpha' = a' \,]\!]_{\pi B}$ and consequently $a(\bar{q}) \mid a'(\bar{q}) = \alpha'(\bar{q})$ in $B/\bar{q}$. Therefore $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$.

Thus, $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$ for all $\bar{q} \in \pi B$, i.e., there exists $c_{\bar{q}} \in B$ such that $a(\bar{q})c_{\bar{q}}(\bar{q}) = w''(\bar{q})$. This shows:

$$\pi B = \bigcup_{\bar{q} \in \pi B} [\![\, ac_{\bar{q}} = w'' \,]\!]_{\pi B}.$$

By compactness, πB is the union of a finite number of these terms, and using the patchwork property of B it is easy to construct an element $c \in B$ such that $ac = w''$, proving that $a \mid w''$ in B, and contradicting that $a \nmid w''$ in B (see above). This contradiction proves that $a + q \mid a' + q$ in B/q implies $a + p \mid a' + p$ in A/p, completing the proof of Theorem 3.1.

■

Let A and B be two models of T^* such that $A \subseteq_{\mathcal{L}} B$, where $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\preceq, |_{\text{loc}}\}$. It is known that $i^* \colon \pi B \to \pi A$ $(q \mapsto q \cap A)$ is a continuous surjective map such that $\pi A \cong \pi B/\sim$, where $\sim$ is the equivalence relation $q \sim q'$ if and only if $i^*(q) = q \cap A = q' \cap A = i^*(q')$. Furthermore, recall that the map $h_{pq} \colon A/p \to B/q$ $(a + p \mapsto a + q$ $p \in \pi A,\ q \in (i^*)^{-1}(\{p\}))$ introduced above is an injective homomorphism of ordered rings preserving divisibility.

Let us denote by $\mathcal{B}(\pi A)$ and $\mathcal{B}(\pi B)$ the Boolean algebras of clopen sets of πA and πB respectively. Then, $j = (i^*)^{-1} \colon \mathcal{B}(\pi A) \to \mathcal{B}(\pi B)$ is an injective homomorphism of Boolean algebras.

We want to show that $A \prec_{\mathcal{L}} B$. Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and $a_1, \ldots, a_n \in A$. By [8, Theorem 1.1], there exists an acceptable sequence $\zeta = \langle \Phi, \theta_1, \ldots, \theta_m \rangle$ of formulas, where $\theta_1, \ldots, \theta_m$ are $\mathcal{L}$-formulas with the same free variables as $\phi(x_1, \ldots, x_n)$ and Φ is a formula in the language of Boolean algebras with m free variables such that:

$$A \models \phi(a_1, \ldots, a_n) \iff \mathcal{B}(\pi A) \models \Phi\Big([\![\, \theta_1(a_1, \ldots, a_n) \,]\!]_A, \ldots, [\![\, \theta_m(a_1, \ldots, a_n) \,]\!]_A \Big),$$

where $[\![\, \theta_j(a_1, \ldots, a_n) \,]\!]_A = \big\{ p \in \pi A : A/p \models \theta_j(a_1 + p, \ldots, a_n + p) \big\}$, for $j = 1, \ldots, m$.

Since A and B are models of T^*, A/p and B/q are real closed valuation rings, for all $p \in \pi A$ and $q \in \pi B$. Therefore, in view of 3.1 and [7], the map $h_{pq} \colon A/p \to B/q$ is an elementary monomorphism, i.e., $h_{pq} \colon A/p \prec B/q$. Then:

$$\begin{aligned} j\Big([\![\, \theta_l(a_1, \ldots, a_n) \,]\!]_A \Big) &= \Big\{ q \in \pi B : B/q \models \theta_l\big(h_{pq}(a_1), \ldots, h_{pq}(a_n)\big) \text{ with } p = q \cap A \Big\} \\ &= [\![\, \theta_l(a_1, \ldots, a_n) \,]\!]_B. \end{aligned}$$

Since $\mathcal{B}(\pi A)$ are $\mathcal{B}(\pi B)$ are atomless Boolean algebras (A and B are models of T^*) $j \colon \mathcal{B}(\pi A) \prec \mathcal{B}(\pi B)$ is an elementary monomorphism. Then one has:

$$\begin{aligned} &\mathcal{B}(\pi A) \models \Phi\Big([\![\, \theta_1(a_1, \ldots, a_n) \,]\!]_A, \ldots, [\![\, \theta_m(a_1, \ldots, a_n) \,]\!]_A \Big) \iff \\ &\mathcal{B}(\pi B) \models \Phi\Big(j\big([\![\, \theta_1(a_1, \ldots, a_n) \,]\!]_A\big), \ldots, j\big([\![\, \theta_m(a_1, \ldots, a_n) \,]\!]_A\big) \Big) \iff \\ &\mathcal{B}(\pi B) \models \Phi\Big([\![\, \theta_1(a_1, \ldots, a_n) \,]\!]_B, \ldots, [\![\, \theta_m(a_1, \ldots, a_n) \,]\!]_B \Big). \end{aligned}$$

By [8, Theorem 1.1] one also has:

$$B \models \phi(a_1,\dots,a_n) \Longleftrightarrow \mathcal{B}(\pi B) \models \Phi\Big([\![\theta_1(a_1,\dots,a_n)]\!]_B, \dots, [\![\theta_m(a_1,\dots,a_n)]\!]_B \Big).$$

Therefore we have just proved:

$$A \models \phi(a_1,\dots,a_n) \text{ if and only if } B \models \phi(a_1,\dots,a_n),$$

showing that $A \prec_{\mathcal{L}} B$. Therefore we can state:

Theorem 3.2 *The theory T^* is model-complete in $\mathcal{L} = \mathcal{L}_{\mathrm{lor}} \cup \{ \preceq, |_{\mathrm{loc}} \}$.* ■

4 Universal theories.

In this section, the universal part of the theory T^* will be formulated and dtudied in several languages. We begin by discussing how a reduced projectable f-ring satisfying the first convexity property can be embedded in a model of T^* in the language $\mathcal{L}_{\mathrm{lor}} \cup \{ \preceq, |, |_{\mathrm{loc}} \}$. Since projectability is not expressed by a universal sentence, later in this section it will be replaced by other universal axioms, one for each of the symbols $|$ and $|_{\mathrm{loc}}$.

Proposition 4.1 *Let A be a reduced f-ring that satisfies the first convexity property. Then there exists $B \models T^*$ such that $A \subseteq_{\mathcal{L}} B$, where $\mathcal{L} = \mathcal{L}_{\mathrm{lor}} \cup \{ \preceq \}$. If, in addition, A is projectable, this inclusion remains valid for the language $\mathcal{L} = \mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |_{\mathrm{loc}}\}$.*

The proof of this proposition will be done in several steps. The first step uses ideas from [17, 6.1].

Step 4.2 *Let A be a reduced f-ring that satisfies the first convexity property. Then there exists $B \models T^*$ such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \mathcal{L}_{\mathrm{lor}}$.*

Proof: Let A be as in the hypothesis. Since A is a reduced f-ring, then $A \subseteq \prod_{p \in \pi A} A/p$, where πA is the space of minimal prime ideals of A and A/p is a totally ordered integral domain, for each $p \in \pi A$. Clearly this inclusion is in the language $\mathcal{L}_{\mathrm{lor}}$.

As A satisfies the first convexity property, the totally ordered rings A/p also satisfy it, for all $p \in \pi A$; see [12, Lemma 2.3]. By [2], A/p is a model of the theory $\mathrm{COVD_D}$ (*Convexely ordered valuation rings*) or a model of $\mathrm{OF_D}$ (*Ordered fields*), for every $p \in \pi A$. By [2, Theorem 1(i)], for each $p \in \pi A$ there exists a real closed valuation ring (not a field), R_p, such that $A/p \subseteq R_p$ in the language $\mathcal{L}_{\mathrm{or}} \cup \{|\}$, and we have $\prod_{p \in \pi A} A/p \subseteq \prod_{p \in \pi A} R_p$, in the language $\mathcal{L}_{\mathrm{lor}}$. Now, for each $p \in \pi A$ take a copy C_p of the Cantor space and observe that $R_p \subseteq R_p^{C_p}$ by the constant inclusion $x \mapsto (x)^{C_p}$. It is clear that this inclusion holds in the language $\mathcal{L}_{\mathrm{or}} \cup \{|\}$. Therefore, one has:

$$A \subseteq \prod_{p \in \pi A} A/p \subseteq \prod_{p \in \pi A} R_p \subseteq \prod_{p \in \pi A} R_p^{C_p}, \qquad (*)$$

in the language $\mathcal{L}_{\mathrm{lor}}$. Since the theory of real closed valuation rings, denoted by RCVR, is complete and model complete in $\mathcal{L}_{\mathrm{or}} \cup \{|\}$ (cf. [7]), by [6, Proposition 3.5.11(ii)] it has the Joint Embedding Property (JEP) in the language $\mathcal{L}_{\mathrm{or}} \cup \{|\}$. Therefore there exists a real closed valuation ring (not a field), R, such that

$$\prod_{p \in \pi A} R_p^{C_p} \subseteq R^C, \qquad (**),$$

where $C = \prod_{p\in\pi A} C_p$ is a product of Cantor spaces. This inclusion can be considered in the language $\mathcal{L}_{\text{lor}}$. By [5, Theorem 2.1.(b)], one has $R^C \in \Gamma^e_{\mathcal{L}'}(\text{RCVR})$. This means that $B = R^C$ is a model of T^* such that $A \subseteq B$ in the language $\mathcal{L}_{\text{lor}}$.

□

Step 4.3 *Let A be a reduced f-ring that satisfies the first convexity property. Then there exists $B \models T^*$ such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\preceq\}$.*

Proof: With notation as in the proof of 4.2, we need to prove that the inclusions in $(*)$ and in $(**)$ preserve the radical relation. Since the radical relation $b \preceq a$ is given by the universal formula $\forall x(ax = 0 \to bx = 0)$ (see section 2), it is clear that this relation is downward preserved at each inclusion. Next, we show that it is upwards preserved. Consider the map:

$$\iota\colon A \to \prod_{p\in\pi A} A/p, \quad a \mapsto (a+p)_{p\in\pi A}.$$

Let $a, b \in A$ and suppose $A \models a \preceq b$. We want to see that $\prod_{p\in\pi A} A/p \models \iota(a) \preceq \iota(b)$. Let $\tilde{c} = (c_p + p)_{p\in\pi A} \in \prod_{p\in\pi A} A/p$ be such that $\iota(b)\tilde{c} = 0$. Then $bc_p + p = 0$, i.e., $bc_p \in p$, for all $p \in \pi A$. Since $a \preceq b$ in A and $c_p \in A$, then $ac_p \preceq bc_p$ for every $p \in \pi A$, whence $ac_p \in p$, for all $p \in \pi A$, i.e., $a\tilde{c} = 0$. This shows that $\forall \tilde{c}\,(\iota(b)\tilde{c} = 0 \Rightarrow \iota(a)\tilde{c} = 0)$ in $\prod_{p\in\pi A} A/p$, i.e., $\iota(b) \preceq \iota(a)$ is valid in $\prod_{p\in\pi A} A/p$. [2]

Let us now see that the radical relation is preserved at the second inclusion of $(*)$. Let $\tilde{a} = (a_p + p)_{p\in\pi A}$ and $\tilde{b} = (b_p + p)_{p\in\pi A}$ in $\prod_{p\in\pi A} A/p$ be such that $\tilde{a} \preceq \tilde{b}$, that is: $\prod_{p\in\pi A} A/p \models \forall x(\tilde{b}x = 0 \to \tilde{a}x = 0)$. To see that $\tilde{a} \preceq \tilde{b}$ is valid in $\prod_{p\in\pi A} R_p$, let $\tilde{x} = (x_p)_{p\in\pi A}$ with $x_p \in R_p$ for all $p \in \pi A$, be such that $\tilde{b}\tilde{x} = 0$. That is, $(b_p + p)x_p = 0$ for all $p \in \pi A$. For a fixed $p \in \pi A$, one has $b_p + p = 0$ or $x_p = 0$, as R_p is an integral domain. If $x_p = 0$, then $(a_p + p)x_p = 0$. If $b_p + p = 0$ then $b_p \in p$. Taking $x \in \prod_{p\in\pi A} A/p$ given by $x_q = \delta_{pq} = \begin{cases} 1 & \text{if} \quad p = q \\ 0 & \text{if} \quad p \neq q \end{cases}$, we have $\tilde{b}x = 0$ in $\prod_{p\in\pi A} A/p$ and hence $\tilde{a}x = 0$ by hypothesis, that is $a_p + p = 0$ and therefore $(a_p + p)x_p = 0$. This is satisfied at all $p \in \pi A$ and hence $\tilde{a}\tilde{x} = 0$, proving that $\prod_{p\in\pi A} R_p \models \tilde{a} \preceq \tilde{b}$.

For the third inclusion in $(*)$, consider $r, s \in \prod_{p\in\pi A} R_p$, given by $r = (r_p)_{p\in\pi A}$ and $s = (s_p)_{p\in\pi A}$, so that $r \preceq s$ in $\prod_{p\in\pi A} R_p$. Take into account that for each $p \in \pi A$, the inclusion $R_p \hookrightarrow R_p^{C_p}$ is given by $r \mapsto (r)^{C_p}$ where $(r)^{C_p}$ is a C_p-uple constantly equal to r. We want to prove that $\forall x(sx = 0 \to rx = 0)$ is true in $\prod_{p\in\pi A} R_p^{C_p}$. Let $x \in \prod_{p\in\pi A} R_p^{C_p}$ be such that $sx = 0$, with $x = (x_p)_{p\in\pi A}$ and $x_p = (x_p^i)_{i\in C_p}$. Note that $sx = 0$ means $s_p x_p^i = 0$ for all $p \in \pi A$ and all $i \in C_p$. Fixing $p \in \pi A$, one has $s_p x_p^i = 0$, for all $i \in C_p$. Since $r \preceq s$ in $\prod_{p\in\pi A} R_p$, with $x^p \in \prod_{p\in\pi A} R_p$ given by $x^p(q) = \delta_{pq} = \begin{cases} 1 & \text{if} \quad p = q \\ 0 & \text{if} \quad p \neq q \end{cases}$, one has $sx^p = 0 \Rightarrow rx^p = 0$, that is, $s_p = 0 \Rightarrow r_p = 0$. Now we have two cases:

- If $s_p = 0$ then $r_p = 0$ and therefore $r_p x_p^i = 0$, for all $i \in C_p$.
- If $s_p \neq 0$ then $x_p^i = 0$ for all $i \in C_p$, and therefore $r_p x_p^i = 0$, for all $i \in C_p$.

We have shown that $s_p x_p^i = 0 \Rightarrow r_p x_p^i = 0$ for all $i \in C_p$ and all $p \in \pi A$. This meanss exactly $r \preceq s$ in $\prod_{p\in\pi A} R_p^{C_p}$. [3]

[2] Observe that in this paragraph the only condition used on A is to be a reduced f-ring.

[3] The only complication that arises in this third inclusion is notational, since everything reduces to prove

Concerning the inclusion in $(**)$, by model-completeness of the theory of real closed valuation rings (cf. [7]) we have $R_p \prec R$, for all $p \in \pi A$. By the Feferman-Vaught theorem, [11], we get:

$$\prod_{p\in\pi A} R_p^{C_p} \prec R^C. \tag{$\dagger$}$$

Then, clearly, the radical relation (and any other definable relation) is preserved by this inclusion.

□

Step 4.4 *Let A be a projectable reduced f-ring that satisfies the first convexity property. Then there exists $B \models T^*$ such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\preceq, |\}$.*

Proof: As in the proof of the Step 4.3, we use the inclusions (*) and (**) in Step 4.2. We begin by proving that the divisibility relation is preserved in the first inclusion of $(*)$. For the implication $(\Rightarrow)$, projectability is not neded. For if $a, b \in A$ are such that $b \mid a$, then there exists $c \in A$ with $bc = a$; therefore $(b+p)(c+p) = bc + p = a + p$, for all $p \in \pi A$. That is $b + p \mid a + p$, for all $p \in \pi A$, i.e.,:

$$(b+p)_{p\in\pi A} \mid (a+p)_{p\in\pi A} \text{ in } \prod_{p\in\pi A} A/p.$$

For the direction $(\Leftarrow)$, if $a, b \in A$ are such that $(b+p)_{p\in\pi A} \mid (a+p)_{p\in\pi A}$, there is $(c_p+p)_{p\in\pi A} \in \prod_{p\in\pi A} A/p$ with $(b+p)_{p\in\pi A} \cdot (c_p+p)_{p\in\pi A} = (a+p)_{p\in\pi A}$. Therefore $bc_p + p = a + p$, for all $p \in \pi A$. Since A is a subdirect product, there exists $\tilde{c_p} \in A$ such that $\tilde{c_p}(p) = c_p$ for every $p \in \pi A$. Considering the set $X_p := [\![b \cdot \tilde{c_p} = a]\!]$, one has $p \in X_p$, for all $p \in \pi A$. Then, $\pi A = \bigcup_{p\in\pi A} X_p$ is a clopen covering. By compactness of πA and the glueing property of A, there exists $c \in A$ such that $\pi A = [\![b \cdot c = a]\!]$, i.e., $(bc)+p = a+p$, for all $p \in \pi A$, that is, $bc - a \in \bigcap_{p\in\pi A} p$. Since A is reduced, $\bigcap_{p\in\pi A} p = \{0\}$, and therefore $bc = a$, i.e., $b \mid a$ in A. Then, the inclusion $A \subseteq \prod_{p\in\pi A} A/p$ holds in the language $\mathcal{L}_{\text{lor}} \cup \{|\}$.

We shall now prove that divisibility is preserved by the other inclusions of $(*)$ and $(**)$. For the inclusion in $(**)$, this is clear, since it is definable by a formula in $\mathcal{L}_{\text{lor}}$. In general, as divisibility is definable by an existential formula, it is upwards preserved under all inclusions. Then we need only prove that divisibility is downwards preserved at the second and third inclusions of $(*)$. Note that in the proof of step 4.2, we remarked that $A/p \subseteq R_p$ and $R_p \subseteq R_p^{C_p}$ are inclusions in the language $\mathcal{L}_{\text{or}} \cup \{|\}$. Pointwise verification shows that divisibility goes down at the second and third inclusions of $(*)$.

□

Step 4.5 *Let A be a projectable reduced f-ring that satisfies the first convexity property. Then there exists $B \models T^*$ such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\preceq, |, |_{\text{loc}}\}$.*

Proof: We are going to prove that, under the given assumptions, the local divisibility relation is preserved by all the inclusions in $(*)$. Since local divisibility is expressed by an existential formula in the language $\mathcal{L}_{\text{lor}}$, it is upwards preserved in any extension. So, it suffices to prove that it is downwards preserved under each inclusion in $(*)$.

For the first inclusion in $(*)$, let $a, b \in A$ be such that $b \mid_{\text{loc}} a$ holds in $\prod_{p\in\pi A} A/p$ and prove that it is true in A. This is clear if $a = 0$. If $a \neq 0$, there exists $w \in \prod_{p\in\pi A} A/p$ such that $w \neq 0$, $w(w-a) = 0$ and $b \mid w$ in $\prod_{p\in\pi A} A/p$. Since $w \neq 0$, there is $p_0 \in \pi A$ such that $w_{p_0} \neq 0$. Therefore $a(p_0) = w_{p_0} \neq 0$ and $b(p_0) \mid w_{p_0} = a(p_0)$ in A/p_0. Hence

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that for each $p \in \pi A$, the inclusion $R_p \hookrightarrow R_p^{C_p}$ preserves the radical relation. That is obvious since the inclusion is given by a constant function.

there is $c_{p_0} \in A/p_0$ so that $b(p_0)c_{p_0} = a(p_0)$. Let $c \in A$ be such that $c(p_0) = c_{p_0}$; then $b(p_0)c(p_0) = a(p_0)$, and hence $p_0 \in [\![bc = a]\!] \cap [\![a \neq 0]\!] = N \neq \emptyset$, a clopen subset of πA. Since A is projectable, let $\tilde{w} \in A$ be defined by $\tilde{w} := a_{\restriction N} \cup 0_{\restriction \pi A \setminus N}$. Then $\tilde{w} \neq 0$, and clearly $\tilde{w}(\tilde{w} - a) = 0$. It is easy to see that $b \mid \tilde{w}$ in A: the element $d \in A$ given by $d = c_{\restriction N} \cup 0_{\restriction \pi A \setminus N} \in A$ satisfies $bd = \tilde{w}$. This shows that $A \models \exists w(w \neq 0 \wedge w(w-a) = 0 \wedge b \mid w)$, i.e., $A \models b \mid_{\mathrm{loc}} a$.

We now turn to the second inclusion: $\prod_{p \in \pi A} A/p \subseteq \prod_{p \in \pi A} R_p$. To ease notation, let $X = \pi A$ with $A_x = A/p$ and $R_x = R_p$ for all $x \in X$. Let $a = (a_x)_{x \in X}$ and $b = (b_x)_{x \in X}$ in $\prod_{x \in X} A_x$ be such that $b \mid_{\mathrm{loc}} a$ in $\prod_{x \in X} R_x$. If $a = 0$ then $b \mid_{\mathrm{loc}} a$ in $\prod_{x \in X} A_x$. If $a \neq 0$, there exists $w = (w_x)_{x \in X} \in \prod_{x \in X} R_x$ such that $w \neq 0$, $w(w-a) = 0$ and $b \mid w$ in $\prod_{x \in X} R_x$. For each $x \in X$, either $w_x = 0$ or $w_x = a_x \in A_x$. Then $w = (w_x)_{x \in X} \in \prod_{x \in X} A_x$. Since $b, w \in \prod_{x \in X} A_x$, then $b \mid w$ in $\prod_{x \in X} R_x$ implies $b \mid w$ in $\prod_{x \in X} A_x$, proving $b \mid_{\mathrm{loc}} a$ in $\prod_{x \in X} A_x$.

For the last inclusion in $(*)$, let $a = (a_x)_{x \in X}$ and $b = (b_x)_{x \in X}$ in $\prod_{x \in X} R_x$ be such that $b \mid_{\mathrm{loc}} a$ in $\prod_{x \in X} R_x^{C_x}$. If $a = 0$, obvioulsy $b \mid_{\mathrm{loc}} a$ in $\prod_{x \in X} R_x$. If $a \neq 0$, there is $w = (w_x^c)_{x \in X, c \in C_x}$ such that $w \neq 0$, $w(w-a) = 0$ and $b \mid w$ in $\prod_{x \in X} R_x^{C_x}$. For each $x \in X$, if there exists $c \in C_x$ with $w_x^c \neq 0$, then $w_x^c = a_x \neq 0$. In this case, redefine $w \in \prod_{x \in X} R_x^{C_x}$ by seting $w_x^c = a_x \neq 0$, for all $c \in C_x$, whence $w_x^c \neq 0$. Otherwise, if for some $x \in X$ one has $w_x^c = 0$ for all $c \in C_x$, there is nothing to redefine. Therefore w is in $\prod_{x \in X} R_x$ and one has $w \neq 0$, $w(w-a) = 0$ and $b \mid w$ in $\prod_{x \in X} R_x^{C_x}$. It was already proved above that $b \mid w$ in $\prod_{x \in X} R_x$, whence $b \mid_{\mathrm{loc}} a$ in $\prod_{x \in X} R_x$. □

Steps 4.2 and 4.5 complete the proof of Proposition 4.1. ■

As noted before, projectability is not adequate for our purposes since it is not expressed by a universal axiom. In the sequel of this section, we replace projectability by a universal axiom scheme for each symbol in $\{\mid, \mid_{\mathrm{loc}}\}$. We begin with:

Lemma 4.6 *Let A be a reduced and projectable f-ring; then A satisfies:*

$$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \rightarrow b \mid a \big),$$

for each $n \in \mathbb{N}$.

Proof: By hypothesis we have $A \in \Gamma^{a}_{\mathcal{L}_{\mathrm{or}}}\big(X, (A_x)_{x \in X}\big)$, where X is a Boolean space and $(A_x)_{x \in X}$ is a family of totally ordered integral domains. Let $a, b, c_1, \ldots, c_n \in A$ be such that $(bc_1 - a) \cdots (bc_n - a) = 0$. Then:

$$\big(b(x)c_1(x) - a(x)\big) \cdots \big(b(x)c_n(x) - a(x)\big) = 0,$$

for all $x \in X$. For each $i \in \{1, \ldots, n\}$, set:

$$N_i := [\![bc_i - a = 0]\!] = \{x \in X : b(x)c_i(x) - a(x) = 0\};$$

these are clopen subsets of X. Since the A_x's are integral domains, then, $X = \bigcup_{i=1}^n N_i$. Without loss of generality, we can suppose that the N_i's are pairwise disjoint and non-empty (if some N_i is empty, eliminate the corresponding c_i). By the patchwork property of A one has:

$$c = c_{1\restriction N_1} \cup \cdots \cup c_{n \restriction N_n} \in A.$$

Clearly $b(x)c(x) - a(x) = 0$, for all $x \in X$, proving that $b \mid a$ in A. ■

Corollary 4.7 *Let B be a reduced and projectable f-ring, and let A be a substructure of B in the language $\mathcal{L}_{\text{lor}} \cup \{|\}$. Then A satisfies:*

$$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \rightarrow b \mid a\big),$$

for all $n \in \mathbb{N}$.

Proof: Immediate from Lemma 4.6. ■

In the same sense one has:

Corollary 4.8 *Let $B \models T^*$ and A be a substructure of B in the language $\mathcal{L}_{\text{lor}} \cup \{|\}$. Then A satisfies:*

$$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \rightarrow b \mid a\big),$$

for each $n \in \mathbb{N}$. ■

In view of our previous results, we establish the following

Definition 4.9 *Let A be any ring. We say that A satisfies the* **divisibility glueing axiom scheme** *if A satisfies:*

$$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \rightarrow b \mid a\big),$$

for all $n \in \mathbb{N}$.

The *divisibility glueing axiom scheme* is a set of universal formulas and the analog of Proposition 4.1 obtained by replacing the projectability assumption by this axiom scheme can be proved for the language $\mathcal{L}_{\text{lor}} \cup \{\preceq, |\}$. In order to carry this conversion out we will need information concerning spectral spaces and irreducible ℓ-ideals in f-rings. For the first the reader is referred to [9]; a suitable reference for the second matter is [3], especially chapters 8 - 10. The notion of an irreducible ℓ-ideal in lattice-ordered rings is defined in [3, (8.4.1)]. We shall denote by $\text{Spec}_\ell(A)$ the set of irreducible ℓ-ideals of A, and prove:

Lemma 4.10 *The set $\text{Spec}_\ell(A)$ of irreducible ℓ-ideals of an f-ring (with unit), A, endowed with the topology defined by the family $\{S(a) : a \in A\}$ as a basis of open sets, where $S(a) = \{p \in \text{Spec}_\ell(A) : a \notin p\}$, is a spectral space.*

Proof: [3, (10.1.6)] proves that if A is an f-ring with unit, then $\text{Spec}_\ell(A)$ is a quasi-compact space. It is clear that $\text{Spec}_\ell(A)$ is T_0. By [3, 10.1.4], the sets $S(a)$ ($a \in A$) are quasi-compact and the family of them is closed under finite intersections. It only remains to prove the soberness axiom for spectral spaces, [9, Axiom (S4), Definition 1.1.5, p. 4]. Using [3, 10.1.7], routine arguments show that if F is a non-empty closed irreducible subset of $\text{Spec}_\ell(A)$ then there exists $p \in \text{Spec}_\ell(A)$ such that $F = H(p) = \overline{\{p\}}$. ■

By [3, (9.1.5)], if A is an f-ring and $p \in \text{Spec}_\ell(A)$, then A/p is a totally ordered ring. Nevertheless, A/p may not be an integral domain. To satisfy this requirement we restrict to the subspace:

$$Y = \big\{p \in \text{Spec}_\ell(A) : p \text{ is prime}\big\},$$

see [3, section (9.2), especially (9.2.5)]. Clearly, A/p is a totally ordered integral domain for $p \in Y$.

Lemma 4.11 *The set Y with the topology induced by $\text{Spec}_\ell(A)$ is a spectral space.*

Proof: By [9, 2.1.3], it is sufficient to prove that Y is proconstructible in $\mathrm{Spec}_\ell(A)$, i.e., that $\mathrm{Spec}_\ell(A)\smallsetminus Y$ is open in the constructible topology of $\mathrm{Spec}_\ell(A)$. Let $p_0 \in \mathrm{Spec}_\ell(A)\smallsetminus Y$. Then p_0 is an irreducible ℓ-ideal that is not a prime ideal: there exists $a, b \in A$ such that $ab \in p_0$ with $a \notin p_0$ and $b \notin p_0$. The set $\mathcal{O} := V(ab) \cap D(a) \cap D(b)$ is open in the constructible topology of $\mathrm{Spec}_\ell(A)$, $p_0 \in \mathcal{O}$ and $\mathcal{O} \subseteq \mathrm{Spec}_\ell(A) \smallsetminus Y$ (no $p \in \mathcal{O}$ is prime). ■

Summarizing, we have proved if A is a (reduced) f-ring, there exists a spectral space $Y \subseteq \mathrm{Spec}_\ell(A)$ such that A/p is a totally ordered integral domain, for all $p \in Y$. By [12, Lemma 2.3], if A is an f-ring satisfying the first convexity property, A/p also satisfies this property, for all $p \in Y$. We are ready to prove the following proposition.

Proposition 4.12 *Let A be a reduced f-ring satisfying the first convexity property and the divisibility glueing axiom scheme. Then there exists $B \models T^*$ such that A is a substructure of B in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|\}$.*

Proof: Let A satisfy the conditions in the hypothesis. Choose any spectral space, Y, with the property of Lemma 4.11 relative to A. Let X be any proconstructible subspace of Y containing πA (by [3, 9.3.2] one has $\pi A \subseteq Y$) (e.g., we can take X to be $\overline{\pi A}^{\mathrm{con}}$, the closure of πA in the constructible topology of Y).

Consider the map $\iota\colon A \to \prod_{p\in X} A/p, \quad a \mapsto (a+p)_{p\in X}$. Clearly, ι is a homomorphism of lattice-ordered rings. Since A is reduced and X contains πA, ι in an embedding, hence a monomorphism of lattice-ordered rings. Clearly, if $a, b \in A$ are such that $b \mid a$ in A then $\iota(b) \mid \iota(a)$ in $\prod_{p\in X} A/p$. We prove the reverse implication.

Let $a, b \in A$ be such that $\iota(b) \mid \iota(a)$ in $\prod_{p\in X} A/p$. Set $A_x := A/x$ for $x \in X$. Then $b(x)$ divides $a(x)$ in A_x, for all $x \in X$, i.e., there is $c_x \in A_x$ such that $b(x)\, c_x = a(x)$, for all $x \in X$. Since A is an f-ring, there exists $\tilde{c}_x \in A$ such that $\tilde{c}_x(x) = c_x$, for all $x \in X$, whence $b(x)\tilde{c}_x(x) = a(x)$, for all $x \in X$. Therefore $x \in [\![\, b\tilde{c}_x = a \,]\!] = N_x$, a clopen set in the constructible topology of X. Therefore: $X = \bigcup_{x\in X} N_x$,, and by compactness of X, there are $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n N_{x_i}$. Set $c_i = \tilde{c}_{x_i}$ and $N_i = N_{x_i}$, for $i = 1, \dots, n$. Then, every $x \in X$ satisfies $b(x)c_i(x) = a(x)$ for some $i \in \{1, \dots, n\}$, whence $\big(b(x)c_1(x) - a(x)\big) \cdots \big(b(x)c_n(x) - a(x)\big) = 0$, for all $x \in X$, i.e., $(bc_1 - a) \cdots (bc_n - a) = 0$. By the divisibility glueing property of A we get $b \mid a$. We have proved:

$$A \models b \mid a \text{ if and only if } \prod_{p\in X} A/p \models \iota(b) \mid \iota(a).$$

Thus, ι preserves the divisibility relation. Note that $\prod_{p\in X} A/p$ is a reduced and projectable f-ring; then by Proposition 4.1, there exists $B \models T^*$ such that $\prod_{p\in X} A/p \subseteq B$ in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|\}$, as asserted. ■

Remark 4.13 Other possible choices of the space X in the previous proof are the whole $Y = \big\{p \in \mathrm{Spec}_\ell(A) : p \text{ is prime}\big\}$, or $X = \overline{\pi A}^{\mathrm{con}}$ where πA is seen as a subspace of $\mathrm{Sper}(A)$, the real spectrum of A, or even a subspace of RCVR-Spec(A) containing πA, cf. [21]. We can now state:

Theorem 4.14 *The universal theory of T^* in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|\}$ is the theory of reduced f-rings satisfying the first convexity property and the divisibility glueing property.*
Proof: Follows at once from Corolary 4.8 and Propostion 4.12. ■

In the remainder of this section, we determine the universal theory of T^* when local divisibilty and the radical relation are added to the language. We begin with:

Proposition 4.15 *Let A be a reduced and projectable f-ring. Then A satisfies:*

$$\forall a \forall b \forall c\,(a \not\preceq bc - a \to b \mid_{\mathrm{loc}} a). \tag{$\star$}$$

Proof: Let A be a reduced and projectable f-ring. Then $A \in \Gamma^{a}_{\mathcal{L}_{\mathrm{or}}}\big(X, (A_x)_{x \in X}\big)$, where $X = \pi A$ is the space of minimal prime ideals and $(A_x)_{x \in X}$ is a family of totally ordered integral domains. Let $a, b, c \in A$ be such that $a \not\preceq bc - a$. By the equivalences in (†), section 2, page 4; there exists $x \in X$ such that $a(x) \neq 0$ and $(bc - a)(x) = 0$. Let $N := [\![\, a \neq 0 \,]\!] \cap [\![\, bc - a = 0 \,]\!]$; N is a non-empty clopen set containing x. By the patchwork property of A, there exists $w \in A$ such that $w = a_{\restriction N} \cup 0_{\restriction X \smallsetminus N}$. Set $c' := c_{\restriction N} \cup 0_{\restriction X \smallsetminus N} \in A$. From the definition of N one has $w \neq 0$, $w(w - a) = 0$ and $bc' = w$, which implies $b \mid_{\mathrm{loc}} a$. ■

Remark 4.16 It is clear that $a \neq 0$ in the previous proof. In general, if the formula ($\star$) in the preceding Proposition is valid for any radical relation $\preceq$, then $a \neq 0$. For if $a = 0$ then $0 \not\preceq bc$, as $0 \preceq d$, for all $d \in A$. We have the following corolaries.

Corollary 4.17 *Let B be reduced and projectable f-ring, and A a substructure of B in the language $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, \mid_{\mathrm{loc}}\}$. Then A satisfies:*

$$\forall a \forall b \forall c\,(a \not\preceq bc - a \to b \mid_{\mathrm{loc}} a).$$

Proof: Follows at once from Proposition 4.15. ■

Corollary 4.18 *Let $B \models T^*$ with A a substructure of B in the language $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, \mid_{\mathrm{loc}}\}$. Then, A satisfies:*

$$\forall a \forall b \forall c\,(a \not\preceq bc - a \to b \mid_{\mathrm{loc}} a).$$

■

The formula $\forall a \forall b \forall c\,(a \not\preceq bc - a \to b \mid_{\mathrm{loc}} a)$ establishes a compatibility condition between the radical relation $\preceq$ and local divisibility.

Definition 4.19 *Let A be any ring. We say that A has the* **local divisibility property** *if A satisfies $\forall a \forall b \forall c\,\big((a \not\preceq bc - a) \to b \mid_{\mathrm{loc}} a\big)$.*

We can now prove:

Proposition 4.20 *Let A be a reduced f-ring satisfying the first convexity property and the local divisibility property. Then there exists $B \models T^*$ such that $A \subseteq B$ as a substructure in the language $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, \mid_{\mathrm{loc}}\}$.*

Proof: Let A be an f-ring with the properties of the statement. As in Proposition 4.12 consider a spectral space $Y \subseteq \mathrm{Spec}_{\ell}(A)$ such that A/p is a totally ordered integral domain for all $p \in Y$, together with a proconstructible subset X of Y containing πA.

As in Proposition 4.12, let $\iota \colon A \to \prod_{x \in X} A_x$ be the lattice-ordered ring homorphism given by $\iota(a) = (a + x)_{x \in X}$. Since X contains πA and A is reduced, ι is a monomorphism preserving the radical relation; hence $\preceq \;=\; \preceq_X$. We want to see that ι preserves local divisibility. Let $a, b \in A$. If $b \mid_{\mathrm{loc}} a$ in A, clearly $\iota(b) \mid_{\mathrm{loc}} \iota(a)$ in $\prod_{x \in X} A_x$. Conversely, assume $\iota(b) \mid_{\mathrm{loc}} \iota(a)$ in $\prod_{x \in X} A_x$. If $\iota(a) = 0$ then $a = 0$, and $b \mid_{\mathrm{loc}} a$ in A. Let $\iota(a) = \big(a(x)\big)_{x \in X} \neq 0$, then there exists $w = (w_x)_{x \in X} \in \prod_{x \in X} A_x$ such that $w \neq 0$, $w\big(w - \iota(a)\big) = 0$ and $\iota(b) \mid w$ in $\prod_{x \in X} A_x$. Let $c = (c_x)_{x \in X} \in \prod_{x \in X} A_x$ be such that $\iota(b)c = w$. Since $w \neq 0$, there is $x_0 \in X$ such that $w_{x_0} \neq 0$ and $w_{x_0} = a(x_0)$ as

$w\big(w - \iota(a)\big) = 0$. With $c \in A$ such that $c(x_0) = c_{x_0}$, we get $b(x_0)c(x_0) = a(x_0)$; thus, $a(x_0) \neq 0$ and $(bc - a)(x_0) = 0$, whence $a \not\preceq_X bc - a$. Since $\preceq\; = \;\preceq_X$ we get $a \not\preceq bc - a$ in A, and then $b \,|_{\mathrm{loc}}\, a$ in A, as A satisfies the local divisibility property. This completes the proof that $\iota\colon A \to \prod_{x\in X} A_x$ preserves local divisibility.

Since A satisfies the first convexity property, then A_x also satisfies it for all $x \in X$, and so does $\prod_{x\in X} A_x$. Since $\prod_{x\in X} A_x$ is also reduced and projectable, Proposition 4.1, shows that there is $B \models T^*$ such that $\prod_{x\in X} A_x \subseteq B$ in the language $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |_{\mathrm{loc}}\}$, whence $A \subseteq B$ in $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |_{\mathrm{loc}}\}$ as well. ■

Theorem 4.14, Corollary 4.18 and Proposition 4.20 yield:

Theorem 4.21 *The universal theory of T^* in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |_{\mathrm{loc}}\}$ is the theory of reduced f-rings satisfying the first convextiy property, the divisibility glueing axiom scheme and the local divisibility property.* ■

By Theorem 3.2, T^* is also model-complete in the language $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |, |_{\mathrm{loc}}\}$. In view of Theorem 4.21, we have:

Theorem 4.22 *The theory T^* is the model-companion of the theory of reduced f-rings satisfying the first convexity property, the divisibility glueing axiom scheme and the local divisibility property in the language $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |, |_{\mathrm{loc}}\}$.* ■

5 The maximal local divisibility relation.

In this section, we study equivalent forms of the local divisibility relation valid in the theory of reduced, projectable and divisible-projectable f-rings (and so in the theory T^*). The main result of this section is the equivalent form of local divisibility given in item (vi) (or (vi)$'$) of Proposition 5.13. This equivalent form of local divisibility is the key to the proof of Proposition 6.9, a generalization of Proposition 3.1 where the ring A can be chosen as a model of the universal part of the theory T^*. In turn, Proposition 6.9 is the main ingredient in the proof of the quantifier elimination theorem 6.14.

We start with a simple fact.

Fact 5.1 *Let A be any ring. Then for all $a, b \in A$ one has:*

$$b \,|_{\mathrm{loc}}\, a \leftrightarrow a = 0 \vee \exists w\big(w \neq 0 \wedge w(a - w) = 0 \wedge w \preceq a \wedge b \mid w\big).$$

Proof: ($\Leftarrow$) is clear.
($\Rightarrow$) This implication is obvious if $a = 0$. Otherwise, there exists $w \neq 0$ such that $w(w - a) = 0$ and $b \mid w$. To see that $w \preceq a$ (i.e., $\forall p \in \pi A\,(a \in p \Rightarrow w \in p)$), let $p \in \pi A$, with $a \in p$; then $wa \in p$. Since $w(w - a) = 0$, we get $w^2 = wa \in p$, and hence $w \in p$. ■

Remark 5.2 *Let A be any ring and let $a, b \in A$ be such that $\exists e(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae)$. Then, $\exists w\big(w \neq 0 \wedge w(a - w) = 0 \wedge w \preceq a \wedge b \mid w\big)$.*

Proof: Set $w := ae \neq 0$. Note that $w(a - w) = ae(a - ae) = a^2e(1 - e) = 0$. Since $x \preceq 1$ for all $x \in A$, then $e \preceq 1$ and, by item (4) in the definition of a radical relation, we get $ea \preceq a$, whence $w \preceq a$. Clearly $b \mid w$. ■

Conversely, we have:

Proposition 5.3 *Let A be a reduced and projectable f-ring. Let $a, b \in A$ be such that $\exists w\big(w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w\big)$. Then $\exists e(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae)$.*

Proof: Assume the statement on the left. The idempotent $e \in A$ is the support of w, constructed using the projectability of A, as follows: let $1 = c + d$, where $c \in w^{\perp}$ and $d \in w^{\perp\perp}$, (see [15, Lema 3.3]). Then $c \cdot w = 0$ and $d \preceq w$; since $w \preceq a$, we get $d \preceq a$, and $w \cdot c = 0$ implies $c \cdot d = 0$. Therefore, $c = c \cdot 1 = c(c+d) = c^2 + cd = c^2 + 0 = c^2$, i.e., c is an idempotent, and so is $d = 1 - c$. Set $e := d$.

First we prove $ad \neq 0$. Assume $ad = 0$; since $w \preceq a$ we get $dw = 0$, and from $d \preceq w$ follows $\mathrm{Ann}(w) \subseteq \mathrm{Ann}(d)$, and then $d = d^2 = 0$. Thus, $c = 1$ and $w = 0$, contradicting the hypothesis $w \neq 0$. Therefore $ad \neq 0$.

Next we show $w \preceq d$, that is: $\mathrm{Ann}(d) \subseteq \mathrm{Ann}(w)$. Given $x \in A$ so that $dx = 0$, from $(1-c)x = 0$ we conclude $x = cx$. From $wc = 0$ we get $wx = wcx = 0$. We have proved $\forall x(dx = 0 \Rightarrow wx = 0)$, that is: $w \preceq d$. Thus, $d \preceq w \wedge w \preceq d$, i.e., $w =_s d$.

Since $w \preceq a$, then $wd \preceq ad$. Observing that $w = w \cdot 1 = w(c+d) = wc + wd = 0 + wd = wd$, we get $w \preceq ad$. Also $d \preceq w$ implies $ad \preceq aw = w^2$. Since $w \preceq 1$ then $w^2 \preceq w$ and, by transitivity, $ad \preceq w$. Therefore $w \preceq ad$ and $ad \preceq w$, that is: $w =_s ad$.

Finally we prove that $w = ad$. We use that the ring A is reduced. Let $p \in \pi A$. If $w \notin p$ then $a - w \in p$, as $w(a-w) = 0 \in p$, and so $(a-w)d \in p$. Observe that $(w-a)d = wd - ad = w - ad \in p$. If $w \in p$, since $w =_s ad$ then $ad \in p$, and $w - ad \in p$. Therefore $w - ad \in \bigcap\big\{p : p \in \pi A\big\} = \{0\}$, because A is reduced. Then $w = ad$, and $b \mid ad$.

Altogether this shows $\exists d\,(d^2 = d \wedge ad \neq 0 \wedge d \preceq a \wedge b \mid ad)$, as required. ∎

Proposition 5.4 *Let A be a reduced and projectable f-ring and $a, b \in A$. The following assertions are equivalent:*

(i) $b \mid_{\mathrm{loc}} a \wedge a \neq 0$,

(ii) $\exists w\,\big(w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w\big)$,

(iii) $\exists e\,(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae)$.

Proof: Comes from Remarks 5.2 and 5.3. ∎

Remark 5.5 By contraposition, Proposition 5.4, proves the equivalence of the following assertions for the stated type of rings:

(i) $a \neq 0 \to b \nmid_{\mathrm{loc}} a$,

(ii) $\forall w\,\big(w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \to b \nmid w\big)$,

(iii) $\forall e\,(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \to b \nmid ae)$. ∎

Proposition 5.6 *Let A be a reduced and projectable f-ring, and $a, b \in A$. The following assertions are equivalent:*

(i) $\exists w\big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge$
$\forall w'(w' \neq 0 \wedge w'(a-w') = 0 \wedge w' \preceq a \wedge b \mid w' \to w' \preceq w)\big]$,

(ii) $\exists e\big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge$
$\forall e'(e'^2 = e' \wedge ae' \neq 0 \wedge e' \preceq a \wedge b \mid ae' \to e' \leqslant e)\big]$.

Proof: (ii)⇒(i). Suppose (ii) and set $w := ae$. Clearly, $w \neq 0$ and $w(a-w) = 0$ are proved as in Remark 5.2 above. Scaling $e \preceq 1$ by a yields $w \preceq a$. Evidently, $b \mid w$. Let $w' \neq 0$ with $w'(a-w') = 0$, $w' \preceq a$ and $b \mid w'$. As in the proof of Proposition 5.3 we set $1 = c' + d'$ with $c' \cdot w' = 0$ and $d' \preceq w'$. Then c' are d' idempotents. From $d' \preceq w'$ and $w' \preceq a$ follows $d' \preceq a$. As in the proof of Proposition 5.3, one deduces that $ad' \neq 0$. It can also be shown that $w' \preceq d'$, and therefore $w' =_s d'$. In a similar way it is shown that $w' = ad'$ and evidently $b \mid ad'$. From (ii) we get $d' \leqslant e$, that is $d' \preceq e$. Then, $ad' \preceq ae$, i.e., $w' \preceq ae = w$.

(i)⇒(ii). Let $w \in A$ satisfy (i). We set $1 = c + d$ with $cw = 0$ and $d \preceq w$. As before, c and d are idempotents such that $w = ad \neq 0$, $d \preceq a$ and $b \mid ad$. In fact, one has $w \preceq d$ and therefore $d =_s w$. The idempotent we are looking for is $e = d$. To finish the proof, let e' be an idempotent such that $ae' \neq 0$, $e' \preceq a$ and $b \mid ae'$. Setting $w' = ae' \neq 0$, we get $w'(a - w') = a^2 e'(1-e') = 0$, $w' \preceq a$ (scaling $e' \preceq 1$ by a), and $b \mid w'$. Then, (i) yields $w' \preceq w$, i.e., $ae' \preceq ad$. To conclude we want to show that $e' \leqslant d$. Let $p \in \pi A$ be such that $e' \notin p$, that is $e' - 1 \in p$. Since $e' \preceq a$ then $a \notin p$. Therefore $w' = ae' \notin p$, whenece $w = ad \notin p$. This means $a \notin p$ and $d \notin p$; in particular $d - 1 \in p$. We have shown that for all $p \in \pi A$, $e' \notin p$ implies $d \notin p$, i.e., $e' \preceq d$, and hence $e' \leqslant d$. ∎

The following proposition gives several statements that turn out to be equivalent to item (ii) in Proposition 5.6:

Proposition 5.7 *Let A be a reduced and projectable f-ring and $a, b \in A$. Then, the following assertions are equivalent on A:*

$$(i)\ \exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$$
$$\Big(a(1-e) \neq 0 \to \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \to b \nmid a(1-e)f\big)\Big)\Big],$$

$$(ii)\ \exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$$
$$\Big(a(1-e) \neq 0 \to \forall f\big(f^2 = f \wedge af \neq 0 \wedge f \preceq a(1-e) \to b \nmid af\big)\Big)\Big],$$

$$(iii)\ \exists w\Big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \big(a - w = 0 \to b \mid a\big) \wedge$$
$$\Big(a - w \neq 0 \to \forall w'\big(w' \neq 0 \wedge w'\big((a-w) - w'\big) = 0 \wedge w' \preceq a - w \to b \nmid w'\big)\Big)\Big].$$

Proof: (i)⇒(ii). Let $e \in A$ satisfy (i). Obviously e satisfies the first five conjuncts of (ii). Let us prove the last one. Suppose $a(1-e) \neq 0$ and let $f \in A$ be such that $f^2 = f$, $af \neq 0$ and $f \preceq a(1-e)$. Then $ef \preceq a(1-e) \cdot e = 0$, and $ef = 0$. Thus, $a(1-e)f = a(f - ef) = af \neq 0$ and $b \nmid af$. Hence e proves that (ii) holds.

(ii)⇒(iii). Let $e \in A$ be an element satisfying all six conjuncts of (ii). Set $w := ae$. Then $w \neq 0$, $w(a-w) = 0$, $w \preceq a$ ($e \preceq 1$ implies $ae \preceq a$) and $b \mid w$.

If $a - w = 0$ then $a - ae = a(1-e) = 0$, and by (ii) one has $b \mid a$. If $a - w \neq 0$ then $a(1-e) \neq 0$ and therefore:

$$(*) \qquad \forall f\big(f^2 = f \wedge af \neq 0 \wedge f \preceq a(1-e) \to b \nmid af\big).$$

We want to prove that $\forall w'\big(w' \neq 0 \wedge w'\big((a-w)-w'\big)=0 \wedge w' \preceq a-w \rightarrow b \nmid w'\big)$ holds in A. Let $w' \in A$ be such that $w' \neq 0$, $w'\big((a-w)-w'\big)=0$ and $w' \preceq a-w$. Since A is projectable, then $1=c'+d'$ with $c' \cdot w'=0$ and $d' \preceq w'$. Since $w' \preceq a-w$, we get $d' \preceq a-w=a-ae=a(1-e)$. Therefore $ed' \preceq ea(1-e)=a(1-e)e=0$, and $ed'=0$; that is $d' \leqslant 1-e$ (or $e \leqslant 1-d'$). One also has $c'd'=0$ and c', d' are idempotents. Also $w' \preceq d'$ because $\operatorname{Ann}(d') \subseteq \operatorname{Ann}(w')$: if $d'x=0$ with $x \in A$, then $xc'=x$, and $xw'=xc'w'=x0=0$, that is, $x \in \operatorname{Ann}(w')$. Then $w'=_s d'$.

Next we prove that $w'=a(1-e)d'$. Let $p \in \pi A$. If $w'+p=0$, then $w' \in p$; consequently $d' \in p$ and $a(1-e)d' \in p$; that is $a(1-e)d'+p=0$, and hence $w'+p=a(1-e)d'+p$. If $w'+p \neq 0$, since $w'\big((a-w)-w'\big)=0 \in p$, then $w'+p=(a-w)+p=a(1-e)+p$; in this case $d'+p \neq 0$ and therefore $d'+p=1+p$. Then $w'+p=a(1-e)d'+p$. This proves that $w'+p=a(1-e)d'+p$, for all $p \in \pi A$. Since A is reduced, $w'=a(1-e)d'$.

We have $d'^2=d'$ and $0 \neq w'=a(1-e)d'=a(d'-ed')=ad'$, since $ed'=0$. Then $ad' \neq 0$ and $d' \preceq a(1-e)$. By (*), $b \nmid ad'$. Since $a(1-e)d'=ad'$, we have $b \nmid a(1-e)d'$, that is $b \nmid w'$.

(iii)$\Rightarrow$(i). Let $w \neq 0$ satisfy (iii). By projectability of A, $1=c+d$ for some $c, d \in A$ such that $c \cdot w=0$ and $d \preceq w$. As in the proof of Proposition 5.3 we get $d=_s w$. It is also shown that $d \preceq a$ and $w=ad$ (A is reduced). Set $e:=d$; one has $w=ad \neq 0$ and $b \mid ad$. Clearly, if $a(1-d)=0$ then $b \mid a$. If $a(1-d) \neq 0$, we need to prove that

$$\forall f\left(f^2=f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid a(1-e)f\right)$$

holds in A. Let $f \in A$ satisfy the first three conjuncts. Setting $w':=a(1-e)f \neq 0$, we have:

$$w'\big((a-w)-w'\big)=w'\big((a-ad)-w'\big)=w'\big(a(1-d)-w'\big)=w'\big(a(1-e)-a(1-e)f\big)=$$
$$w'\big(a(1-e)(1-f)\big)=a(1-e)fa(1-e)(1-f)=\big(a(1-e)\big)^2 f(1-f)=\big(a(1-e)\big)^2 0=0.$$

Scaling $f \preceq 1$ by $a(1-e)$, we get $w'=a(1-e)f \preceq a(1-e)=a-w$. By (iii), $b \nmid w'$, that is, $b \nmid a(1-e)f$, as needed. ∎

Corollary 5.8 *Let A be a reduced and projectable f-ring and $a, b \in A$. Then, the following assertions are equivalent in A:*

(i) $$\exists e\Big[e^2=e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e)=0 \rightarrow b \mid a\big) \wedge \big(a(1-e) \neq 0 \rightarrow b \nmid_{\text{loc}} a(1-e)\big)\Big],$$

(ii) $$\exists w\Big[w \neq 0 \wedge w(a-w)=0 \wedge w \preceq a \wedge b \mid w \wedge \big(a-w=0 \rightarrow b \mid a\big) \wedge \big(a-w \neq 0 \rightarrow b \nmid_{\text{loc}} a-w\big)\Big].$$

Proof: Rewrite the last implication in statements (i) and (iii) of Proposition 5.7 using Remark 5.5, . ∎

Next we show that the assertions in Proposition 5.7 are equivalent to item (ii) in Proposition 5.6.

Proposition 5.9 *Let A be any ring, and $a, b \in A$. The following assertions are equivalent:*

(i) $\exists e\big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge$
$$\forall e'(e'^2 = e' \wedge ae' \neq 0 \wedge e' \preceq a \wedge b \mid ae' \to e' \leqslant e)\big],$$

(ii) $\exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$
$$\Big(a(1-e) \neq 0 \to \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \to b \nmid a(1-e)f\big)\Big)\Big].$$

(iii) $\exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$
$$\Big(a(1-e) \neq 0 \to b \nmid_{\mathrm{loc}} a(1-e)\Big)\Big].$$

Proof: (i)$\Rightarrow$(ii). Let $e \in A$ satisfy (i). We only need prove the last two conjuncts of (ii).

• If $a(1-e) = 0$ then $a = ae$, and $b \mid a$.

• If $a(1-e) \neq 0$, we need to prove the consequent of the last implication in (ii). For this, let $f \in A$ be such that $f^2 = f$, $a(1-e)f \neq 0$ and $f \preceq a(1-e)$. Suppose, by contradiction, that $b \mid a(1-e)f$. Scaling $f \preceq a(1-e)$ by e we get $ef = 0$. Then, $a(1-e)f = af$. Therefore $af \neq 0$ and $b \mid af$. Scaling $f \preceq a(1-e)$ by f one gets $f^2 \preceq a(1-e)f$, i.e., $f \preceq af$. Since $f \preceq 1$, then $af \preceq a$ and, by transitivity, $f \preceq a$. Thus, we have $f^2 = f$, $af \neq 0$, $b \mid af$ and $f \preceq a$. With $e' := f$, (i) yields $f \leqslant e$, i.e., $f^2 \leqslant ef = 0$. So, $f = 0$. But this contradicts our initial assumption that $af \neq 0$. So, we must have $b \nmid a(1-e)f$.

(ii)$\Rightarrow$(i). Let $e \in A$ satisfy (ii). Then e satisfies the first four conjuncts of (i); we prove that it satisfies the fifth one. Let $e' \in A$ be such that $e'^2 = e'$, $ae' \neq 0$, $e' \preceq a$ and $b \mid ae'$. We shall see that $e' \leqslant e$ or, equivalently, $e'(1-e) = 0$.

• If $a(1-e) = 0$ then $1 - e \in \mathrm{Ann}(a)$. Since $e' \preceq a$, we have $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(e')$ and hence $1 - e \in \mathrm{Ann}(e')$, that is: $e'(1-e) = 0$.

• If $a(1-e) \neq 0$, by (ii) we have:
$$\forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \to b \nmid a(1-e)f\big)\Big).$$
Suppose, towards a contradiction, that $e'(1-e) \neq 0$. First note that $a(1-e)e' \neq 0$: indeed, if $a(1-e)e' = 0$, then $(1-e)e' \in \mathrm{Ann}(a)$, and since $e' \preceq a$, we get $(1-e)e' \in \mathrm{Ann}(e')$, whence $(1-e)e' \cdot e' = (1-e)e' = 0$, contradicting $a(1-e)e' \neq 0$. Set $f := (1-e)e'$; obviously, $f^2 = f$. Note also that $a(1-e)f = a(1-e)e' \neq 0$. Since $e' \preceq a$, then $f = e'(1-e) \preceq a(1-e)$. Thus, $b \nmid a(1-e)f$. On the other hand, $a(1-e)f = a(1-e)(1-e)e' = a(1-e)e'$. Since $b \mid ae'$, we get $b \mid a(1-e)e'$. Thus, $b \mid a(1-e)e'$ and $b \nmid a(1-e)e'$, a contradiction. Therefore, $e'(1-e) = 0$, and $e' \leqslant e$, as required.

(ii)$\Leftrightarrow$(iii) is clear by Remark 5.5. ■

Remark 5.10 Alternatively, Proposition 5.9 can be formulated as follows: in any ring A and for any $a, b, e \in A$, the following assertions are equivalent:

(i) $e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \forall e'(e'^2 = e' \wedge ae' \neq 0 \wedge e' \preceq a \wedge b \mid ae' \to e' \leqslant e)$,

(ii) $e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$
$$\Big(a(1-e) \neq 0 \to \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \to b \nmid a(1-e)f\big)\Big).$$

(iii) $e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$
$$\Big(a(1-e) \neq 0 \to b \nmid_{\mathrm{loc}} a(1-e)\Big).$$

Observe that, by the proof of Proposition 5.9 the same idempotent $e \in A$ satisfies all three conditions (i), (ii) and (iii) above. Observe also that the element e given by Proposition 5.9 (i) is unique; for if $\bar{e}_1$ and $\bar{e}_2$ are idempotents satisfying 5.9 (i), then $\bar{e}_1 \leqslant \bar{e}_2$ and $\bar{e}_2 \leqslant \bar{e}_1$, i.e., $\bar{e}_1 = \bar{e}_2$.

■

Corollary 5.11 *Let A be a reduced and projectable f-ring, and let and $a, b \in A$. Then the following assertions are equivalent:*

(i) $\exists w \big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge$
$$\forall w'(w' \neq 0 \wedge w'(a-w') = 0 \wedge w' \preceq a \wedge b \mid w' \to w' \preceq w)\big],$$

(ii) $\exists w \Big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \big(a - w = 0 \to b \mid a\big) \wedge$
$$\Big(a - w \neq 0 \to \forall w'\big(w' \neq 0 \wedge w'\big((a-w) - w'\big) = 0 \wedge w' \preceq a - w \to b \nmid w'\big)\Big)\Big].$$

Proof: Follows from Propositions 5.7, 5.9 and 5.6.

■

Corollary 5.11 was proved under the hypotheses of projectability and reducibility of the ring. The author thinks this equivalence may be proved under more general assumptions. The following proposition uses divisible-projectability of the ring and proves that local divisibility is equivalent to an apparently stronger form.

Proposition 5.12 *Let B be a reduced and divisible-projectable f-ring, and let $a, b \in B$. The following assertions are equivalent:*

(i) $b \mid_{\mathrm{loc}} a$ *and* $a \neq 0$*;*

(ii) $\exists w \Big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \big(a - w = 0 \to b \mid a\big) \wedge$
$$\Big(a - w \neq 0 \to \forall w'\big(w' \neq 0 \wedge w'\big((a-w) - w'\big) = 0 \wedge w' \preceq a - w \to b \nmid w'\big)\Big)\Big].$$

Proof: (ii)⇒(i) is obvious.
(i)⇒(ii). Suppose $b \mid_{\mathrm{loc}} a$ and $a \neq 0$; then $b \neq 0$. Since B is divisible-projectable, there are $a_1, a_2 \in B$ such that $a = a_1 + a_2$, $a_1 \cdot a_2 = 0$, $b \mid a_1$ and $b \nmid_{\mathrm{loc}} a_2$ or $a_2 = 0$. If $a_1 = 0$, then $a_2 = a$, and $b \nmid_{\mathrm{loc}} a$, since $a \neq 0$. Therefore $a_1 \neq 0$, and $a_1 \cdot a_2 = 0$ says that $a_1(a - a_1) = 0$ with $b \mid a_1$. From $a_1(a - a_1) = 0$ comes $a_1 \preceq a$. Then there exists $w = a_1 \neq 0$ such that $w(a - w) = 0$, $w \preceq a$ and $b \mid w$. If $a_2 = 0$, then $a - a_1 = a - w = 0$ and $a_1 = a$ with $b \mid a$. If $a_2 \neq 0$, then $a_2 = a - a_1 \neq 0$ and one gets $b \nmid_{\mathrm{loc}} a - a_1$. By Fact 5.1, this non local divisibility means that $\forall w'\big(w' \neq 0 \wedge w'\big((a-w) - w'\big) = 0 \wedge w' \preceq a - w \to b \nmid w'\big)$, which proves (ii).

■

The following Proposition summarizes all equivalent forms of local divisibility in the class of reduced, projectable and divisible-projectable f-rings proved above.

Proposition 5.13 *Let A be a reduced projectable and divisible-projectable f-ring ; and let $a, b \in A$. The following assertions are equivalent:*

(i) $b \mid_{\mathrm{loc}} a \wedge a \neq 0$,

(ii) $\exists w\big(w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w\big)$,

(iii) $\exists e(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae)$,

(iv) $\exists e\big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge$
$$\forall e'(e'^2 = e' \wedge ae' \neq 0 \wedge e' \preceq a \wedge b \mid ae' \rightarrow e' \leqslant e)\big],$$

(v) $\exists w\big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge$
$$\forall w'(w' \neq 0 \wedge w'(a-w') = 0 \wedge w' \preceq a \wedge b \mid w' \rightarrow w' \preceq w)\big],$$

(vi) $$\exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \rightarrow b \mid a\big) \wedge \Big(a(1-e) \neq 0 \rightarrow \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid a(1-e)f\big)\Big)\Big],$$

(vi)′ $$\exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \rightarrow b \mid a\big) \wedge \Big(a(1-e) \neq 0 \rightarrow b \nmid_{\mathrm{loc}} a(1-e)\Big)\Big],$$

(vii) $$\exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \rightarrow b \mid a\big) \wedge \Big(a(1-e) \neq 0 \rightarrow \forall f\big(f^2 = f \wedge af \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid af\big)\Big)\Big],$$

(viii) $$\exists w\Big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \big(a-w = 0 \rightarrow b \mid a\big) \wedge \Big(a-w \neq 0 \rightarrow \forall w'\big(w' \neq 0 \wedge w'\big((a-w)-w'\big) = 0 \wedge w' \preceq a-w \rightarrow b \nmid w'\big)\Big)\Big],$$

(viii)′ $$\exists w\Big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \big(a-w = 0 \rightarrow b \mid a\big) \wedge \Big(a-w \neq 0 \rightarrow b \nmid_{\mathrm{loc}} a-w\Big)\Big].$$

■

6 Quantifier elimination.

In the previous section we exhibited several equivalents ways of expressing the local divisibility property in reduced projectable and divisible-projectable f-rings, and in particular in models of the theory T^*. One of these equivalences, the *maximal local divisibility* relation will give us more control over the fibers where the divisibility is carried out, a control necessary to prove our main quantifier elimination result. We define:

Definition 6.1 *Let A be any ring. We define a binary relation called* **maximal local divisibility** *by: for $a, b \in A$,*

$$b \mid^{\mathrm{m}}_{\mathrm{loc}} a \leftrightarrow a = 0 \vee \exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \rightarrow b \mid a\big) \wedge \Big(a(1-e) \neq 0 \rightarrow \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid a(1-e)f\big)\Big)\Big].$$

By Proposition 5.13, if A is any reduced projectable divisible-projectable f-ring, the maximal local divisibility relation can be expressed as follows:

$$A \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a \leftrightarrow a = 0 \vee \exists e \Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge \Big(a(1-e) \neq 0 \to b \nmid_{\mathrm{loc}} a(1-e)\Big)\Big].$$

In any ring A, we clearly have:

$$A \models \forall a \forall b \Big(b \mid^{\mathrm{m}}_{\mathrm{loc}} a \to a = 0 \vee \exists e \big(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \big)\Big);$$

Remark 5.2 gives:

$$A \models \forall a \forall b (b \mid^{\mathrm{m}}_{\mathrm{loc}} a \to b \mid_{\mathrm{loc}} a).$$

If, in addition, A is a reduced, projectable and divisible-projectable f-ring, then the equivalence of (i) and (vii) in Proposition 5.13 yields:

$$A \models \forall a \forall b (b \mid^{\mathrm{m}}_{\mathrm{loc}} a \leftrightarrow b \mid_{\mathrm{loc}} a).$$

Note that this equivalence is valid in any model of T^*, whence:

$$T^* \vdash \forall a \forall b (b \mid^{\mathrm{m}}_{\mathrm{loc}} a \leftrightarrow b \mid_{\mathrm{loc}} a).$$

In this section we consider the language $\mathcal{L} = \mathcal{L}_{\mathrm{lor}} \cup \{\mid, \preceq, \mid^{\mathrm{m}}_{\mathrm{loc}}\}$. First of all, we are going to adapt the characterizations of the universal theories given in section 4 to this new language $\mathcal{L}$. We have:

Proposition 6.2 *Let B be reduced, projectable and divisible-projectable f-ring. Then:*

$$B \models \forall a \forall b \forall c (a \not\preceq bc - a \to b \mid^{\mathrm{m}}_{\mathrm{loc}} a).$$

Proof: By the preceding remarks this follows from Proposition 4.15. ∎

Corollary 6.3 *Let B be a reduced, projectable and divisible-projectable f-ring and let A be any ring such that $A \subseteq B$ as a substructure in the language $\mathcal{L}$. Then:*

$$A \models \forall a \forall b \forall c (a \not\preceq bc - a \to b \mid^{\mathrm{m}}_{\mathrm{loc}} a).$$

Proof: Evident, since the formula in the statement is universal. ∎

Corollary 6.4 *Let $B \models T^*$ and A be a ring such that $A \subseteq B$ as a substructure in the language $\mathcal{L}$. Then:*

$$A \models \forall a \forall b \forall c (a \not\preceq bc - a \to b \mid^{\mathrm{m}}_{\mathrm{loc}} a).$$

Proof: In particular, B satisfies the hypothesis of Corollary 6.3. ∎

Definition 6.5 *Let A be any ring. We say that A has the* **maximal local divisibility property** *if*

$$A \models \forall a \forall b \forall c (a \not\preceq bc - a \to b \mid^{\mathrm{m}}_{\mathrm{loc}} a). \qquad (\star\star)$$

A version of Theorem 4.21 in the language $\mathcal{L}$ will be proved uponreplacing the local divisibility property by the *maximal* local divisibility property $(\star\star)$. We first prove:

Proposition 6.6 *Let A be a reduced f-ring satisfying the first convexity property, the divisibility glueing axiom scheme and the maximal local divisibility property. Then there exists $B \models T^*$ such that $A \subseteq B$ in the language $\mathcal{L}$.*

Proof: The maximal local divisibility property together with $\forall a \forall b (b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a \to b \,|_{\mathrm{loc}}\, a)$, implies the local divisibility property. By Theorem 4.21, there exists $B \models T^*$ such that $A \subseteq B$ in the language $\mathcal{L}_{\mathrm{lor}} \cup \{ \preceq, |, |_{\mathrm{loc}} \}$. We show that for all $a, b \in A$, $A \models b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a$ if and only if $B \models b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a$.

($\Rightarrow$) Suppose $A \models b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a$. Then $A \models b \,|_{\mathrm{loc}}\, a$. By the construction of B, $B \models b \,|_{\mathrm{loc}}\, a$. Since B is a model of T^*, $B \models b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a$.

($\Leftarrow$) Assume $B \models b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a$. If $a = 0$, then $A \models b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a$. Let $a \neq 0$. We know that $B \models b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a$ implies $B \models b \,|_{\mathrm{loc}}\, a$. Since A is a substructure of B in the language $\mathcal{L}_{\mathrm{lor}} \cup \{ \preceq, |, |_{\mathrm{loc}} \}$, then $A \models b \,|_{\mathrm{loc}}\, a$. Since $a \neq 0$, there exists $w \in A$, $w \neq 0$, $w(a - w) = 0$ and $b \mid w$, i.e., $bc = w$ for some $c \in A$. Since $w \neq 0$ and A is reduced, there exists $p \in \pi A$ such that $w + p \neq 0$, whence $w + p = a + p \neq 0$. Therefore, $bc + p = w + p = a + p$. Thus, $bc - a \in p$ and $a \notin p$, showing $a \not\preceq bc - a$ in A, for $a, b, c \in A$. Since A satisfies the maximal local divisibility property, we conclude that $A \models b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a$. ■

Theorem 6.7 *The universal theory of T^* in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$ is the theory of reduced f-rings satisfying the first convexity property, the divisibility glueing axiom scheme and the maximal local divisibility property.*

Proof: Follows from Propositios 6.6, Corollary 6.4 and Theorem 4.21. ■

We introduce a ternary relation expressing in a concise way the relationship between an idempotent e and elements a and b in Proposition 5.9 and in Remark 5.10:

$$\begin{aligned} \mathrm{Divloc}(b, a, e) \quad \leftrightarrow_{\mathrm{def}} \quad & (e^2 = e \wedge ae \neq 0 \wedge b \mid ae \wedge e \preceq a) \\ & \wedge \forall e' (e'^2 = e' \wedge ae' \neq 0 \wedge b \mid ae' \wedge e' \preceq a \to e' \leqslant e). \end{aligned}$$

Recall that the idempotent e satisfying $\mathrm{Divloc}(b, a, e)$ is unique, cf. Remark 5.10. The following Proposition states the fact that the support of the idempotent e satisfying $\mathrm{Divloc}(b, a, e)$ is determined by the fibers where b divides a and $a \neq 0$.

Proposition 6.8 *Let B be a reduced, projectable and divisible-projectable f-ring. Let $a, b, e \in B$ be such that $B \models \mathrm{Divloc}(b, a, e)$. Let $B \in \Gamma^a_{\mathcal{L}_{\mathrm{or}} \cup \{|\}}\big(X, (B_x)_{x \in X}\big)$, where X is a Boolean space and $(B_x)_{x \in X}$ is a family of totally ordered integral domains. Then:*

$$[\![\, e = 1 \,]\!] = [\![\, b \mid a \,]\!] \cap [\![\, a \neq 0 \,]\!] .$$

Proof: Let $x \in X$ be such that $x \in [\![\, e = 1 \,]\!]$. Then $e(x) = 1$, and since $e \preceq a$ then $a(x) \neq 0$. Also, $b \mid ae$ implies $b(x) \mid a(x)e(x) = a(x)$. Therefore, $x \in [\![\, b \mid a \,]\!] \cap [\![\, a \neq 0 \,]\!]$, proving the inclusion $\subseteq$. For the converse, assume $[\![\, e = 1 \,]\!] \subsetneq [\![\, b \mid a \,]\!] \cap [\![\, a \neq 0 \,]\!]$. Set

$$M := [\![\, b \mid a \,]\!] \cap [\![\, a \neq 0 \,]\!] \smallsetminus [\![\, e = 1 \,]\!] \neq \emptyset,$$

and let $e' \in B$ be defined by $e' = 1_{\restriction M} \cup 0_{\restriction X \smallsetminus M}$. Clearly $e'^2 = e'$ and $e' \preceq a$; moreover, $ae' \neq 0$, since $M \neq \emptyset$. By compactness of X and the patchwork property of B, one has $b \mid ae'$. Since $B \models \mathrm{Divloc}(b, a, e)$, we get $e' \leqslant e$. But $x_0 \in M$ entails $e'(x_0) = 1$ and $e(x_0) \neq 1$, i.e., $e(x_0) = 0$, contradicting $e' \leqslant e$ and the assumed strict inclusion. This proves Proposition 6.8. ■

The following Theorem shows that bringing the relation symbol $|^{\mathrm{m}}_{\mathrm{loc}}$ into the language entails that divisibility is preserved by local morphisms.

Theorem 6.9 *Let $A \models T^*_\forall$ in the language $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |^{\text{m}}_{\text{loc}}\}$ and let B be a reduced, projectable and divisible-projectable f-ring. Let $f\colon A \to B$ be a monomorphism in the language $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |^{\text{m}}_{\text{loc}}\}$. For $q \in \overline{\pi B}^{\text{con}}$ and $p \in \overline{\pi A}^{\text{con}}$ such that $p = f^{-1}(q)$, the map $f_{pq}\colon A/p \to B/q$ induced by f is a monomorphism in the language $\mathcal{L}' = \{0, 1, +, \cdot, \leqslant, |\}$.*

Proof. Clearly, $f_{pq}\colon A/p \to B/q, f_{pq}(a + p) = f(a) + q$, is a monomorphism of totally ordered integral domains. We need only show that f_{pq} preserves the divisibility relation, i.e., $A/p \models b+p \mid a+p$ if and only if $B/q \models f(b)+q \mid f(a)+q$, for $a, b \in A$. The direction $(\Rightarrow)$ is clear.
$(\Leftarrow)$ If $f(a) + q = 0$, this implication is clear, as $f(a) \in q$ and $a \in f^{-1}(q) = p$ imply $a + p = 0$. Assume $f(a) + q \neq 0$. Since $B/q \models f(b) + q \mid f(a) + q$, there is $c \in B$ so that $(f(b) + q)(c + q) = f(a) + q$, i.e., $f(b)c - f(a) \in q$; this, together with $f(a) \notin q$, yields $f(a) \not\preceq f(b)c - f(a)$. By Proposition 6.2 we get $f(b) \mid^{\text{m}}_{\text{loc}} f(a)$ and $f(a) \neq 0$. By Definition 6.1, there exists $\bar{e} \in B$ such that $\bar{e}^2 = \bar{e}$, $f(a)\bar{e} \neq 0$, $f(b) \mid f(a)\bar{e}$ and $\bar{e} \preceq f(a)$. The implications $\big(a(1 - \bar{e}) = 0 \to b \mid a\big)$ and $\Big(a(1 - \bar{e}) \neq 0 \to \forall f\big(f^2 = f \wedge\ a(1 - \bar{e})f \neq 0 \wedge f \preceq a(1-\bar{e}) \to b \nmid a(1-\bar{e})f\big)\Big)$ are also valid in B. The conclusion of this last implication is that $f(b) \nmid_{\text{loc}} f(a)(1 - \bar{e})$ holds in B. Since $B/q \models f(b) + q \mid f(a) + q \wedge f(a) + q \neq 0$, by Proposition 6.8 one has $\bar{e} - 1 \in q$, i.e., $\bar{e} + q = 1 + q$.

On the other hand, since $|^{\text{m}}_{\text{loc}}$ is in the language, we get $A \models b \mid^{\text{m}}_{\text{loc}} a$ and $a \neq 0$. Then there is an idempotent $e_A \in A$ such that $e_A^2 = e_A$, $ae_A \neq 0$, $b \mid ae_A$, $e_A \preceq a$, $\big(a(1 - e_A) = 0 \to b \mid a\big)$ and:

$$\Big(a(1 - e_A) \neq 0 \to \forall e'\big(e'^2 = e' \wedge a(1 - e_A)e' \neq 0 \wedge e' \preceq a(1 - e_A) \to b \nmid a(1 - e_A)e'\big)\Big).$$

Observe that the last formula can be written as $a(1 - e_A) \neq 0 \to b \nmid_{\text{loc}} a(1 - e_A)$, in A. Aplying the monomorphism f to the first five formulas, we obtain $f(e_A)^2 = f(e_A)$, $f(a)f(e_A) \neq 0$, $f(e_A) \preceq f(a)$, $f(b) \mid f(a)f(e_A)$ and $f(a)\big(1 - f(e_A)\big) = 0 \to f(b) \mid f(a)$. Next, we show that the formula $f(a)\big(1 - f(e_A)\big) \neq 0 \to f(b) \nmid_{\text{loc}} f(a)\big(1 - f(e_A)\big)$ holds in B. Otherwise, we have $f(a)\big(1 - f(e_A)\big) \neq 0$ and $f(b) \mid_{\text{loc}} f(a)\big(1 - f(e_A)\big)$ in B. Since B is a reduced, projectable and divisible-projectable f-ring, Proposition 5.13 entails $f(b) \mid^{\text{m}}_{\text{loc}} f(a)\big(1 - f(e_A)\big)$ in B. Since the relation symbol $|^{\text{m}}_{\text{loc}}$ is in the language, then $b \mid^{\text{m}}_{\text{loc}} a(1 - e_A)$ holds in A. Therefore $b \mid_{\text{loc}} a(1 - e_A)$ is valid in A with $a(1 - e_A) \neq 0$, contradicting that the formula $a(1 - e_A) \neq 0 \to b \nmid_{\text{loc}} a(1 - e_A)$ is true in A. It follows that the implication $f(a)\big(1 - f(e_A)\big) \neq 0 \to f(b) \nmid_{\text{loc}} f(a)\big(1 - f(e_A)\big)$ is valid in B. We have shown that both the elements $\bar{e}$ and $f(e_A)$ of B satisfy statement (ii) in Remark 5.10, at the values $f(a)$ and $f(b)$ (instead of a and b). Since this formula is equivalent to $\text{Divloc}(f(b), f(a), \cdot)$ in B, by uniqueness (cf. Remark 5.10 or Proposition 6.8) we conclude that $f(e_A) = \bar{e}$. Since $\bar{e} - 1 \in q$, then $f(e_A) - 1 = f(e_A - 1) \in q$, whence $e_A - 1 \in f^{-1}(q) = p$, i.e., $e_A + p = 1 + p$. Recalling that $b \mid ae_A$ in A, we get $(b + p) \mid (a + p)(e_A + p) = (a + p)(1 + p) = a + p$ in A/p, proving $A/p \models b + p \mid a + p$, as needed. ∎

Theorem 6.9 will be the main tool in the proof of quantifier elimination for the theory T^* in the language $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |^{\text{m}}_{\text{loc}}\}$. This will be done using the amalgamation property for models of the universal theory $T^*_\forall$.

The result in the next Lemma, needed below, is well known. For completeness'sake we include its proof.

Lemma 6.10 *If $(A_i)_{i\in I}$ is a family of totally ordered integral domains, then $A = \prod_{i\in I} A_i$ is a reduced, projectable and divisible-projectable f-ring.*

Proof: Clearly A is an f-ring. Since the A_i's are integral domains, for all $i \in I$, A is reduced. Next, we prove that A s projectable. Let $a = (a_i)_{i\in I} \in A$ and $b = (b_i)_{i\in I} \in A$. Let $I_0 := \{i \in I : a_i = 0\}$ and $I_1 := I \setminus I_0$. Define c and d in A by

$$c(i) = \begin{cases} b(i) & \text{if} \quad i \in I_0 \\ 0 & \text{if} \quad i \in I_1, \end{cases} \qquad \text{and} \qquad d(i) = \begin{cases} 0 & \text{if} \quad i \in I_0 \\ b(i) & \text{if} \quad i \in I_1. \end{cases}$$

Clearly, $b = c + d$ and $c \cdot a = 0$. It is easy to see that $d \preceq a$, as needed.

Now we show that A is divisible-projectable. Let $x, y \in A$ with $y \neq 0$. Set $I_0 := \{i \in I : y(i) \mid x(i)\}$ and $I_1 := I \setminus I_0 = \{i \in I : y(i) \nmid x(i)\}$, and define $z, w \in A$ by:

$$z(i) = \begin{cases} x(i) & \text{if} \quad i \in I_0 \\ 0 & \text{if} \quad i \in I_1, \end{cases} \qquad \text{and} \qquad w(i) = \begin{cases} 0 & \text{if} \quad i \in I_0 \\ x(i) & \text{if} \quad i \in I_1. \end{cases}$$

Clearly, $z \cdot w = 0$, $x = z + w$ and $y \mid z$.

Finally, let us see that $\forall w'(w' \neq 0 \wedge w'(w' - w) = 0 \to y \nmid w')$. Let $w' \in A$ be such that $w' \neq 0$ and $w'(w' - w) = 0$. There exists $i \in I$ such that $w'(i) \neq 0$. Then $w'(i) = w(i) \neq 0$, since A_i is an integral domain and $i \in I_1$ we get $y(i) \nmid x(i) = w(i)$. This yields $y \nmid w'$, showing that A is divisible-projectable. ■

Theorem 6.11 *The theory $T^*_\forall$ has the amalgamation property in $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |^{\text{m}}_{\text{loc}}\}$.*

Proof: Let $A, B, C \models T^*_\forall$ together with monomorphisms $f : A \to B$ and $g : A \to C$ for the language $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |^{\text{m}}_{\text{loc}}\}$. By Theorem 6.7 we may replace B and C by extensions that are models of T^*. Thus, we may assume without loss of generality, that B and C are reduced, projectable and divisible-projectable f-rings.

We need show that there exists $D \models T^*_\forall$ and monomorphisms $h : B \to D$ and $k : C \to D$ such that the following diagram is conmutative:

$$\begin{tikzcd}
 & B \arrow[dr, dashed, "h"] & \\
A \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\circlearrowright"] & & D, \\
 & C \arrow[ur, dashed, "k"'] &
\end{tikzcd}$$

that is: $h \circ f = k \circ g$.

As the radical relation is in the language, by [20, Theorem, p. 23, and Proposition (a) and (b), p. 22] there are continuous, surjective functions $\tilde{f} : \overline{\pi B}^{\text{con}} \to \overline{\pi A}^{\text{con}}$ and $\tilde{g} : \overline{\pi C}^{\text{con}} \to \overline{\pi A}^{\text{con}}$; that is:

$$\begin{tikzcd}
\overline{\pi B}^{\text{con}} \arrow[dr, "\tilde{f}"] & \\
 & \overline{\pi A}^{\text{con}}. \\
\overline{\pi C}^{\text{con}} \arrow[ur, "\tilde{g}"'] &
\end{tikzcd}$$

Since we assume that B and C are (unitary) projectable f-rings, by [16, 6.12] πB and πC are Boolean spaces; in particular, they are quasi-compact, hence proconstructible subsets of the Zariski spectra of B and C, cf. [26, Corollary 2.7]. The same arguments were done at the beginning of section 3, second paragraph, p. 8. Then $\overline{\pi B}^{\mathrm{con}} = \pi B$ and $\overline{\pi C}^{\mathrm{con}} = \pi C$. Thus, we have:

$$\begin{tikzcd}
\pi B \arrow[dr, "\tilde{f}"] & \\
& \overline{\pi A}^{\mathrm{con}}. \\
\pi C \arrow[ur, "\tilde{g}"'] &
\end{tikzcd}$$

In order to complete (dually) this diagram, we use the pullback of πB and πC over $\overline{\pi A}^{\mathrm{con}}$, given by:

$$X = \pi B \times_{\overline{\pi A}^{\mathrm{con}}} \pi C = \big\{(q_1, q_2) \in \pi B \times \pi C : \tilde{f}(q_1) = \tilde{g}(q_2)\big\}.$$

We then have the following diagram:

$$\begin{tikzcd}
& \pi B \arrow[dr, "\tilde{f}"] & \\
X \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rr, phantom, "\circlearrowleft"] & & \overline{\pi A}^{\mathrm{con}}. \\
& \pi C \arrow[ur, "\tilde{g}"'] &
\end{tikzcd}$$

All three spaces $\overline{\pi A}^{\mathrm{con}}, \pi B$ and πC are Boolean; hence so is $\pi B \times \pi C$, cf. [18, Corollary 3.14, p. 1249]. It is straighforward to prove that $X = \pi B \times_{\overline{\pi A}^{\mathrm{con}}} \pi C$ is a Boolean space. [4] Then, every element $\mathfrak{q} \in X$ is of the form $\mathfrak{q} = (q_1, q_2) \in \pi B \times \pi C$, with $\tilde{f}(q_1) = \tilde{g}(q_2)$, where $\tilde{f}\colon \pi B \to \overline{\pi A}^{\mathrm{con}}$ and $\tilde{g}\colon \pi C \to \overline{\pi A}^{\mathrm{con}}$ are continuous, surjective functions. Set $p := \tilde{f}(q_1) = \tilde{g}(q_2) \in \overline{\pi A}^{\mathrm{con}}$.

Any $p \in \overline{\pi A}^{\mathrm{con}} \subseteq \mathrm{Spec}(A)$ is a prime ideal. In order to prove that p is an ℓ-ideal, by [3, (8.2.1) and (2.2.1)(5)] it is suficient to see that p is convex and closed under absolute value. Since A satisfies the first convexity property, p is convex. It is easily seen that p is closed under absolute value: since A is an f-ring, $\big(x - |x|\big)\big(x + |x|\big) = 0$ holds for all $x \in p$. Since p is prime, either $x - |x| \in p$ or $x + |x| \in p$. In either case, $|x| \in p$. By [3, (9.2.5)], A/p is a totally ordered integral domain.

By Theorem 6.9, $f_{pq_1}\colon A/p \to B/q_1$ and $g_{pq_2}\colon A/p \to C/q_2$ are monomorphisms for the language $\mathcal{L}_{\mathrm{or}} \cup \{|\}$. Since the reduced f-ring A satisfies the first convexity property, then $A/p \models \mathrm{COVR} \cup \mathrm{OF}$, where COVR is the theory of convexely ordered valuation rings and OF is the theory of ordered fields, both in the language $\mathcal{L}_{\mathrm{or}} \cup \{|\}$ (cf. [2, Theorem 1]). Observe that B/q_1 and C/q_2 are real closed valuation rings (RCVR), as B and C are models of T^* (cf. [12, Corollary 2.11 and Proposition 2.4]). By quantifier elimination of RCVR in $\mathcal{L}_{\mathrm{or}} \cup \{|\}$ (cf. [7]), there exists a real closed valuation ring $R_{\mathfrak{q}}$ and monomorphisms $h_{\mathfrak{q}}\colon B/q_1 \to R_{\mathfrak{q}}$ and $k_{\mathfrak{q}}\colon C/q_2 \to R_{\mathfrak{q}}$ for the language $\mathcal{L}_{\mathrm{or}} \cup \{|\}$ such that the diagram:

[4] Even in the case where B and C are not projectable, it is possible to prove that $\overline{\pi B}^{\mathrm{con}} \times_{\overline{\pi A}^{\mathrm{con}}} \overline{\pi C}^{\mathrm{con}} \subseteq \overline{\pi B}^{\mathrm{con}} \times \overline{\pi C}^{\mathrm{con}}$ is a Boolean space.

$$\begin{tikzcd}[column sep=small] & B/q_1 \arrow[dr, "h_{\mathfrak{q}}"] & \\ A/p \arrow[ur, "f_{pq_1}"] \arrow[dr, "g_{pq_2}"'] \arrow[rr, phantom, "\circlearrowright"] & & R_{\mathfrak{q}}, \\ & C/q_2 \arrow[ur, "k_{\mathfrak{q}}"'] & \end{tikzcd}$$

is conmutative, i.e., $h_{\mathfrak{q}} \circ f_{pq_1} = k_{\mathfrak{q}} \circ g_{pq_2}$.

Now consider the ring $D = \prod_{\mathfrak{q}\in X} R_{\mathfrak{q}}$, and let $h\colon B \to D$ and $k\colon C \to D$ be given by:

$$h(b) = (h_{\mathfrak{q}}(b + q_1))_{\mathfrak{q}\in X} \quad \text{and} \quad k(c) = (k_{\mathfrak{q}}(c + q_2))_{\mathfrak{q}\in X},$$

for $b \in B$ and $c \in C$, where $\mathfrak{q} = (q_1, q_2) \in X$.

Claim 6.12 *h is a monomorphism in the language $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |^{\rm m}_{\rm loc}\}$.*

Proof: By Lemma 6.10, D is a reduced, projectable and divisible-projectable f-ring, and it is easily seen h is a morphism of lattice-ordered rings. Note also that when $\mathfrak{q}$ runs over X, then q_1 runs over πB, as $\tilde{f}$ is surjective. Therefore h is injective, whence a monomorphism of lattice-ordered rings (or of f-rings).

Next, we prove that h preserves the radical relation, namely: for all $b, b' \in B$, $b \preceq b'$ if and only if $h(b) \preceq h(b')$. The direction ($\Leftarrow$) is clear as $\preceq$ it is expressed by a universal formula. Conversely, suppose that $b \preceq b'$. Let $x = (x_{\mathfrak{q}})_{\mathfrak{q}\in X}$ be such that $h(b')x = 0$. Then $h_{\mathfrak{q}}(b' + q_1)x_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in X$. Fix an arbitrary $\mathfrak{q} \in X$. We have $h_{\mathfrak{q}}(b' + q_1) = 0$ or $x_q = 0$. Since $h_{\mathfrak{q}}$ is injective, $b' \in q_1$ or $x_{\mathfrak{q}} = 0$. From $b \preceq b'$ in B comes $b \in q_1$ or $x_{\mathfrak{q}} = 0$, whence $h_{\mathfrak{q}}(b + q_1)x_{\mathfrak{q}} = 0$. Since $\mathfrak{q} \in X$ is arbitrary we conclude that $h(b)x = 0$, and hence $h(b) \preceq h(b')$.

As our next order of business we prove that divisibility is preserved by h, i.e., for all $b, b' \in B$, $B \models b \mid b'$ if and only if $D \models h(b) \mid h(b')$. The implication ($\Rightarrow$) is clear since the formula defining divisibility is existential. Conversely, suppose that $h(b) \mid h(b')$ in D. This means that $h_{\mathfrak{q}}(b + q_1) \mid h_{\mathfrak{q}}(b' + q_1)$ in $R_{\mathfrak{q}}$, for all $\mathfrak{q} \in X$. Recall that whenever $\mathfrak{q} = (q_1, q_2)$ runs over all of X, then q_1 runs over all of πB. Therefore $h_{\mathfrak{q}}(b + q_1) \mid h_{\mathfrak{q}}(b' + q_1)$ in $R_{\mathfrak{q}}$ for all $q_1 \in \pi B$. Since each $h_{\mathfrak{q}}$ preserves divisibility, we have $b + q_1 \mid b' + q_1$ in B/q_1, for all $q_1 \in \pi B$. As B is projectable, compactness of πB and the patchwork property of B entail that $b \mid b'$ holds in B.

We show now that maximal local divisibility is preserved by h, that is: for $b, b' \in B$, $B \models b \;|^{\rm m}_{\rm loc}\; b'$ if and only if $D \models h(b) \;|^{\rm m}_{\rm loc}\; h(b')$. We first deal with the implication ($\Leftarrow$). Since $h(b) \;|^{\rm m}_{\rm loc}\; h(b')$, we have $h(b) \;|_{\rm loc}\; h(b')$. If $h(b') = 0$ then $b' = 0$, and hence $b \;|^{\rm m}_{\rm loc}\; b'$ in B. If $h(b') \neq 0$, there is $w = (w_{\mathfrak{q}})_{\mathfrak{q}\in X} \in D$ such that $w \neq 0$, $w\big(h(b') - w\big) = 0$ and $h(b) \mid w$ in D. Hence, there exists $\mathfrak{q} \in X$ such that $w_{\mathfrak{q}} \neq 0$. Then $w_{\mathfrak{q}} = h(b')_{\mathfrak{q}} = h_{\mathfrak{q}}(b' + q_1) \neq 0$, where $\mathfrak{q} = (q_1, q_2) \in X$. Since $h(b) \mid w$ in D, there is $c = (c_{\mathfrak{q}})_{\mathfrak{q}\in X}$ such that $h(b)c = w$. Then $h(b)_{\mathfrak{q}} c_{\mathfrak{q}} = w_{\mathfrak{q}} = h(b')_{\mathfrak{q}}$, that is: $h_{\mathfrak{q}}(b + q_1)c_{\mathfrak{q}} = h_{\mathfrak{q}}(b' + q_1)$. This shows that $h_{\mathfrak{q}}(b + q_1) \mid h_{\mathfrak{q}}(b' + q_1)$ in $R_{\mathfrak{q}}$. Since $h_{\mathfrak{q}}$ preserves divisiblity, we get $b + q_1 \mid b' + q_1$ in B/q_1, i.e., there is $c' \in B$ such that $bc' - b' \in q_1$. Remark that $b' \notin q_1$. Therefore $b' \not\preceq bc' - b'$ in B. Since $B \models T^*_\forall$, then B has the maximal local divisibility property: i.e. B satisfies

$\forall a \forall b \forall c (a \not\preceq bc - a \rightarrow b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a)$. Therefore $b \,|^{\mathrm{m}}_{\mathrm{loc}}\, b'$ in B, and we have proved that maximal local divisibility is downward preserved from D to B.

In particular, we showed that if $h(b) \,|^{\mathrm{m}}_{\mathrm{loc}}\, h(b')$ in D then $b \,|_{\mathrm{loc}}\, b'$ in B. Since D is a reduced projectable and divisible-projectable f-ring, $D \models h(b) \,|_{\mathrm{loc}}\, h(b') \rightarrow h(b) \,|^{\mathrm{m}}_{\mathrm{loc}}\, h(b')$. Therefore we have also proved that local divisibility is downward preserved from D to B.

As a last step, we prove that maximal local divisibility is upwards preserved from B to D, i.e., $b \,|^{\mathrm{m}}_{\mathrm{loc}}\, b'$ in B implies $h(b) \,|^{\mathrm{m}}_{\mathrm{loc}}\, h(b')$ in D. By hypothesis, there exists $e \in B$ such that $e^2 = e$, $b'e \neq 0$, $e \preceq b'$, $b \mid b'e$, $b'(1-e) = 0 \rightarrow b \mid b'$, and $b'(1-e) \neq 0 \rightarrow b \nmid_{\mathrm{loc}} b'(1-e)$ (see 5.13 (vi)$'$). Since h is an f-ring monomorphism that preserves the radical relation $\preceq$ and the divisibility relation $|$, we obtain $h(e)^2 = h(e)$, $h(b')h(e) \neq 0$, $h(e) \preceq h(b')$, $h(b) \mid h(b')h(e)$ and $h(b')\big(1 - h(e)\big) = 0 \rightarrow h(b) \mid h(b')$. Let us prove that the last implication is preserved by h, i.e., $h(b')\big(1-h(e)\big) \neq 0 \rightarrow h(b) \nmid_{\mathrm{loc}} h(b')\big(1-h(e)\big)$. Suppose $h(b')\big(1-h(e)\big) \neq 0$. Since h is a morphism, $b'(1-e) \neq 0$. By our assumption $b \nmid_{\mathrm{loc}} b'(1-e)$. Since local divisibility is downwards preserved from D to B, its negation is upwards preserved from B to D. Then, $h(b) \nmid_{\mathrm{loc}} h(b')\big(1-h(e)\big)$. Summarizing, we obtained: $h(e)^2 = h(e)$, $h(b')h(e) \neq 0$, $h(e) \preceq h(b')$, $h(b) \mid h(b')h(e)$, $h(b')\big(1-h(e)\big) = 0 \rightarrow h(b) \mid h(b')$ and $h(b')\big(1-h(e)\big) \neq 0 \rightarrow h(b) \nmid_{\mathrm{loc}} h(b')\big(1-h(e)\big)$. Then $h(b) \,|^{\mathrm{m}}_{\mathrm{loc}}\, h(b')$ holds in D, completing the proof that h preserves maximal local divisibility. □

A similar argument shows that the map $k\colon C \rightarrow D$ is a monomorphism in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$. It remains to be proved that $h \circ f = k \circ g$. Given $a \in A$, we have:

$$\begin{aligned}
(h \circ f)(a) = h\big(f(a)\big) &= \Big(h_{\mathfrak{q}}\big(f(a) + q_1\big)\Big)_{\mathfrak{q} \in X} &&= \Big(h_{\mathfrak{q}}\big(f_{pq_1}(a+p)\big)\Big)_{\mathfrak{q} \in X} \\
&= \Big(\big(h_{\mathfrak{q}} \circ f_{pq_1}\big)(a+p)\Big)_{\mathfrak{q} \in X} &&= \Big(\big(k_{\mathfrak{q}} \circ g_{pq_2}\big)(a+p)\Big)_{\mathfrak{q} \in X} \\
&= \Big(k_{\mathfrak{q}}\big(g_{pq_2}(a+p)\big)\Big)_{\mathfrak{q} \in X} &&= \Big(k_{\mathfrak{q}}\big(g(a) + q_2\big)\Big)_{\mathfrak{q} \in X} \\
&= k\big(g(a)\big) &&= (k \circ g)(a).
\end{aligned}$$

By Lenma 6.10, D is a reduced, projectable and divisible-projectable f-ring. It is easy to see that D satisfies the first convexity property. By Lemma 4.6, D satisfies the divisibility glueing axiom scheme. By Proposition 6.2, D satisfies the maximal local divisibility property. Then, we have $D \models T^*_\forall$ and that the following diagram conmutes:

$$\begin{tikzcd}
 & B \arrow[dr, "h"] & \\
A \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\circlearrowright"] & & D, \\
 & C \arrow[ur, "k"'] &
\end{tikzcd}$$

thus proving that $T^*_\forall$ has the amalgamation property in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$. ■

We can then state our main result:

Theorem 6.13 *T^* admits quantifier elimination in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$.*

Proof: By Theorem 3.2, T^* is model-complete in the language $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |_{\mathrm{loc}}\}$, hence also in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |_{\mathrm{loc}}\}$. Since $T^* \vdash \forall a \forall b (b \,|_{\mathrm{loc}}\, a \leftrightarrow b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a)$, then T^* is model-complete in $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |_{\text{loc}}^{\text{m}}\}$. By Theorem 6.11, $T_\forall^*$ has the amalgamation property in $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |_{\text{loc}}^{\text{m}}\}$. The conclusion follows from [6, Proposition 3.5.19.]. ■

Theorem 6.14 *T^* admits quantifier elimination in the language $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |_{\text{loc}}\}$.*

Proof: Clear from Theorem 6.13, using $T^* \vdash \forall a \forall b(b \mid_{\text{loc}} a \leftrightarrow b \mid_{\text{loc}}^{\text{m}} a)$. ■

Theorem 6.15 *The theory T^* is the model-completion of the theory of reduced f-rings satisfying the first convexity property, the divisibility glueing axiom scheme and the local divisibility property.*

Proof: Follows at once from the above and [6, Proposition 3.5.19.]. ■

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